Recap
• Rooms adjacent to pits will have breezes

• Socrates is a person
  All people are mortal

• Anybody’s grandmother is either their mother’s or their father’s mother
Elements of First-Order Logic

- Objects (in the world)
- Relations among (tuples of) objects
- Mappings from (tuples of) objects to objects (i.e., objects identified in terms of other objects)
- Connectives
- Variables and Quantifiers
Syntax of First-Order Logic

- Constants
- Functional expressions
- Atomic sentences
- Connectives
- Variables and Quantifiers

Terms
Atomic Sentences

• Predicate symbol + list of terms (arguments)
  • \textit{Brother}(R, J)
  • \textit{Married}(\textit{Father}(\textit{Richard}), \textit{Mother}(\textit{John}))
Syntax of First-Order Logic

- Constants
- Functional expressions
- Atomic sentences
- Connectives
- Variables and Quantifiers

Terms
Complex Sentences

- Connectives combine sentences
  - $\neg \text{Brother}(\text{LeftLeg}(R), J)$
  - $\text{Brother}(R, J) \land \text{Brother}(J, R)$
  - $\neg \text{King}(R) \Rightarrow \text{King}(J)$
Syntax of First-Order Logic

- Constants
- Functional expressions
- Atomic sentences
- Connectives
- Variables and Quantifiers

Terms
Quantifiers

- If $\alpha$ is a sentence, then so are
  - $\forall x \, \alpha$
  - $\exists x \, \alpha$
Syntax of First-Order Logic

- Constants
- Functional expressions
- Atomic sentences
- Connectives
- Variables and Quantifiers
Semantics of First-Order Logic
Possible Worlds for PL

- Assignment of truth values to propositional variables
### Semantics of PL

<table>
<thead>
<tr>
<th>$B_{1,1}$</th>
<th>$P_{1,2}$</th>
<th>$P_{2,1}$</th>
<th>$P_{1,2} \lor P_{2,1}$</th>
<th>$B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})$</th>
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Possible Worlds for FOL

- **Domain**: Non-empty set of objects in the world
- **Relations**: Sets of tuples of objects
- **Functions**: Total mappings from tuples of objects to objects
Interpretation

• Mapping from elements a sentence to elements of a possible world

• Specifies exactly which objects, relations, and functions are referred to by the constant, predicate, and function symbols
Interpretation

- Domain of objects $D$

- Mapping from constant symbols to objects:
  $$I(c) \in D$$

- Mapping from predicate symbols to relations:
  $$I(p) \subseteq D^n$$  for $p$ of arity $n$

- Mapping from function symbols to (total) functions:
  $$I(f) : D^n \rightarrow D$$  for $f$ of arity $n$
Semantics of Unquantified Sentences

$P(\langle x_1, \ldots, x_n \rangle)$ is true in interpretation $I$ if

$\langle I(x_1), \ldots, I(x_n) \rangle \in I(P)$

Connectives: use truth tables (as for PL)
Semantics of Quantified Sentences
Extended Interpretation

- Adds mapping from variables to objects:

  \[ I(v) \in D \]
Semantics of Quantified Sentences

\( \forall x \alpha \) is true in interpretation \( I \) if

\( I'(\alpha) \) is true in every extended interpretation \( I' \)

\( \exists x \alpha \) is true in interpretation \( I \) if

\( I'(\alpha) \) is true in some extended interpretation \( I' \)
Semantics of FOL

• Atomic sentences: Use interpretation

• Complex sentences: Use semantics of connectives

• $\forall x \alpha$ : True in every extended interpretation

• $\exists x \alpha$ : True in at least one extended interp.


**Entailment**

- If $\alpha$ entails $\beta$: $\alpha \models \beta$
- Whenever $\alpha$ is true, so is $\beta$
- Every model of $\alpha$ is a model of $\beta$
- $\text{Models}(\alpha) \subseteq \text{Models}(\beta)$
- $\alpha$ is at least as strong an assertion as $\beta$
- Rules out no fewer possible worlds
137,506,194,466 models with \( \leq 6 \) objects!
Computing Entailment

- Number of models (probably) unbounded
- And anyway hard to evaluate truth in a model
- Can’t do model checking
- Look for inference rules, do theorem proving
First-Order Inference
• $\forall x \text{ King}(x) \Rightarrow \text{ Evil}(x)$

• $\text{King}(J)$
\[ \forall x \ King(x) \Rightarrow Evil(x) \]

- King(J)
- Evil(J)

“Modus Ponens”
Universal Instantiation

\[ \forall x \ P(x) \]
\[ \frac{}{P(a), \ P(b), \ P(c), \ P(f(a)), \ P(f(b)), \ P(g(a)), \ldots} \]
Propositionalization

\( \forall x \text{ King}(x) \Rightarrow \text{ Evil}(x) \)

\text{King}(R) \Rightarrow \text{Evil}(R)

\text{King}(J) \Rightarrow \text{Evil}(J)

\text{King}(\text{Father}(R)) \Rightarrow \text{Evil}(\text{Father}(R))

...
Propositionalization

- Convert FOL sentences to PL sentences and do PL inference on them
- Seems kind of indirect
- Technical problem enumerating all the ground terms (could be infinite; see book)
• $\forall x \text{ King}(x) \Rightarrow \text{ Evil}(x)$

• $\text{King}(J)$
• $\forall x \ King(x) \Rightarrow Evil(x)$

• $King(J)$
• King(J) $\Rightarrow$ Evil(J)

• King(J)
• King(J) ⇒ Evil(J)

• King(J)

• Evil(J)
• $\forall x \ King(x) \land Greedy(x) \Rightarrow Evil(x)$

• $King(J)$

• $Greedy(J), Greedy(R)$
\[ \forall x \ (\text{King}(x) \land \text{Greedy}(x) \Rightarrow \text{Evil}(x)) \]

- King(J)
- Greedy(J), Greedy(R)
- Evil(J)
• \( \forall x \text{ King}(x) \land \text{ Greedy}(x) \Rightarrow \text{ Evil}(x) \)

• \( \text{King}(J) \)

• \( \text{Greedy}(J), \text{Greedy}(R) \)

• \( \text{Evil}(J) \)
• $\forall x \ King(x) \land Greedy(x) \Rightarrow Evil(x)$

• $King(J)$

• $Greedy(J), Greedy(R)$

• $Evil(J)$
• \( \forall x \ King(x) \land Greedy(x) \Rightarrow Evil(x) \)

• King(J)

• Greedy(J), Greedy(R)

• Evil(J)
Substitution

- Replacement (binding) of variables by terms
  - \{ x/J \}
  - \{ x/R, y/Father(J) \}
\begin{itemize}
\item $\forall x \ King(x) \land Greedy(x) \Rightarrow Evil(x)$
\item $King(J)$ \hspace{1cm} \{ $x/J$ \}
\item $Greedy(J), Greedy(R)$
\end{itemize}

If there is some substitution that makes the premises true, then the conclusion with the same substitution is also true.
• \( \forall x \text{ King}(x) \land \text{ Greedy}(x) \Rightarrow \text{ Evil}(x) \)

• \( \text{King}(J) \)

• \( \forall y \text{ Greedy}(y) \)

\{ x/J, y/J \}
Generalized Modus Ponens

\[ p'_1, p'_2, \ldots, p'_n, (p_1 \land p_2 \land \cdots \land p_n \Rightarrow q) \]
\[ \text{Subst}(\Theta, q) \]

\[ p_i, p'_i, q \] are atomic sentences
\[ \Theta \] is a substitution such that:
\[ \text{Subst}(\Theta, p'_i) = \text{Subst}(\Theta, p_i) \]
Lifted Inference Rule

• Inference rule lifted from ground (variable-free) propositional logic to first-order logic

• “The key advantage of lifted inference rules over propositionalization is that they make only those substitutions that are required to allow particular inferences to succeed”
Inference Rules

\[ \frac{\alpha \land \beta}{\alpha} \]

And-elimination

\[ \frac{\neg \neg \alpha}{\alpha} \]

Double negation

\[ \frac{\neg (\alpha \land \beta)}{\neg \alpha \lor \neg \beta} \]

DeMorgan’s Laws

\[ \frac{(\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)}{(\neg \neg \alpha)^{\land} (\beta \Rightarrow \alpha)} \]

Modus Ponens

\[ \frac{\alpha \leftrightarrow \beta}{\alpha \leftrightarrow \beta} \]

Definition of biconditional
Unit Resolution

\[ l_1 \lor \cdots \lor l_i \lor \cdots \lor l_k, \quad m \]

\[ l_1 \lor \cdots \lor l_{i-1} \lor l_{i+1} \cdots \lor l_k \]

\( l_i \), ..., \( l_k \) are literals
\( l_i \) and \( m \) are complementary

Positive literal: \( P \)
Negative literal: \( \neg P \)

Complementary literals
Resolution

\[
\begin{align*}
&l_1 \lor \cdots \lor l_i \lor \cdots \lor l_k, \quad m_1 \lor \cdots \lor m_j \lor \cdots \lor m_n \\
&\overline{l_1 \lor \cdots \lor l_{i-1} \lor l_{i+1} \cdots \lor l_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \cdots \lor m_n}
\end{align*}
\]

\[
l_1, \ldots, l_k, m_1, \ldots, m_n \text{ are literals} \\
l_i \text{ and } m_j \text{ are complementary}
\]

Technical note: Resulting clause must be factored to contain only one copy of each literal.
Resolution

- Sound
- Complete
- Can it be lifted to FOL?
CNF for FOL

• Every sentence of first-order logic can be converted into an inferentially equivalent sentence in conjunctive normal form (CNF)
CNF for FOL

- Eliminate implications
- Move negation inwards
- Standardize variables apart
- Skolemize existential variables
- Drop universal variables
- Distribute $\lor$ over $\land$
CNF for FOL

- Every sentence of first-order logic can be converted into an inferentially equivalent sentence in conjunctive normal form (CNF)
- Not logically equivalent
- Inferentially equivalent: (un)satisfiable iff original sentence is (un)satisfiable
Resolution

\[
\begin{align*}
& l_1 \lor \cdots \lor l_i \lor \cdots \lor l_k, \quad m_1 \lor \cdots \lor m_j \lor \cdots \lor m_n \\
\hline
& l_1 \lor \cdots \lor l_{i-1} \lor l_{i+1} \cdots \lor l_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \cdots \lor m_n
\end{align*}
\]

\[
l_1, \ldots, l_k, m_1, \ldots, m_n \text{ are literals} \\
l_i \text{ and } m_j \text{ are complementary}
\]

Technical note: Resulting clause must be \textit{factored} to contain only one copy of each literal.
• $\forall x \text{ King}(x) \land \text{ Greedy}(x) \implies \text{ Evil}(x)$

• \text{ King}(J)

• $\forall y \text{ Greedy}(y)$

\[
\{ \ x/J, \ y/J \ \}
\]
Unification

\[ \text{Unify}(p, q) = \Theta \text{ where } \text{Subst}(\Theta, p) = \text{Subst}(\Theta, q) \]
• Knows(John,x) and Knows(John,Jane)
  • \{ x/Jane \}
• Knows(John,x) and Knows(y, Bill)
  • \{x/Bill, y/John \}
• Knows(John,x) and Knows(y, Mother(y))
  • \{ y/John, x/Mother(John) \}
• Knows(John,x) and Knows(x,Elizabeth)
  • Fails
Unification

\[ \text{Unify}(p, q) = \Theta \text{ where } \text{Subst}(\Theta, p) = \text{Subst}(\Theta, q) \]

- Variables need to be standardized apart
- Occurs check
- Most general unifier: places the fewest restrictions on the variables
  - Is unique
  - AIMA Fig 9.1
FOL Resolution

\[
\begin{align*}
&\quad l_1 \lor \cdots \lor l_i \lor \cdots \lor l_k, \quad m_1 \lor \cdots \lor m_j \lor \cdots \lor m_n \\
\text{Subst}(\Theta, l_1 \lor \cdots \lor l_{i-1} \lor l_{i+1} \cdots \lor l_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \cdots \lor m_n) \\
\text{\large $\Theta = Unify(l_i, \neg m_j)$}
\end{align*}
\]
Proof by Resolution

- Convert KB to CNF
- Convert $\neg \alpha$ to CNF and add to KB
- Apply resolution rule to complementary clauses (with unification)
- Until you derive the empty clause
The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all its missiles were sold to it by Colonel West, who is American.

To prove: West is a criminal
It is a crime for an American to sell weapons to hostile nations.
\( \forall x, y, z \ American(x) \land Weapon(y) \land Sells(x, y, z) \land Hostile(z) \Rightarrow Criminal(x) \)

Nono ... has some missiles.
\( \exists x \ Owns(Nono, x) \land Missile(x) \)

All Nono’s missiles were sold to it by Colonel West.
\( \forall x \ Missile(x) \land Owns(Nono, x) \Rightarrow Sells(West, x, Nono) \)

Missiles are weapons.
\( \forall x \ Missile(x) \Rightarrow Weapon(x) \)

An enemy of America counts as “hostile.”
\( \forall x \ Enemy(x, America) \Rightarrow Hostile(x) \)

West is an American. Nono is an enemy of America.
\( American(West) \quad Enemy(Nono, America) \)
∀x, y, z American(x) ∧ Weapon(y) ∧ Sells(x, y, z) ∧ Hostile(z) ⇒ Criminal(x)

∃x Owns(Nono, x) ∧ Missile(x)

∀x Missile(x) ∧ Owns(Nono, x) ⇒ Sells(West, x, Nono)

∀x Missile(x) ⇒ Weapon(x)

∀x Enemy(x, America) ⇒ Hostile(x)

American(West)  Enemy(Nono, America)
Proof by Resolution

- Convert KB to CNF
- Convert \( \neg \alpha \) to CNF and add to KB
- Apply resolution rule to complementary clauses (with unification)
- Until you derive the empty clause
Convert to CNF

- Eliminate implications
- Move negation inwards
- Standardize variables apart
- Skolemize existential variables
- Drop universal variables
- Distribute $\lor$ over $\land$
∀x, y, z American(x) ∧ Weapon(y) ∧ Sells(x, y, z) ∧ Hostile(z) ⇒ Criminal(x)

Eliminate implications:
∀x, y, z ¬[American(x) ∧ Weapon(y) ∧ Sells(x, y, z) ∧ Hostile(z)] ∨ Criminal(x)

Move negation inwards (DeMorgan’s Laws)
∀x, y, z [¬American(x) ∨ ¬Weapon(y) ∨ ¬Sells(x, y, z) ∨ ¬Hostile(z)] ∨ Criminal(x)

Standardize variables apart

Skolemize existential variables

Drop universal quantifiers
¬American(x) ∨ ¬Weapon(y) ∨ ¬Sells(x, y, z) ∨ ¬Hostile(z) ∨ Criminal(x)

Distribute ∨ over ∧
\( \forall x, y, z \ American(x) \land Weapon(y) \land Sells(x, y, z) \land Hostile(z) \Rightarrow Criminal(x) \)

\( \neg American(x) \lor \neg Weapon(y) \lor \neg Sells(x, y, z) \lor \neg Hostile(z) \lor Criminal(x) \)
\[ \forall x, y, z \text{ American}(x) \land \text{ Weapon}(y) \land \text{ Sells}(x, y, z) \land \text{ Hostile}(z) \Rightarrow \text{ Criminal}(x) \]

\[ \neg \text{ American}(x) \lor \neg \text{ Weapon}(y) \lor \neg \text{ Sells}(x, y, z) \lor \neg \text{ Hostile}(z) \lor \text{ Criminal}(x) \]

\[ \exists x \text{ Owns}(\text{Nono}, x) \land \text{ Missile}(x) \]

\[ \text{Owns}(\text{Nono}, M_1), \text{Missile}(M_1) \]

\[ \forall x \text{ Missile}(x) \land \text{ Owns}(\text{Nono}, x) \Rightarrow \text{ Sells}(\text{West}, x, \text{Nono}) \]

\[ \neg \text{ Missile}(x) \lor \neg \text{ Owns}(\text{Nono}, x) \lor \text{ Sells}(\text{West}, x, \text{Nono}) \]

\[ \forall x \text{ Missile}(x) \Rightarrow \text{ Weapon}(x) \]

\[ \neg \text{ Missile}(x) \lor \text{ Weapon}(x) \]

\[ \forall x \text{ Enemy}(x, \text{America}) \Rightarrow \text{ Hostile}(x) \]

\[ \neg \text{ Enemy}(x, \text{America}) \lor \text{ Hostile}(x) \]

\[ \text{American}(\text{West}) \]

\[ \text{American}(\text{West}) \]

\[ \text{Enemy}(\text{Nono}, \text{America}) \]

\[ \text{Enemy}(\text{Nono}, \text{America}) \]
\neg American(x) \lor \neg Weapon(y) \lor \neg Sells(x, y, z) \lor \neg Hostile(z) \lor Criminal(x)

Owns(Nono, \text{M}_1), \text{Missile}(\text{M}_1)

\neg \text{Missile}(x) \lor \neg \text{Owns}(\text{Nono}, x) \lor Sells(\text{West}, x, \text{Nono})

\neg \text{Missile}(x) \lor \text{Weapon}(x)

\neg \text{Enemy}(x, \text{America}) \lor \text{Hostile}(x)

\text{American(\text{West})}

\text{Enemy(Nono, \text{America})}
Proof by Resolution

- Convert KB to CNF
- Convert $\neg \alpha$ to CNF and add to KB
- Apply resolution rule to complementary clauses (with unification)
- Until you derive the empty clause
“West is a criminal”

\[ \text{Criminal}(\text{West}) \]

Negate and convert to CNF

\[ \neg \text{Criminal}(\text{West}) \]
\neg American(x) \lor \neg Weapon(y) \lor \neg Sells(x, y, z) \lor \neg Hostile(z) \lor Criminal(x)

Owns(Nono, M_1), Missile(M_1)

\neg Missile(x) \lor \neg Owns(Nono, x) \lor Sells(West, x, Nono)

\neg Missile(x) \lor Weapon(x)

\neg Enemy(x, America) \lor Hostile(x)

American(West)

Enemy(Nono, America)

\neg Criminal(West)
Proof by Resolution

- Convert KB to CNF
- Convert $\neg \alpha$ to CNF and add to KB
- Apply resolution rule to complementary clauses (with unification)
- Until you derive the empty clause
\neg \text{American}(x) \lor \neg \text{Weapon}(y) \lor \neg \text{Sells}(x, y, z) \lor \neg \text{Hostile}(z) \lor \text{Criminal}(x)

\text{Owns}(\text{Nono}, M_1), \text{Missile}(M_1)

\neg \text{Missile}(x) \lor \neg \text{Owns}(\text{Nono}, x) \lor \text{Sells}(\text{West}, x, \text{Nono})

\neg \text{Missile}(x) \lor \text{Weapon}(x)

\neg \text{Enemy}(x, \text{America}) \lor \text{Hostile}(x)

\text{American}(\text{West})

\text{Enemy}(\text{Nono}, \text{America})

\neg \text{Criminal}(\text{West})
\neg \text{American}(x) \lor \neg \text{Weapon}(y) \lor \neg \text{Sells}(x, y, z) \lor \neg \text{Hostile}(z) \lor \text{Criminal}(x)

\text{Owns}(\text{Nono}, M_1), \text{Missile}(M_1)

\neg \text{Missile}(x) \lor \neg \text{Owns}(\text{Nono}, x) \lor \text{Sells}(\text{West}, x, \text{Nono})

\neg \text{Missile}(x) \lor \text{Weapon}(x)

\neg \text{Enemy}(x, \text{America}) \lor \text{Hostile}(x)

\text{American}(\text{West})

\text{Enemy}(\text{Nono}, \text{America})

\neg \text{Criminal}(\text{West})
¬American(x) ∨ ¬Weapon(y) ∨ ¬Sells(x, y, z) ∨ ¬Hostile(z) ∨ Criminal(x)

¬Criminal( West )

{ x/West }

¬American( West ) ∨ ¬Weapon(y) ∨ ¬Sells( West, y, z) ∨ ¬Hostile(z)
Resolution Proof

Since $KB \cup \{\neg \text{Criminal}(\text{West})\}$ is unsatisfiable,

$$KB \models \text{Criminal}(\text{West})$$
Forward Chaining

• Knowledge base of definite clauses
• Starting from known facts, trigger rules whose premises are satisfied
• Using substitution to match
• Add their conclusions to the KB
• Until the query is answered or no new facts can be generated
Backward Chaining

• Work backward from the goal, chaining through rules to find known facts that support the proof
• Allow substitutions when matching facts and rules
• DFS => incomplete
• The basis of logic programming (Prolog)
FOL Inference

- Semantics of first-order sentences
- Entailment: same as PL!
- Lifted inference rules
- Resolution (first-order CNF, unification)
  - Proof by contradiction
- Forward and backward chaining
For Next Time:
Review for Midterm
AIMA Chapters 1–9
(per syllabus)

Exam issues: email me ASAP