

Hamiltonian Cycle

Hamiltonian Cycle Problem

Hamiltonian Cycle

Given a directed graph G , is there a cycle that visits every vertex exactly once?

Such a cycle is called a **Hamiltonian cycle**.

Hamiltonian Cycle is NP-complete

Theorem

Hamiltonian Cycle is NP-complete.

Proof. First, $\text{HamCycle} \in \text{NP}$. Why?

Second, we show $3\text{-SAT} \leq_P \text{Hamiltonian Cycle}$.

Suppose we have a black box to solve Hamiltonian Cycle, how do we solve 3-SAT?

In other words: how do we encode an instance I of 3-SAT as a graph G such that I is satisfiable exactly when G has a Hamiltonian cycle.

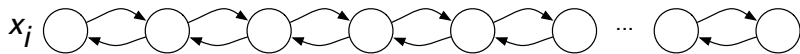
Consider an instance I of 3-SAT, with variables x_1, \dots, x_n and clauses C_1, \dots, C_k .

Reduction Idea

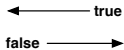
Reduction Idea (very high level):

- Create some graph structure (a “gadget”) that represents the variables
- And some graph structure that represents the clauses
- Hook them up in some way that encodes the formula
- Show that this graph has a Ham. cycle iff the formula is satisfiable.

Gadget Representing the Variables

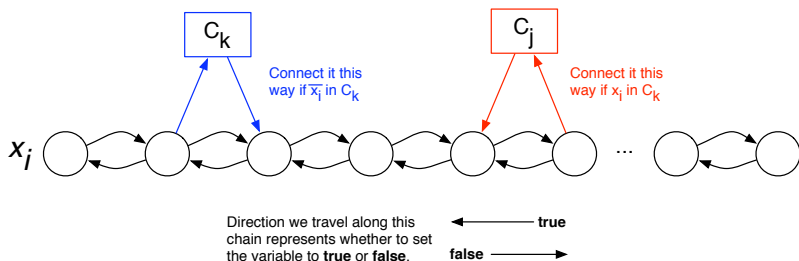


Direction we travel along this chain represents whether to set the variable to **true** or **false**.

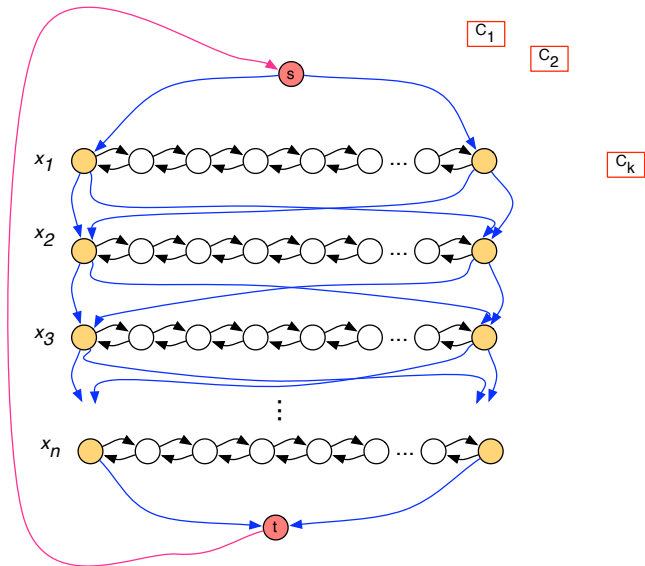


Hooking in the Clauses

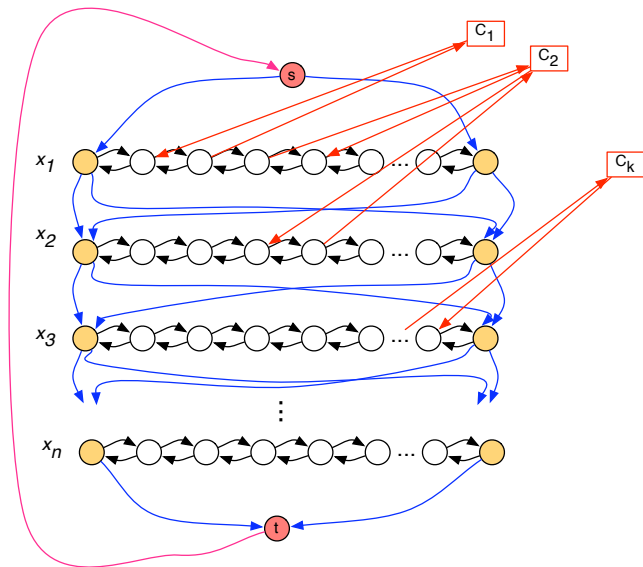
Add a new node for each clause:



Connecting up the paths



Connecting up the paths



Hamiltonian Cycle is NP-complete

- A Hamiltonian path encodes a truth assignment for the variables (depending on which direction each chain is traversed)
- For there to be a Hamiltonian cycle, we have to visit every clause node
- We can only visit a clause if we satisfy it (by setting one of its terms to true)
- Hence, if there is a Hamiltonian cycle, there is a satisfying assignment

Hamiltonian Path

Hamiltonian Path: Does G contain a **path** that visits every node exactly once?

How could you prove this problem is NP-complete?

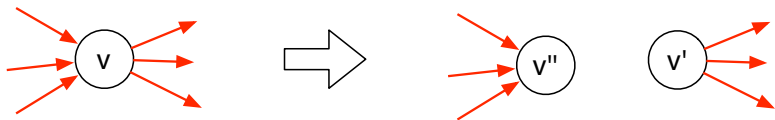
Hamiltonian Path

Hamiltonian Path: Does G contain a **path** that visits every node exactly once?

How could you prove this problem is NP-complete?

Reduce Hamiltonian Cycle to Hamiltonian Path.

Given instance of Hamiltonian Cycle G , choose an arbitrary node v and split it into two nodes to get graph G' :



Now any Hamiltonian Path must start at v' and end at v'' .

Hamiltonian Path

G'' has a Hamiltonian Path $\iff G$ has a Hamiltonian Cycle.

\implies If G'' has a Hamiltonian Path, then the same ordering of nodes (after we glue v' and v'' back together) is a Hamiltonian cycle in G .

\impliedby If G has a Hamiltonian Cycle, then the same ordering of nodes is a Hamiltonian path of G' if we split up v into v' and v'' . \square

Hence, **Hamiltonian Path** is NP-complete.

Traveling Salesman Problem

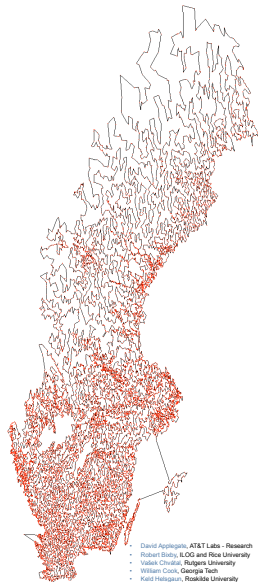
Traveling Salesman Problem

Given n cities, and distances $d(i, j)$ between each pair of cities, does there exist a path of length $\leq k$ that visits each city?

Notes:

- We have a distance between every pair of cities.
- In this version, $d(i, j)$ doesn't have to equal $d(j, i)$.
- And the distances don't have to obey the triangle inequality ($d(i, j) \leq d(i, k) + d(k, j)$ for all i, j, k).

TSP large instance



• David Applegate, AT&T Labs - Research
• Robert Bixby, ILOG and Rice University
• Vašek Chvátal, Rutgers University
• William Cook, Georgia Tech
• Keld Helsgaun, Roskilde University

<http://www.tsp.gatech.edu/sweden/index.html>

- TSP visiting 24,978 (all) cities in Sweden.
- Solved by David Applegate, Robert Bixby, Vašek Chvátal, William Cook, and Keld Helsgaun
- <http://www.tsp.gatech.edu/sweden/index.html>
- Lots more cool TSP at <http://www.tsp.gatech.edu/>

Traveling Salesman is NP-complete

Thm. Traveling Salesman is NP-complete.

TSP seems a lot like Hamiltonian Cycle. We will show that

$$\text{HAMILTONIAN CYCLE} \leq_P \text{TSP}$$

To do that:

Given: a graph $G = (V, E)$ that we want to test for a Hamiltonian cycle,

Create: an instance of TSP.

Creating a TSP instance

A TSP instance D consists of n cities, and $n(n - 1)$ distances.

Cities We have a city c_i for every node v_i .

Distances Let $d(c_i, c_j) = \begin{cases} 1 & \text{if edge } (v_i, v_j) \in E \\ 2 & \text{otherwise} \end{cases}$

TSP Reduction

Theorem

G has a Hamiltonian cycle $\iff D$ has a tour of length $\leq n$.

Proof. If G has a Ham. Cycle, then this ordering of cities gives a tour of length $\leq n$ in D (only distances of length 1 are used).

Suppose D has a tour of length $\leq n$. The tour length is the sum of n terms, meaning each term must equal 1, and hence cities that are visited consecutively must be connected by an edge in G . \square

Also, TSP \in **NP**: a certificate is simply an ordering of the n cities.

TSP is NP-complete

Hence, TSP is NP-complete.

Even TSP restricted to the case when the $d(i, j)$ values come from actual distances on a map is NP-complete.