

## 1 Schedule

Problem session: Mon, Nov. 22, 5pm-6pm, CSB 601.

The homework is **Tuesday, Nov 23**.  
The **QUIZ** will be on **Tuesday, Nov 30**.

## 2 List of algorithms covered in the class

(B-basic, I-intermediate, A-advanced):

- I: Gaussian elimination (p. 219, DSV).
- I: Computing the dual of a linear program (p. 206, DSV).
- A: The simplex algorithm (p. 213, DSV).
- B: Zero-sum games using linear programming (p. 208, DSV).

## 3 Basic material

**Important concepts, problems, theorems, and algorithms:**

- system of linear equations,
- linear program, dual linear program.

**Basic problems (solve, do NOT hand in):**

**5.1** Solve the following linear program:

$$\begin{aligned} \max x_1 + x_2 \\ x_1 + 2x_2 &\leq 3 \\ 3x_1 + x_2 &\leq 4 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

**5.2** Write the following system of equations in the matrix form  $Ax = b$ :

$$\begin{aligned} y_1 + y_2 + y_4 &= 4, \\ y_1 + y_3 &= 10, \\ y_3 + 4y_4 &= 3, \\ y_2 + y_4 &= 1. \end{aligned}$$

**5.3** Compute the rank of the following two matrices

$$\begin{pmatrix} 2 & 4 & 9 \\ 1 & -3 & 5 \\ 0 & 10 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 9 & 3 \\ 1 & -3 & 5 & -1 \\ 0 & 10 & -1 & 5 \end{pmatrix}$$

Does the system  $2x + 4y + 9z = 3, x - 3y + 5z = -1, 10y - z = 5$  have a solution?

5.4 Compute the value of the zero-sum two person game given by the following payoff matrix:

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 4 & 2 \end{pmatrix}$$

5.5 Construct the linear program dual to the following linear program:

$$\begin{aligned} \max x_1 + 2x_2 + 3x_3 + 4x_4 \\ x_2 + x_3 + x_4 &\leq 1 \\ x_1 + x_3 + x_4 &\leq 2 \\ x_1 + x_2 + x_4 &\leq 3 \\ x_1 + x_2 + x_3 &\leq 4 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0. \end{aligned}$$

Find the optimal solution of the primal and the dual problem. Use a linear programming solver to obtain the solutions (for example you can use freeware `lpsolve` or function `Maximize` in MATHEMATICA (installed in most labs)).

5.6 Solve the following linear program:

$$\begin{aligned} \max \sum_{i=1}^9 x_i, \\ x_1 + x_2 &\leq 1, \\ x_2 + x_3 &\leq 1, \\ x_3 + x_4 &\leq 1, \\ x_4 + x_5 &\leq 1, \\ x_5 + x_6 &\leq 1, \\ x_6 + x_7 &\leq 1, \\ x_7 + x_8 &\leq 1, \\ x_8 + x_9 &\leq 1, \\ x_9 + x_1 &\leq 1, \\ x_i \geq 0, i = 1, \dots, 9. \end{aligned}$$

Use a linear programming solver to obtain the solution.

## 4 Homework

5.7 (due Nov 23) Krusty the Clown purchased a new super-efficient SUV with two engines. The first engine runs on a 2:1 mixture of gasoline and ethanol and gets 12 mpg. The second engine runs on a 1:2 mixture of gasoline and ethanol and gets 22 mpg. Krusty has 12 gallons of gasoline and 14 gallons of ethanol. How far can he get?

- Write a linear program for the problem. You are allowed to have only two variables in your linear program.
- Solve the linear program (you do not need to use the simplex method).
- Write down the dual of your linear program from part (a).
- Solve the dual program (you do not need to use the simplex method).

**5.8 (due Nov 23)** There are  $n$  bottles which contain different mixtures of three chemicals called  $A, B, C$ . The  $i$ -th bottle contains the chemicals in ratio  $a_i : b_i : c_i$  (thus,  $a_i/(a_i + b_i + c_i)$  fraction of the  $i$ -th bottle is chemical  $A$ ,  $b_i/(a_i + b_i + c_i)$  fraction of the  $i$ -th bottle is chemical  $B$ , and  $c_i/(a_i + b_i + c_i)$  fraction of the  $i$ -th bottle is chemical  $C$ ). We want to know whether it is possible to obtain a mixture containing the chemicals  $A, B, C$  in ratio  $a : b : c$  by mixing various amounts from the bottles. Give an efficient algorithm for this problem.

For example, if the input is  $n = 2$ , the ratios in the bottles are  $1 : 1 : 2$  and  $3 : 3 : 1$ , and we want to obtain mixture with ratio  $1 : 1 : 1$  then the answer is YES (we can take 2 parts from the first bottle and 1 part from the second bottle).

**5.9 (due Nov 23)** We are given  $n$  pairs of real numbers  $(x_1, y_1), \dots, (x_n, y_n)$ . We want to find real numbers  $a, b$  which minimize

$$\max_{i \in \{1, \dots, n\}} |ax_i + y_i - b|.$$

Write a linear program for this problem.

**5.10 (due Nov 23)** We are given  $n$  pairs of real numbers  $(x_1, y_1), \dots, (x_n, y_n)$ . We want to find real numbers  $a, b$  which minimize

$$\sum_{i \in \{1, \dots, n\}} |ax_i + y_i - b|.$$

Write a linear program for this problem.

**5.11 (due Nov 23)** We are given  $n$  pairs of real numbers  $(x_1, y_1), \dots, (x_n, y_n)$ . We want to find real numbers  $a, b$  which minimize

$$\sum_{i \in \{1, \dots, n\}} (ax_i + y_i - b)^2.$$

Find a formula for  $a$  and  $b$  (you do NOT need linear programming for this one).

## 5 Additional problems from the book (do not turn in)

Try to solve the following problems. A few of them will be on the quiz. We will go over the ones that you choose in the problem sessions.

- 7.1, 7.2, 7.3, 7.4, 7.6, 7.7, 7.8, 7.9, 7.11, 7.12, 7.13, 7.14, 7.15, 7.19, 7.27.