Probabilistic Reasoning
Hunt The Wumpus

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A = Agent
B = Breeze
G = Glitter, Gold
OK = Safe square
P = Pit
S = Stench
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W = Wumpus

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1,1 & \text{V} & 2,1 & \text{B} \\
\end{array} \]

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• A pit causes a breeze in all neighboring squares.
• Each square other than [1,1] contains a pit with probability 0.2.
Diagnostic rule: Toothache $\Rightarrow$ Cavity
Diagnostic rule: \( \text{Toothache} \Rightarrow \text{Cavity} \)

\( \text{Toothache} \Rightarrow \text{Cavity} \lor \text{GumProblem} \lor \text{Abcess} \lor \ldots \)
Diagnostic rule:  \( \text{Toothache} \Rightarrow \text{Cavity} \)

Toothache \( \Rightarrow \) Cavity \( \lor \) GumProblem \( \lor \) Abcess \( \lor \) ...

Causal rule:  \( \text{Cavity} \Rightarrow \text{Toothache} \)
Approach

- Representing uncertain information
- Reasoning with uncertain information
Uncertainty
Sources of Uncertainty

Partial Observability

Nondeterminism
Sources of Uncertainty

Partial Observability and Nondeterminism
Probability
WARNING!

MATH AHEAD
Possible Worlds

Hungry=true, Cranky=false

Hungry=false, Cranky=true

Hungry=true, Cranky=true

Hungry=false, Cranky=false
Possible Worlds

Hungry $\lor$ Cranky

- Hungry=true, Cranky=false
- Hungry=false, Cranky=true
- Hungry=true, Cranky=true
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Possible Worlds

Hungry $\Rightarrow$ Cranky

- Hungry=true, Cranky=false
- Hungry=false, Cranky=true
- Hungry=false, Cranky=false
Possible Worlds

• Logical assertions rule out possible worlds
Possible Worlds

• Logical assertions rule out possible worlds
• Probabilistic assertions talk about how probable (likely) the possible worlds are
Sample Space

• In probability theory, the set of all possible worlds is called the sample space:
  \[ \Omega = \{ \omega_i \} \]

• Possible worlds \( \omega_i \) are:
  • Mutually exclusive
  • Exhaustive
\[ \Omega = \{ \omega_i \} = \{ (1,1), (1,2), (2,1), (3,1), \ldots \} \]
Probability Model

• Assigns a numerical probability $P(\omega)$ to each possible world*, such that:

\[
0 \leq P(\omega) \leq 1 \\
\sum_{\omega \in \Omega} P(\omega) = 1
\]

*Discrete, countable set of worlds
Where Do Probabilities Come From?

- Assigns a numerical probability $P(\omega)$ to each possible world*, such that:

$$0 \leq P(\omega) \leq 1$$

$$\sum_{\omega \in \Omega} P(\omega) = 1$$

*Discrete, countable set of worlds
Degrees of Belief

- The degree to which an agent believes a possible world is the actual world
  - From 0: certainly not the case (i.e., false)
  - To 1: certainly is the case (i.e., true)
- Could come from statistical data, general principles, combination of evidence, ...
Probability Model

• Assigns a numerical probability $P(\omega)$ to each possible world*, such that:

\[ 0 \leq P(\omega) \leq 1 \]

\[ \sum_{\omega \in \Omega} P(\omega) = 1 \]

*Discrete, countable set of worlds
$P(\omega_i) = 1/36$ for all $\omega_i \in \Omega$
\[ P(\omega_i) = \frac{2}{36} \text{ if } \omega_i \in \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\} \]

\[ P(\omega_i) = \frac{2}{90} \text{ otherwise} \]
Propositions (Events)

• A proposition (event) corresponds to the set of possible worlds in which the proposition holds

\[ P(\phi) = \sum_{\omega \in \phi} P(\omega) \]
\[ P(\text{Total} = 11) = P((5, 6)) + P((6, 5)) = \frac{1}{36} + \frac{1}{36} = \frac{1}{18} \]
\[ P(Doubles) = \frac{1}{4} \]

\[ P((1, 1)) + P((2, 2)) + P((3, 3)) + P((4, 4)) + P(5, 5) + P((6, 6)) = \frac{1}{4} \]
Probability (so far)

- Sample space (possible worlds)
- Probability model (degrees of belief in worlds)
- Propositions (subsets of worlds in which proposition holds)
Programming Language for Uncertainty

Propositions + Probabilities $\implies$ Probability Statements
Unconditional (Prior) Probabilities

• Degrees of belief in propositions in the absence of any other information

\[ P(Doubles) = \frac{1}{4} \]

\[ P(cavity) = 0.2 \]
Conditional (Posterior) Probability

- Degree of belief in a proposition given some information (evidence)

\[ P(SnakeEyes \mid Die_1 = 1) = \frac{1}{4} \]
\[ P(\text{cavity} \mid \text{toothache}) = 0.6 \]

- Whenever evidence is true and we have no further information, conclude probability of proposition
Conditional Probability

\[ P(a \mid b) = \frac{P(a \land b)}{P(b)} \quad \text{when } P(b) > 0 \]
20 equally likely possible worlds
12 possible worlds where \( b \) is true \( P(b) = 12/20 \)
\( a \) is true in 6 of those 12 \( P(a \mid b) = 6/12 = 1/2 \)
20 equally likely possible worlds
12 possible worlds where \( b \) is true \( P(b) = 12/20 \)
6 worlds where \( a \land b \) is true: \( P(a \land b) = 6/20 \)

\[
P(a \mid b) = \frac{P(a \land b)}{P(b)} = \frac{6/20}{12/20} = \frac{1}{2}
\]
Conditional Probability

\[ P(a \mid b) = \frac{P(a \land b)}{P(b)} \quad \text{when } P(b) > 0 \]
Product Rule

\[ P(a \land b) = P(a \mid b)P(b) \]
Probability

Probability: Degree of belief in possible world

0 \leq P(\omega) \leq 1

\sum_{\omega \in \Omega} P(\omega) = 1

Probability of a proposition: Conditional probability:

P(\phi) = \sum_{\omega \in \phi} P(\omega)

P(a \mid b) = \frac{P(a \land b)}{P(b)} \text{ when } P(b) > 0

P(a \land b) = P(a \mid b)P(b)
Incomplete Information

Factored Representation

Variables & Values (domains)
Random Variables

- Take on values from a domain
- Boolean random variables have domain \{ true, false \}
- Domains can be finite or infinite
  - Discrete, infinite: integers
  - Continuous, infinite: reals
Random Variables

\[ \text{Die}_1 : \{1, 2, 3, 4, 5, 6\} \]

\[ \text{Total} : \{2, \ldots, 12\} \]

\[ \text{Doubles} : \{\text{true, false}\} \]

\[ \text{Weather} : \{\text{sunny, rain, cloudy, snow}\} \]
Atomic Propositions

• Restriction on possible values of a random variable (a.k.a. constraint)

• Including statement that a random variable takes on a particular value (i.e., domain restricted to a single value)
Atomic Propositions

Boolean random variables:

\[ \text{Doubles} = \text{true} \rightarrow \text{doubles} \]
\[ \text{Doubles} = \text{false} \rightarrow \neg \text{doubles} \]

Symbolic (unambiguous) value:

\[ \text{Weather} = \text{sunny} \rightarrow \text{sunny} \]

Ordered domains:

\[ 50 \leq \text{Weight} < 100 \]
\[ \text{NumberOfAtomsInUniverse} \geq 10^{70} \]
Connectives

Same connectives as propositional logic:

\( \land, \lor, \Rightarrow, \Leftrightarrow \)

\( Cavity = \{ \text{true, false} \} \)
\( Toothache = \{ \text{true, false} \} \)
\( Age = \{ \text{baby, child, teen, adult, senior} \} \)

\( P(cavity \mid \neg toothache \land \text{teen}) = 0.1 \)

\( P(Cavity = \text{true} \mid Toothache = \text{false} \land Age = \text{teen}) = 0.1 \)
The Language of Probability Assertions

- Random variables (and domains)
- Atomic propositions:
  - $Var = value$ (or $<, \leq, >, \geq$)
- Connectives
Probabilities and Propositions

\[ P(\phi) = \sum_{\omega \in \phi} P(\omega) \]
The Language of Probability Assertions

Weather : \{sunny, rain, cloudy, snow\}

\[ P(Weather = \text{sunny}) = 0.6 \]
\[ P(Weather = \text{rain}) = 0.1 \]
\[ P(Weather = \text{cloudy}) = 0.29 \]
\[ P(Weather = \text{snow}) = 0.01 \]
Probability Distributions

- Describe the probabilities of all the possible values of a random variable

\[ P(\text{Weather} = \text{sunny}) = 0.6 \]
\[ P(\text{Weather} = \text{rain}) = 0.1 \]
\[ P(\text{Weather} = \text{cloudy}) = 0.29 \]
\[ P(\text{Weather} = \text{snow}) = 0.01 \]

\[ \mathbf{P}(\text{Weather}) = \langle 0.6, 0.1, 0.29, 0.01 \rangle \]
Joint Distributions

- Distributions over multiple variables
- Describe probabilities of all combinations of the values of the variables
## Joint Distributions

\[ P(\text{Weather, Cavity}) \]

<table>
<thead>
<tr>
<th>Weather</th>
<th>Cavity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>true</td>
</tr>
<tr>
<td>sunny</td>
<td></td>
</tr>
<tr>
<td>rain</td>
<td></td>
</tr>
<tr>
<td>cloudy</td>
<td></td>
</tr>
<tr>
<td>snow</td>
<td></td>
</tr>
</tbody>
</table>
Joint Distributions

\[ P(\text{sunny}, \text{Cavity}) \]

<table>
<thead>
<tr>
<th>Weather</th>
<th>Sunny</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cavity</td>
<td>True</td>
</tr>
</tbody>
</table>

\[ P(\text{sunny}, \text{Cavity}) \]
Joint Distributions

\[ P(\text{sunny, cavity}) \]

<table>
<thead>
<tr>
<th>Weather</th>
<th>sunny</th>
<th>Cavity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>true</td>
</tr>
</tbody>
</table>

\[ P(\text{sunny, cavity}) = P(\text{sunny, cavity}) = P(\text{sunny} \land \text{cavity}) \]
Joint Distributions

- Distributions over multiple variables
- Describe probabilities of all combinations of the values of the variables
Conditional Distribution

\[ P(X \mid Y) \] gives the values of \[ P(X = x_i \mid Y = y_i) \]
for each \( i, j \) pair
Continuous Distributions

- Probability density functions (PDFs)

\[ P(x) = \lim_{dx \to 0} \frac{P(x \leq X \leq x + dx)}{dx} \]
Full Joint Probability Distribution

- Joint probability distribution over all the random variables
- Probabilities for every possible combination of values assigned to random variables
- Probabilities for every possible world
### Full Joint Probability Distribution

\[ P(Cavity, Toothache, Weather) \]

<table>
<thead>
<tr>
<th></th>
<th>toothache</th>
<th>¬toothache</th>
<th>cavity</th>
<th>¬cavity</th>
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</tr>
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<tbody>
<tr>
<td>sunny</td>
<td></td>
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</table>
Programming Language for Knowledge 3.0

- Probabilities
- Language of probability assertions
- Distributions, conditional distributions, joint distributions, full joint distributions
SO WHAT!
THE GOOD, THE MAD AND THE UGLY

EDITED BY STEFAN SCHRAJER
Probabilistic Inference
Probabilistic Inference

• Computing posterior probabilities for propositions given prior probabilities and observed evidence
\[ P(\text{Cavity}, \text{Toothache}, \text{Catch}) \]

<table>
<thead>
<tr>
<th></th>
<th>toothache</th>
<th>$\neg$toothache</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neg$catch</td>
<td>0.108</td>
<td>0.012</td>
</tr>
<tr>
<td>$\neg$catch</td>
<td>0.016</td>
<td>0.064</td>
</tr>
<tr>
<td>$\neg$cavity</td>
<td>0.072</td>
<td>0.008</td>
</tr>
<tr>
<td>$\neg$cavity</td>
<td>0.144</td>
<td>0.576</td>
</tr>
</tbody>
</table>
Probability of a Proposition

\[ P(\text{cavity} \lor \text{toothache}) = 0.28 \]

<table>
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Probability of a Proposition

\[ P(\text{cavity} \land \text{toothache}) = 0.12 \]

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Here, the probability of having a cavity and toothache together is 0.12.
Conditional Probability

\[
P(cavity \mid toothache) = \frac{P(cavity \land toothache)}{P(toothache)}
\]

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## Conditional Probability

\[
P(\text{cavity} \mid \text{toothache}) = \frac{P(\text{cavity} \land \text{toothache})}{P(\text{toothache})} = 0.12
\]

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Conditional Probability

\[ P(\text{cavity} \mid \text{toothache}) = \frac{P(\text{cavity} \land \text{toothache})}{P(\text{toothache})} = \frac{0.12}{0.2} \]

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Conditional Probability

\[ P(\text{cavity} \mid \text{toothache}) = \frac{P(\text{cavity} \land \text{toothache})}{P(\text{toothache})} = 0.6 \]

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Conditional Probability

\[
P(\text{cavity} \mid \text{toothache}) = \frac{P(\text{cavity} \land \text{toothache})}{P(\text{toothache})} = 0.6
\]

\[
P(\neg\text{cavity} \mid \text{toothache}) = \frac{P(\neg\text{cavity} \land \text{toothache})}{P(\text{toothache})}
\]
### Conditional Probability

\[
P(\neg \text{cavity} \mid \text{toothache}) = \frac{P(\neg \text{cavity} \land \text{toothache})}{P(\text{toothache})} = 0.4
\]

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\[ P(\text{cavity} \mid \text{toothache}) = \frac{P(\text{cavity} \land \text{toothache})}{P(\text{toothache})} = 0.6 \]

\[ P(\neg \text{cavity} \mid \text{toothache}) = \frac{P(\neg \text{cavity} \land \text{toothache})}{P(\text{toothache})} = 0.4 \]

\[ P(\text{Cavity} \mid \text{toothache}) = \langle 0.6, 0.4 \rangle \]
Probabilistic Inference

\[ P(\text{cavity} \mid \text{toothache}) = \frac{P(\text{cavity} \land \text{toothache})}{P(\text{toothache})} = 0.6 \]

\[ P(\neg\text{cavity} \mid \text{toothache}) = \frac{P(\neg\text{cavity} \land \text{toothache})}{P(\text{toothache})} = 0.4 \]
Normalization

\[ P(\text{cavity} | \text{toothache}) = \alpha P(\text{cavity} \land \text{toothache}) = \alpha \ 0.12 \]

\[ P(\neg\text{cavity} | \text{toothache}) = \alpha P(\neg\text{cavity} \land \text{toothache}) = \alpha \ 0.08 \]
Normalization

\[ P(\text{cavity} \mid \text{toothache}) = \alpha P(\text{cavity} \land \text{toothache}) = \alpha 0.12 \]

\[ P(\neg \text{cavity} \mid \text{toothache}) = \alpha P(\neg \text{cavity} \land \text{toothache}) = \alpha 0.08 \]

\[ P(\text{Cavity} \mid \text{toothache}) = \alpha \langle 0.12, 0.08 \rangle \]
Normalization

\[ P(\text{cavity} \mid \text{toothache}) = \alpha P(\text{cavity} \land \text{toothache}) = \alpha 0.12 \]

\[ P(\neg \text{cavity} \mid \text{toothache}) = \alpha P(\neg \text{cavity} \land \text{toothache}) = \alpha 0.08 \]

\[ P(\text{Cavity} \mid \text{toothache}) = \frac{1}{0.12 + 0.08} \langle 0.12, 0.08 \rangle \]
Normalization

\[ P(\text{cavity} \mid \text{toothache}) = \alpha P(\text{cavity} \land \text{toothache}) = \alpha 0.12 \]

\[ P(\neg\text{cavity} \mid \text{toothache}) = \alpha P(\neg\text{cavity} \land \text{toothache}) = \alpha 0.08 \]

\[ P(\text{Cavity} \mid \text{toothache}) = \langle 0.6, 0.4 \rangle \]

We didn’t know \( P(\text{toothache}) \)!
Inference
(Single Variable)

\[
P(X \mid e) = \alpha P(X, e) = \alpha \sum_y P(X, e, y)
\]

Query variable \(X : \text{Domain}(X) = \{x_1, \ldots, x_m\}\)

Evidence variables \(E : \{E_1, \ldots, E_k\}\)

Observations \(e : \{e_1, \ldots, e_k\} \text{ s.t. } E_i = e_i\)

Unobserved variables \(Y : \{Y_1, \ldots, Y_l\}\)

\(\text{Domain}(Y_i) = \{y_{i,1}, \ldots, y_{i,n_i}\}\)
Inference (Single Variable)

\[ P(X \mid e) = \alpha P(X, e) = \alpha \sum_y P(X, e, y) \]

For each possible value \( x_i \) for \( X \)

For each possible combination of values \( y \) for \( Y \)

Add \( P(x_i, e, y) \)

Result: vector \( P(X \mid e) = \langle P(x_i \mid e) \rangle = \langle P(X = x_i \mid e) \rangle \)
Inference (Single Variable)

\[ P(X \mid e) = \alpha P(X, e) = \alpha \sum_y P(X, e, y) \]

For each possible value \( x_i \) for \( X \)

For each possible combination of values \( y \) for \( Y \)

Add \( P(x_i, e, y) \)

\[ P(X = x_i, E_1 = e_1, \ldots, E_k = e_k, \ldots, Y_1 = y_{1, i_1}, \ldots, Y_l = y_{l, i_l}) \]

Result: vector \( P(X \mid e) = \langle P(x_i \mid e) \rangle = \langle P(X = x_i \mid e) \rangle \)
Inference (Single Variable)

$$P(X \mid e) = \alpha P(X, e) = \alpha \sum_y P(X, e, y)$$

```c
int m;    // Number of values in domain of query variable X
int k;    // Number of evidence variables E
int[k] e; // e[i]: index of i’th evidence value in domain of Ei
int[l] n; // n[i]: number of values in domain of Yi
double[l][n[l]] D; // D[i][j]: j’th value of domain for Yi

for i from 1 to m
    PXe[i] = 0
    for i1 from 1 to n[1]
        for i2 from 1 to n[2]
            ...
            for il from 1 to n[l]
                PXe[i] += JPDF[i][e[1]]...[e[k]][D[1][i1]]...[D[l][il]]
return PXe // vector of length m
```

Full joint prob. dist.

$l$ nested loops
Inference
(Single Variable)

\[ P(X \mid e) = \alpha P(X, e) = \alpha \sum_y P(X, e, y) \]

Time Complexity  \( O(m^n) \)

Space Complexity  \( O(m^n) \)
Summary

- Uncertainty arises from partial observability and/or non-determinism
- Probability model (for us) assigns a degree of belief to possible worlds; satisfies axioms of probability theory
- Probabilities:
  - Unconditional (prior)
  - Conditional (posterior, given evidence)
Summary

• Factored representation: Random variables, $Var=value$ (atomic props), connectives

• Distributions: probabilities for all the values of a variable

• Set of variables: Joint distribution

• Inference: Computing posterior (conditional) probabilities for propositions given observed evidence
For Next Time:
AIMA Ch 14-14.5