

## Hamiltonian Cycle

# Hamiltonian Cycle Problem

## Hamiltonian Cycle

Given a directed graph  $G$ , is there a cycle that visits every vertex exactly once?

Such a cycle is called a **Hamiltonian cycle**.

# Hamiltonian Cycle is NP-complete

## Theorem

*Hamiltonian Cycle is NP-complete.*

*Proof.* First,  $\text{HamCycle} \in \text{NP}$ . Why?

Second, we show  $3\text{-SAT} \leq_P \text{Hamiltonian Cycle}$ .

Suppose we have a black box to solve Hamiltonian Cycle, how do we solve 3-SAT?

In other words: how do we encode an instance  $I$  of 3-SAT as a graph  $G$  such that  $I$  is satisfiable exactly when  $G$  has a Hamiltonian cycle.

Consider an instance  $I$  of 3-SAT, with variables  $x_1, \dots, x_n$  and clauses  $C_1, \dots, C_k$ .

# Reduction Idea

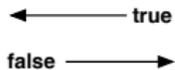
## Reduction Idea (very high level):

- Create some graph structure (a “gadget”) that represents the variables
- And some graph structure that represents the clauses
- Hook them up in some way that encodes the formula
- Show that this graph has a Ham. cycle iff the formula is satisfiable.

# Gadget Representing the Variables

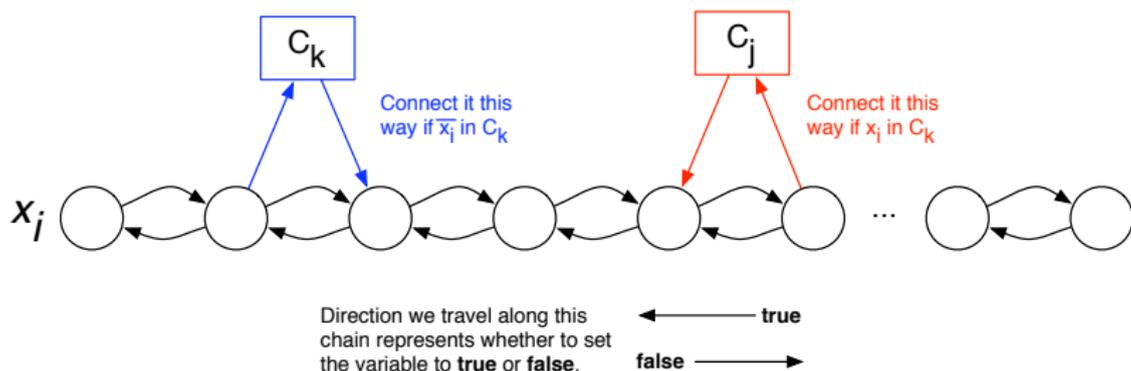


Direction we travel along this chain represents whether to set the variable to **true** or **false**.

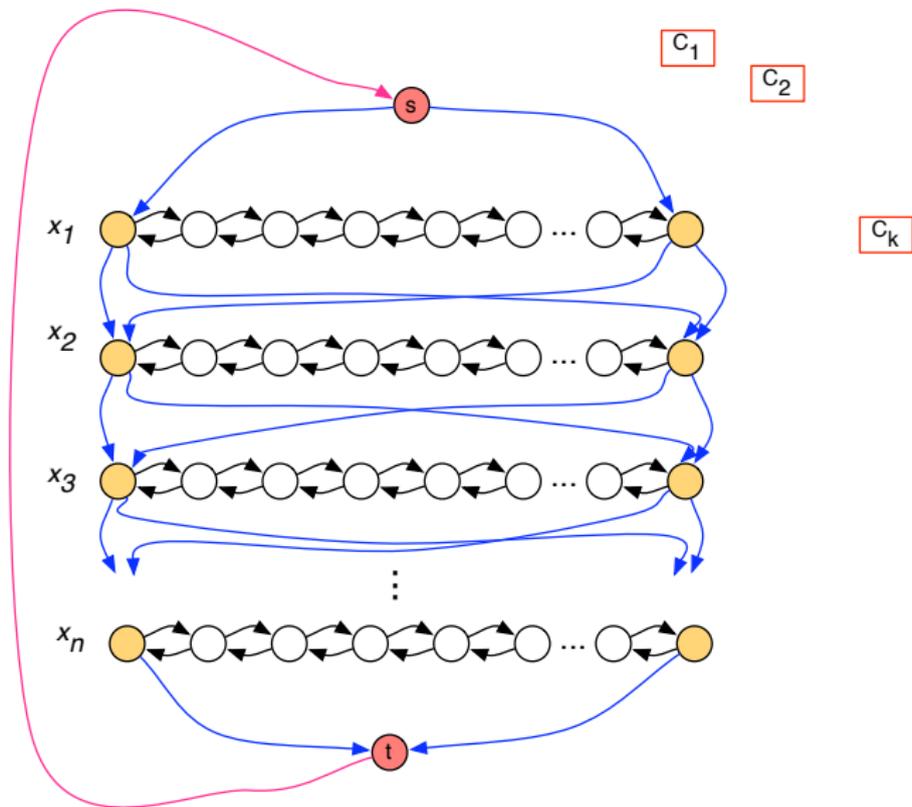


# Hooking in the Clauses

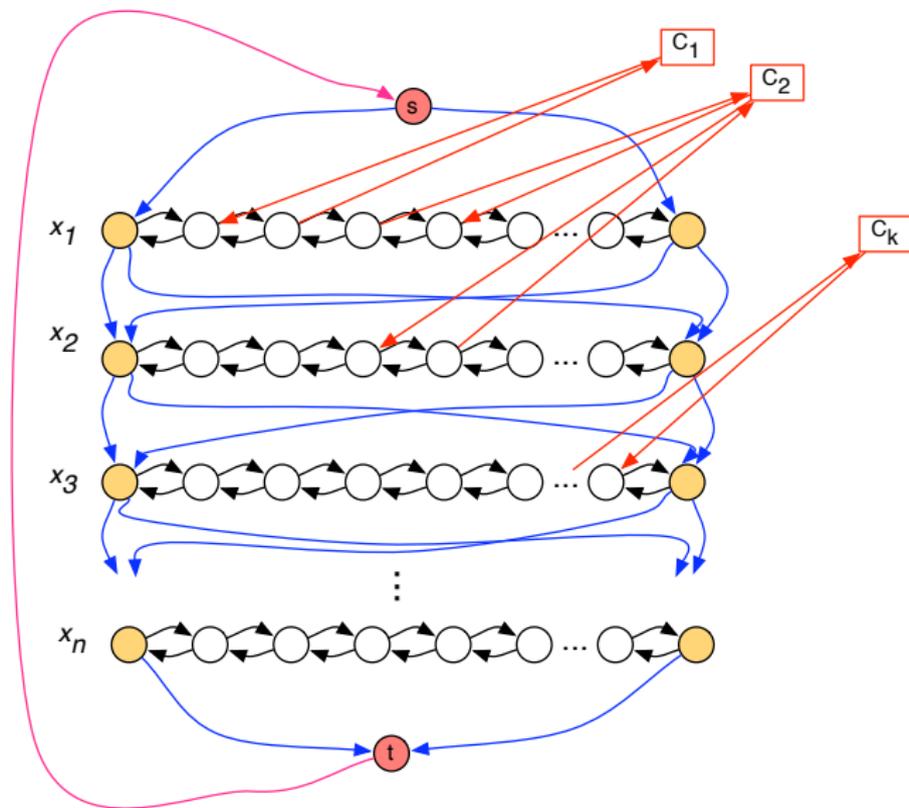
Add a new node for each clause:



# Connecting up the paths



# Connecting up the paths



# Hamiltonian Cycle is NP-complete

- A Hamiltonian path encodes a truth assignment for the variables (depending on which direction each chain is traversed)
- For there to be a Hamiltonian cycle, we have to visit every clause node
- We can only visit a clause if we satisfy it (by setting one of its terms to true)
- Hence, if there is a Hamiltonian cycle, there is a satisfying assignment

# Hamiltonian Path

**Hamiltonian Path:** Does  $G$  contain a **path** that visits every node exactly once?

How could you prove this problem is NP-complete?

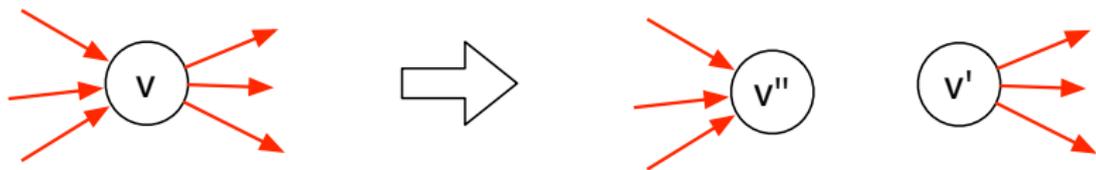
# Hamiltonian Path

**Hamiltonian Path:** Does  $G$  contain a **path** that visits every node exactly once?

How could you prove this problem is NP-complete?

Reduce Hamiltonian Cycle to Hamiltonian Path.

Given instance of Hamiltonian Cycle  $G$ , choose an arbitrary node  $v$  and split it into two nodes to get graph  $G'$ :



Now any Hamiltonian Path must start at  $v'$  and end at  $v''$ .

# Hamiltonian Path

$G''$  has a Hamiltonian Path  $\iff G$  has a Hamiltonian Cycle.

$\implies$  If  $G''$  has a Hamiltonian Path, then the same ordering of nodes (after we glue  $v'$  and  $v''$  back together) is a Hamiltonian cycle in  $G$ .

$\impliedby$  If  $G$  has a Hamiltonian Cycle, then the same ordering of nodes is a Hamiltonian path of  $G'$  if we split up  $v$  into  $v'$  and  $v''$ .  $\square$

Hence, **Hamiltonian Path** is NP-complete.

# Traveling Salesman Problem

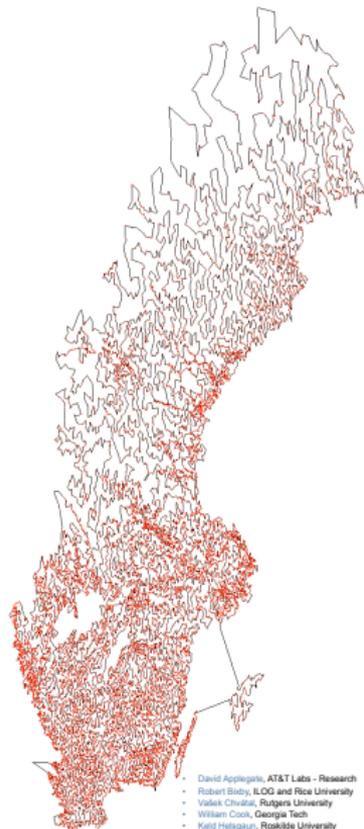
## Traveling Salesman Problem

Given  $n$  cities, and distances  $d(i, j)$  between each pair of cities, does there exist a path of length  $\leq k$  that visits each city?

### Notes:

- We have a distance between every pair of cities.
- In this version,  $d(i, j)$  doesn't have to equal  $d(j, i)$ .
- And the distances don't have to obey the triangle inequality ( $d(i, j) \leq d(i, k) + d(k, j)$  for all  $i, j, k$ ).

# TSP large instance



• David Applegate, AT&T Labs - Research  
• Robert Bixby, ILOG and Rice University  
• Vašek Chvátal, Rutgers University  
• William Cook, Georgia Tech  
• Keld Helsgaun, Roskilde University

<http://www.tsp.gatech.edu/sweden/index.html>

- TSP visiting 24,978 (all) cities in Sweden.
- Solved by David Applegate, Robert Bixby, Vašek Chvátal, William Cook, and Keld Helsgaun
- <http://www.tsp.gatech.edu/sweden/index.html>
- Lots more cool TSP at <http://www.tsp.gatech.edu/>

# Traveling Salesman is NP-complete

**Thm.** Traveling Salesman is NP-complete.

TSP seems a lot like Hamiltonian Cycle. We will show that

$$\text{HAMILTONIAN CYCLE} \leq_P \text{TSP}$$

To do that:

**Given:** a graph  $G = (V, E)$  that we want to test for a Hamiltonian cycle,

**Create:** an instance of TSP.

# Creating a TSP instance

A TSP instance  $D$  consists of  $n$  cities, and  $n(n - 1)$  distances.

**Cities** We have a city  $c_i$  for every node  $v_i$ .

**Distances** Let  $d(c_i, c_j) = \begin{cases} 1 & \text{if edge } (v_i, v_j) \in E \\ 2 & \text{otherwise} \end{cases}$

# TSP Reduction

## Theorem

$G$  has a Hamiltonian cycle  $\iff D$  has a tour of length  $\leq n$ .

*Proof.* If  $G$  has a Ham. Cycle, then this ordering of cities gives a tour of length  $\leq n$  in  $D$  (only distances of length 1 are used).

Suppose  $D$  has a tour of length  $\leq n$ . The tour length is the sum of  $n$  terms, meaning each term must equal 1, and hence cities that are visited consecutively must be connected by an edge in  $G$ .  $\square$

Also, TSP  $\in$  **NP**: a certificate is simply an ordering of the  $n$  cities.

# TSP is NP-complete

Hence, TSP is NP-complete.

Even TSP restricted to the case when the  $d(i, j)$  values come from actual distances on a map is NP-complete.