# Graph Algorithms (part 3 of CSC 282),

http://www.cs.rochester.edu/~stefanko/Teaching/10CS282

## 1 Schedule

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| Homework is **due Thursday, Oct 21**. |
| The **QUIZ** will be on **Tuesday, Oct. 26**. |

## 2 List of algorithms covered in the class

(B-basic, I-intermediate, A-advanced):

B: DFS, BFS (p.85, DSV; p.105, DSV).
B: Topological sort (p.90, DSV).
A: Strongly connected components (p.91, DSV).
B: Dijkstra's algorithm (p.108, DSV).
I: Bellman-Ford algorithm (p.117, DSV).
A: Floyd-Warshall algorithm (p.173, DSV).
B: Prim's algorithm (p.137, DSV) (a.k.a., Borůvka's algorithm).
I: Kruskal's algorithm (p.131, DSV).
I: Union-find (p.132, DSV).
A: Maximum matching in bipartite graphs.
A: Maximum-weight matching in bipartite graphs.

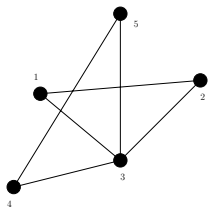
## 3 Basic material

**Important concepts, problems, theorems, and algorithms:**

- Graph, directed graph, representing graphs (adjacency list representation, adjacency matrix).
- Connected component, strongly connected component.
- Topological sorting.
- Shortest path.
- Minimum spanning tree.
- Matching, maximum matching, maximal matching, augmenting path.

**Mechanical problems (<u>solve, but do NOT turn in</u>):**

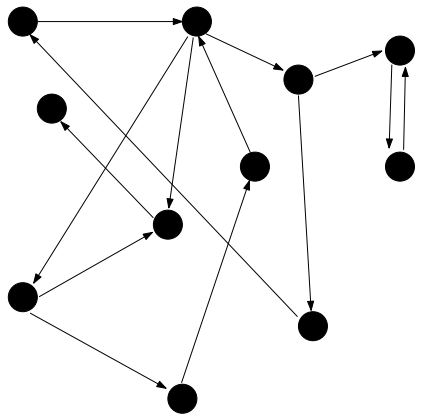
**3.1** Give the adjacency matrix representation and the adjacency lists representation of the following graph:



**3.2** Draw the graph with the following adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

**3.3** Find the strongly connected components of the following graph:



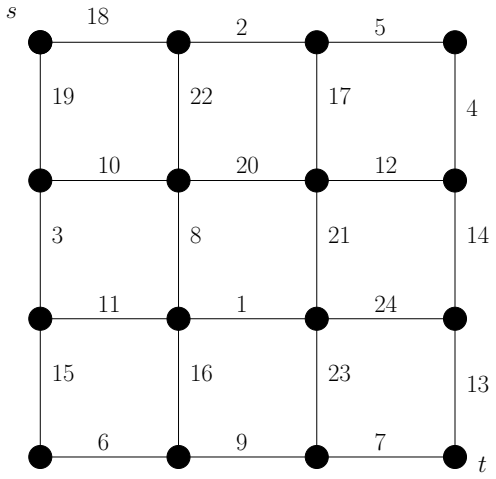
**3.4** Find a topological sorting of the following directed acyclic graph $G = (V, E)$ where

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},$$

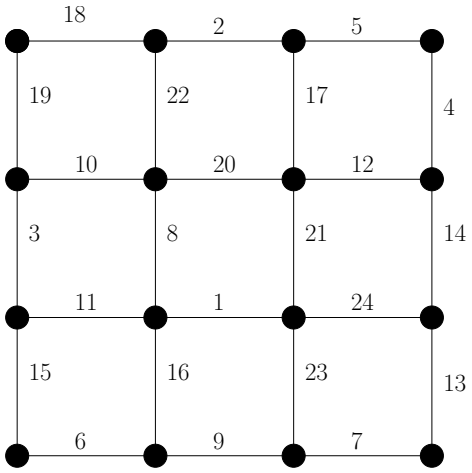
and

$$E = \{(7, 9), (9, 8), (6, 8), (10, 6), (2, 3), (3, 6), (1, 3), (4, 10), (5, 2), (7, 5)\}.$$

**3.5** Find the shortest $s - t$ path in the following graph:



**3.6** Find the minimum spanning tree of the following graph:




# 4 Homework

**3.7** (**due Thursday, Oct 21**) Let $G = (V, E)$ be a digraph given in the adjacency-list representation (i. e., for each vertex $v \in V$ we have a (linked) list of out-neighbors of $v$). Assume that $V = \{1, \ldots, n\}$.

1. Write pseudocode for a procedure which outputs an adjacency-list representation of the reverse digraph (i. e., $G$ with each edge reversed). The procedure should run in time $O(|V| + |E|)$.
2. Write pseudocode for a procedure which outputs the adjacency-list representation of $G$ in which the out-neighbors of each vertex are listed in the increasing order. The procedure should run in time $O(|V| + |E|)$.
3. Write pseudocode for a procedure which checks if $G$ is undirected (i. e., the reverse of every $e \in E$ is also in $E$). The procedure should run in time $O(|V| + |E|)$.

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**Definitions needed for problems 3.8 - 3.11 below:**

Let $G = (V, E)$ be an undirected graph. A *matching* of $G$ is a set of edges $E' \subseteq E$ such that every vertex of $G$ is in *at most one* edge of $E'$. A *perfect matching* of $G$ is a set of edges $E' \subseteq E$ such that every vertex of $G$ is in *exactly one* edge of $E'$. A matching $E'$ of $G = (V, E)$ is *maximal* if there does not exist $e \in E \setminus E'$ such that $E' \cup \{e\}$ is a matching. A matching $E'$ of $G = (V, E)$ is *maximum* if it has the largest cardinality (that is, the number of elements) among the matchings of $G$. To understand the definitions, try to answer the following two questions (do not turn in):

- Is every maximum matching maximal matching?
- Is every maximal matching maximum matching?

Let $G = (V, E, w)$ be an undirected graph. Let $E' \subseteq E$ be a matching of $G$. The weight of $E'$ is

$$\sum_{e \in E'} w(e).$$

We will also consider the following two problems:

Max-Weight Matching:
INPUT: weighted graph $G = (V, E, w)$
OUTPUT: a matching $E' \subseteq E$ of $G$ with the maximum weight.

**FACT (useful for some of the problems):** there is a polynomial-time algorithm to find the maximum-weight matching.

MAX-WEIGHT PERFECT MATCHING:
INPUT: weighted graph $G = (V, E, w)$
OUTPUT: a perfect matching $E' \subseteq E$ of $G$ with the maximum weight.

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**3.8** (**due Thursday, Oct 21**) Consider the following algorithm to find a maximal matching of a given input graph $G = (V, E)$:

- $E' \leftarrow \{\}$; $V' \leftarrow \{\}$;
- For each edge $\{u, v\} \in E$ do
  - if $u \notin V'$ and $v \notin V'$ then $E' \leftarrow E' \cup \{\{u, v\}\}$; $V' \leftarrow V' \cup \{u, v\}$

Give an example of a graph for which the algorithm does not find a maximum matching. (The algorithm is a little underspecified — it does not specify the order in which the edges are considered. You can pick any order.)

**3.9** (**due Thursday, Oct 21**) Prove that the algorithm from the previous problem always returns a matching whose size is at least half of the maximum.

**3.10** (**due Thursday, Oct 21**) Suppose that you have an algorithm (let's call it $A$) which solves the MAX-WEIGHT MATCHING problem on any weighed graph $G = (V, E, w)$ in time $O(E + V)$. Show how you can use algorithm $A$ to solve the MAX-WEIGHT PERFECT MATCHING problem in time $O(E + V)$. (HINT: consider the weighted graph $G' = (V, E, w')$ where $w'(e) = w(e) + C$ where $C$ is some large number.)

**3.11** (**due Thursday, Oct 21**)[**BONUS PROBLEM**] Suppose that you have an algorithm (let's call it $B$) which solves the MAX-WEIGHT PERFECT MATCHING problem on any weighed graph $G = (V, E, w)$ in time $O(E + V)$. Show how you can use the algorithm $B$ to solve the MAX-WEIGHT MATCHING problem in time $O(E + V)$.

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**3.12** (**due Thursday, Oct 21**) Let $G = (V, E)$ be an undirected graph. A 2-coloring of $G$ is an assignment $\phi : V \to \{1, 2\}$ (that is, each vertex gets one of two colors) such that for all $\{u, v\} \in E$ we have $\phi(u) \neq \phi(v)$ (that is, endpoints of each edge get different colors). Give an $O(m + n)$ algorithm which finds either a 2-coloring of a graph or an odd cycle.

**3.13** (**due Thursday, Oct 21**) Let $G = (V, E)$ be an undirected graph. Give an $O(m + n)$ algorithm that decides whether $G$ contains a cycle and if it does then it outputs one (the output should be a sequence of vertices in the order they occur on the cycle).

**3.14** (**due Thursday, Oct 21**) Show that if we exchange the lines "for $k = 1$ to $n$:" and "for $i = 1$ to $n$:" of Floyd-Warshall algorithm (DPV (draft), p. 188) the resulting algorithm does not correctly compute all-pairs-shortest paths. Give an example graph for which the modified algorithm fails.

**3.15** (**due Thursday, Oct 21**)[**BONUS PROBLEM**] The *max-weight* of a spanning tree $T$ is the maximum weight of an edge of $T$. A *min-max-weight spanning tree* is a spanning tree with the minimum max-weight. Give $O(E + V)$ algorithm which finds min-max-weight spanning tree of a given input graph $G$.

**3.16** (**due Thursday, Oct 21**) We are given $n$ rectangles of sizes $a_1 \times b_1, \ldots, a_n \times b_n$. We want to build the highest tower out of the rectangles. In a tower, if a rectangle of width $w$ is on top of a rectangle of width $w'$ then we require $w \leq w'$. We are allowed to rotate the rectangles (i. e., an $a \times b$ rectangle can be changed into a $b \times a$ rectangle). Give an $O(n)$ algorithm which finds the height of the highest tower.

(For example if the input is $11 \times 11, 8 \times 2, 1 \times 10$ then the solution is a tower of height $29 = 11 + 8 + 10$.)

**3.17** (**due Thursday, Oct 21**) [**BONUS PROBLEM**] Suppose that in Problem 3.16 we change the requirement $w \leq w'$ to $w < w'$, i. e., a rectangle on top of another rectangle has to be strictly thinner. Note that now it can happen that not all rectangles get used (e. g., if we have two squares of the same size). Give a polynomial-time algorithm for the modified problem.

(For example if the input is $2 \times 11, 2 \times 10, 10 \times 10$ then the solution is a tower of height $22 = 2 + 10 + 10$.)

**3.18** (**due Thursday, Oct 21**) [**BONUS PROBLEM**] We have $n$ jobs and $m$ machines. We are given $n \times m$ table $T$ where $T_{ij}$ is the time to complete job $i$ on machine $j$. Our task is to schedule the jobs on the machines to minimize the **average completion time**. Give a polynomial-time algorithm for the problem.

(For example if we have 3 jobs and 2 machines and

$$T = \begin{pmatrix} 1 & 3 \\ 3 & 2 \\ 4 & 1 \end{pmatrix},$$

then the following schedule is optimal:

- machine 1: 1;
- machine 2: 3, 2.

The completion times of jobs are: 1, 3, 1, the average completion time is 5/3.)

# 5 Additional problems from the book

Try to solve the following problems. A few of them will be on the quiz. We will go over the ones that you choose in the problem sessions.

- 3.1, 3.2, 3.3, 3.4, 3.6, 3.7, 3.8, 3.9, 3.11, 3.12, 3.13, 3.15, 3.18, 3.19, 3.21, 3.22, 3.23, 3.24, 3.25, 3.26, 3.27,
- 4.1, 4.2, 4.3, 4.4, 4.5, 4.7, 4.8, 4.10, 4.11, 4.13, 4.15, 4.17, 4.18, 4.19, 4.20, 4.21,
- 5.1, 5.2, 5.3, 5.7.