The Self-Reducibility Technique

Qi Ge Xiaoming Gu

Department of Computer Science
University of Rochester

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Outline

Motivation

- Is there a sparse coNP-complete language?
- Is there a sparse NP-complete language?
- Is there a sparse language in NP$-$P?

Self-Reducibility

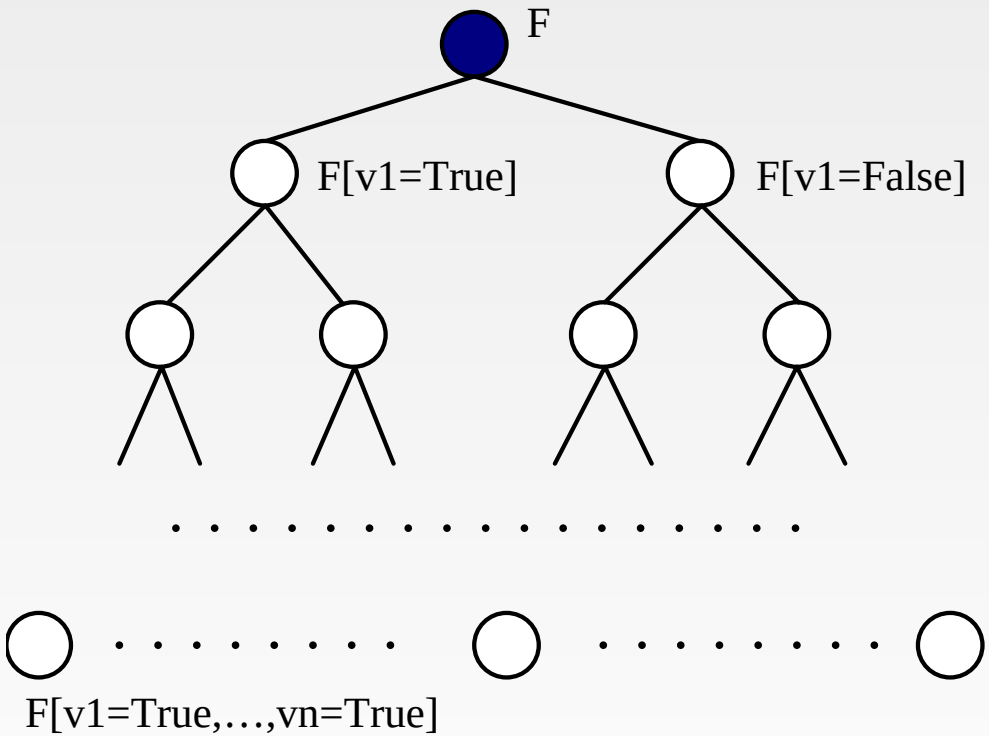
Definition

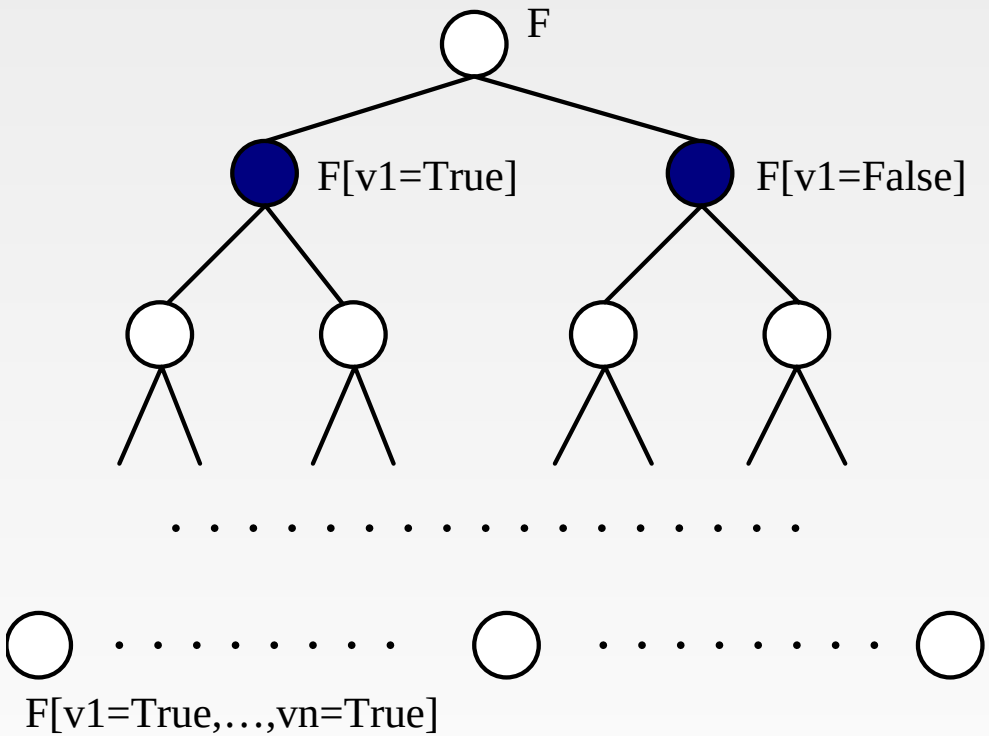
A language L is said to be self-reducible, if there is a deterministic polynomial-time TM M such that $L = L(M^L)$ and, for each input of length n, $M^L(x)$ queries the oracle for words of length, at most, $n - 1$.

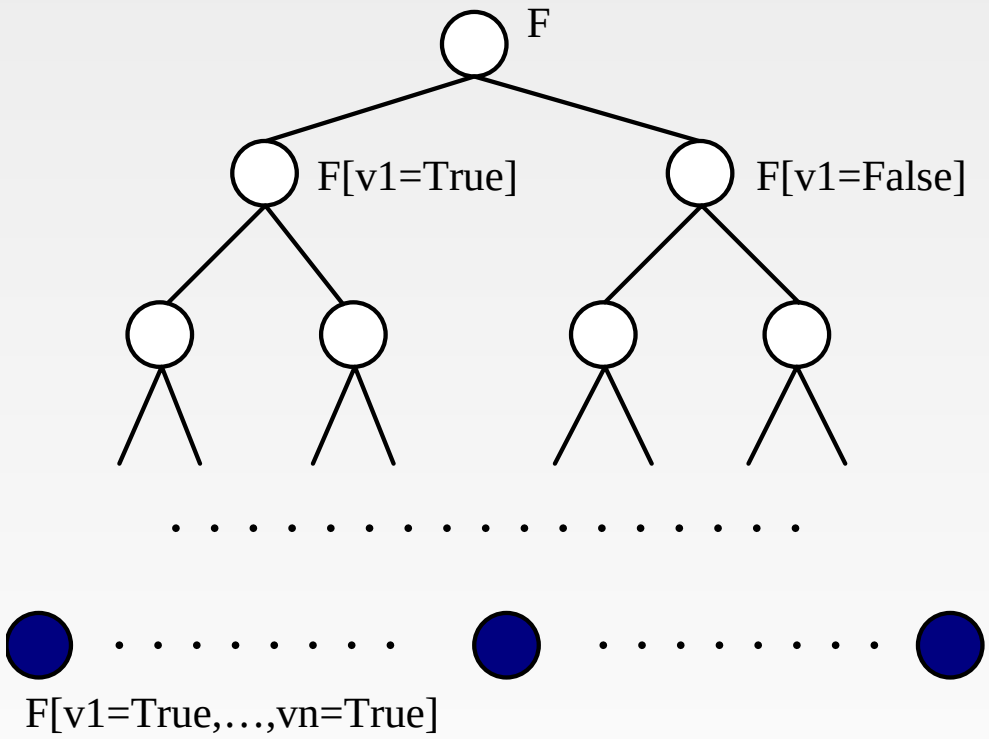
Example

SAT:

- $SAT = \{F | F \text{ is satisfiable.}\}$
- A formula F with variables $v_1, v_2, \ldots, v_n$ is satisfiable iff either $F[v_i = True]$ or $F[v_i = False]$ is satisfiable, for each $1 \leq i \leq n$.


F
F[v1=True]
F[v1=False]
F[v1=True,…,vn=True]


F
F[v1=True]
F[v1=False]
F[v1=True,…,vn=True]


F
F[v1=True]
F[v1=False]
F[v1=True,…,vn=True]

Sparse Sets

Definition

A set S is sparse if there is some polynomial p such that $|S^{\leq n}| = |\{x | x \in S \wedge |x| \leq n\}| \leq p(n)$.

Example

Tally set:

- $T \subseteq \{1\}^*$
- $p(n) = n + 1$
- There is some tally set which is even undecidable, e.g., $T_{HP} = \{1^{(1x)_{binary}} | x \in HP\}$.

Tally NP-complete Languages

- An easier problem: is there any tally NP-complete language?
- Tally set: $T \subseteq \{1\}^*$.

Tally NP-complete Languages (Cont.)

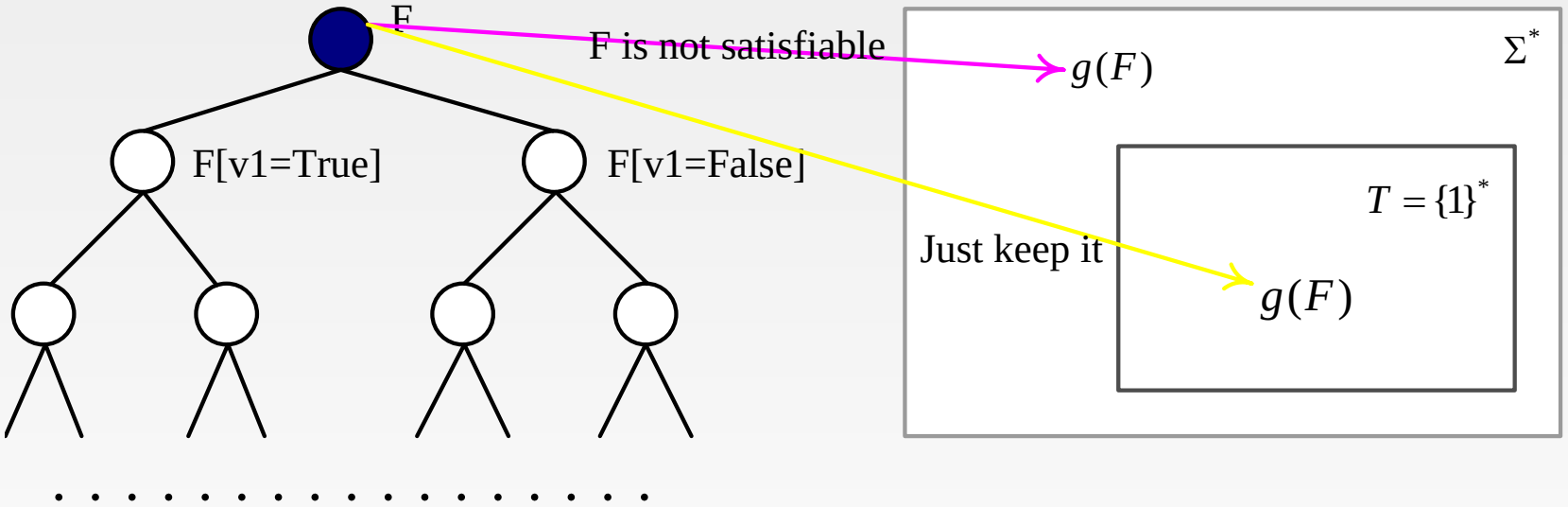
Theorem

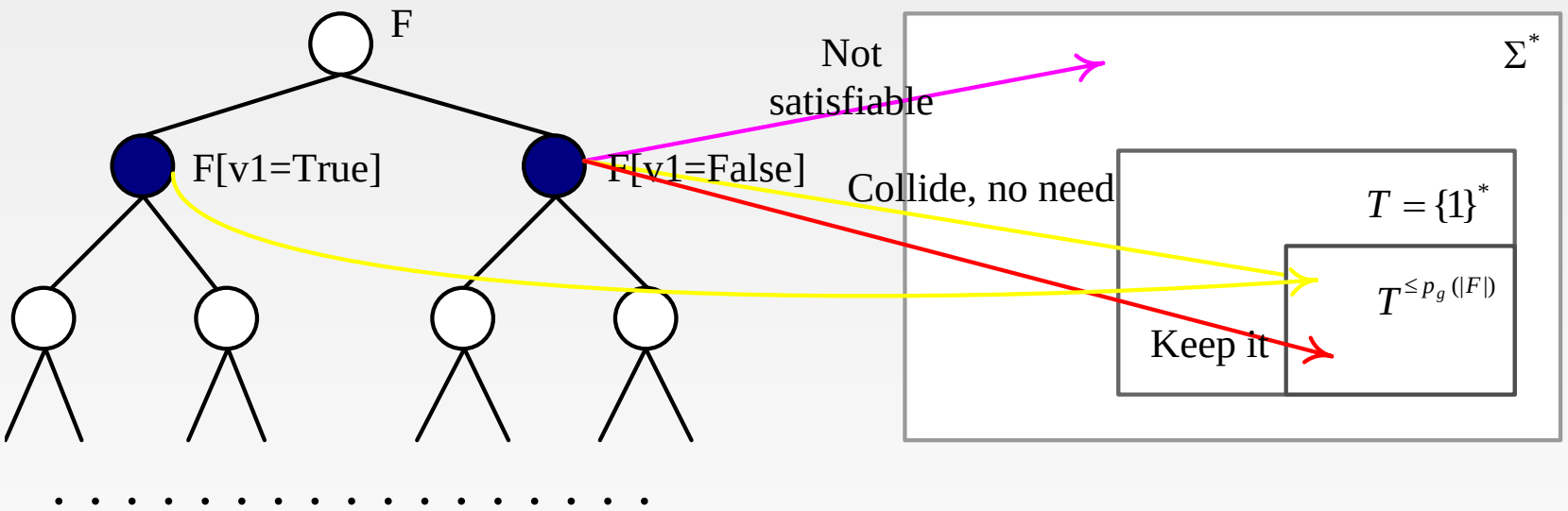
If there is a tally set that is $\leq_m^p$-hard for NP, then P=NP.

Tally NP-complete Languages (Cont.)

How to prove?

- $SAT \leq_m^p T$, that is there is a polynomial-time deterministic function g many-one reducing SAT to T.
- $T \subseteq \{1\}^*$.
- SAT is 2-disjunctive-self-reducible.


F
F is not satisfiable
$g(F)$
Σ^*
F[v1=True]
F[v1=False]
$T = \{1\}^*$
Just keep it
$g(F)$


F
F[v1=True]
F[v1=False]
Not satisfiable
Collide, no need
Keep it
Σ^*
$T = \{1\}^*$
$T^{\leq p_g(|F|)}$

Tally NP-complete Languages (Cont.)

Algorithm

Input: F with $v_1, v_2, \ldots, v_n$ as its variables

1. Initialization: $C = \{F\}$;
2. Do $i = 1$ to n
3. Expand C;
4. Prune C;
5. Pass C to the next phase;
6. F is satisfiable iff there is some formula $f \in C$ which evaluates to True;

Tally NP-complete Languages (Cont.)

Correctness and time complexity of the algorithm:

- At the beginning of each stage, there is some satisfiable formula in C iff at the end of this stage, there is some satisfiable formula in C.
- The size of C at the beginning of each stage is at most $p_g(|F|) + 1$.

Sparse coNP-complete Language

- Is there any sparse coNP-complete language?
- Sparse set S: there is a polynomial p such that for all n, $|S^{\leq n}| \leq p(n)$.

Sparse coNP-complete Language (Cont.)

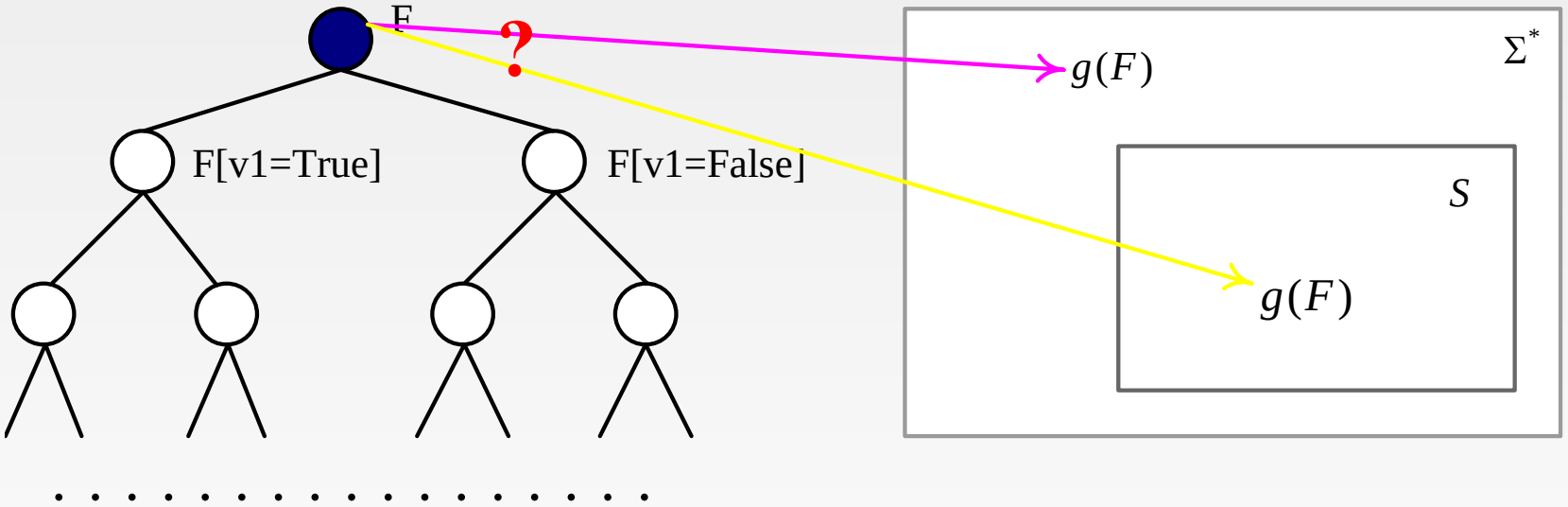
Theorem

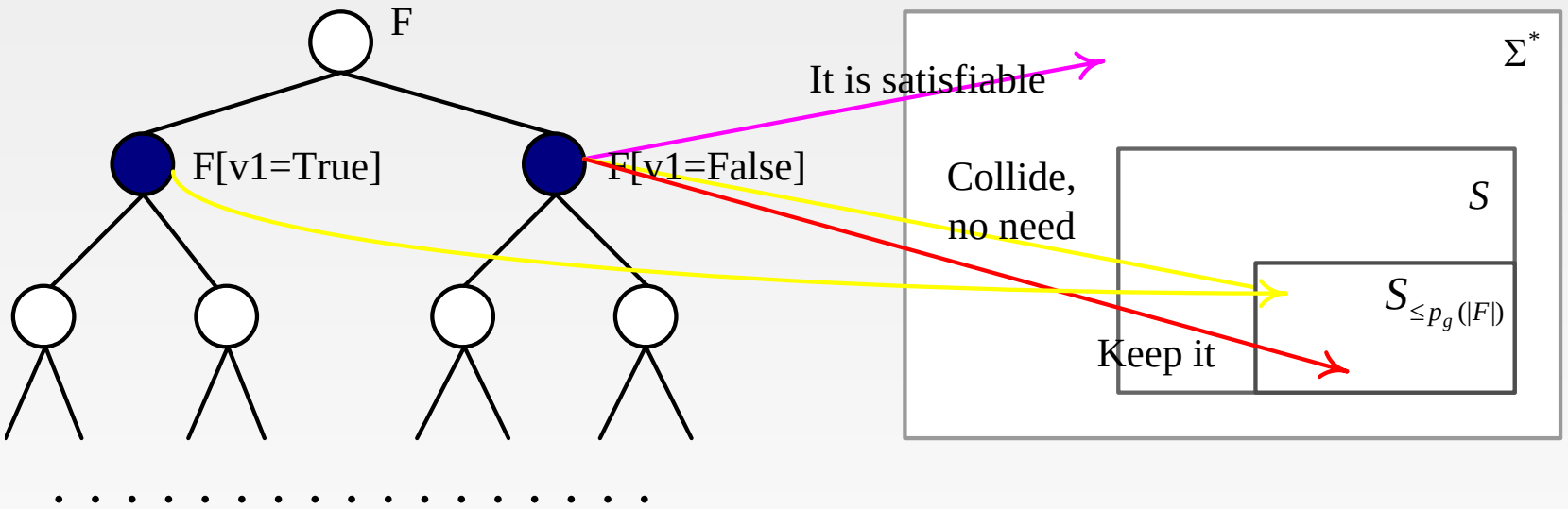
If there is a sparse set that is $\leq_m^p$-hard for coNP, then P=NP.

Sparse coNP-complete Language (Cont.)

How to prove? Can we generalize the above method?

- $SAT \leq_m^p S^c$, that is there is a polynomial-time deterministic function g many-one reducing SAT to S^c.
- S is a sparse set.
- SAT is 2-disjunctive-self-reducible.


F
?
F[v1=True]
F[v1=False]
$g(F)$
Σ^*
S
$g(F)$


F
F[v1=True]
F[v1=False]
It is satisfiable
Collide,
no need
Keep it
Σ^*
S
$S_{\leq p_g(|F|)}$

Sparse coNP-complete Language (Cont.)

Algorithm

Input: F with $v_1, v_2, \ldots, v_n$ as its variables

1. Initialization: $C = \{F\}$;
2. Do $i = 1$ to n
3. Expand C;
4. Prune C;
5. If C contains at least $p_d(p_k(|F|)) + 1$ elements, stop and declare $F \in SAT$;
6. Pass C to next phase;
7. F is satisfiable iff there is some formula $f \in C$ which evaluates to True;

Sparse coNP-complete Languages (Cont.)

Correctness and time complexity of the algorithm:

- At the beginning of each stage, there is some satisfiable formula in C iff at the end of this stage, there is some satisfiable formula in C.
- The size of C at the beginning of each stage is at most $p_d(p_k(|F|)) + 1$.

Sparse NP-complete Language

- Is there a sparse NP-complete language?

Sparse NP-complete Language (Cont.)

Theorem

If there is a sparse set that is $\leq_m^p$-complete for NP, then P=NP.

Sparse NP-complete Language (Cont.)

What do we have now?

- S is a sparse set.
- $SAT \leq_m^p S$, that is there is a polynomial-time deterministic function g many-one reducing SAT to S.
- SAT is 2-disjunctive-self-reducible.

What's the problem?

- Is $SAT \leq_m^p S$ enough to get the result?
- Can we find a relationship between S^c and S, e.g., $S^c \leq_m^p S$?

Sparse NP-complete Language (Cont.)

Definition

The census function c_S of set S is defined to be $c_S(n) = |\{x | x \in S \wedge |x| \leq n\}|$.

For each sparse set S, we have that the census function of S is bounded by some polynomial.

Sparse NP-complete Language (Cont.)

Why we choose polynomial-time computable census function?

- If the census function of some sparse NP-complete language S is polynomial-time computable, then $S^c \in NP$.
- $SAT \leq_m^p S$ and $S^c \leq_m^p S \Rightarrow SAT^c \leq_m^p S$.
- Then we have P=NP (by our previous result about sparse coNP-hard sets).

Sparse NP-complete Language (Cont.)

Theorem

If an NP-complete sparse language exists such that its census function c_S is computable in polynomial time, then P=NP.

Sparse NP-complete Language (Cont.)

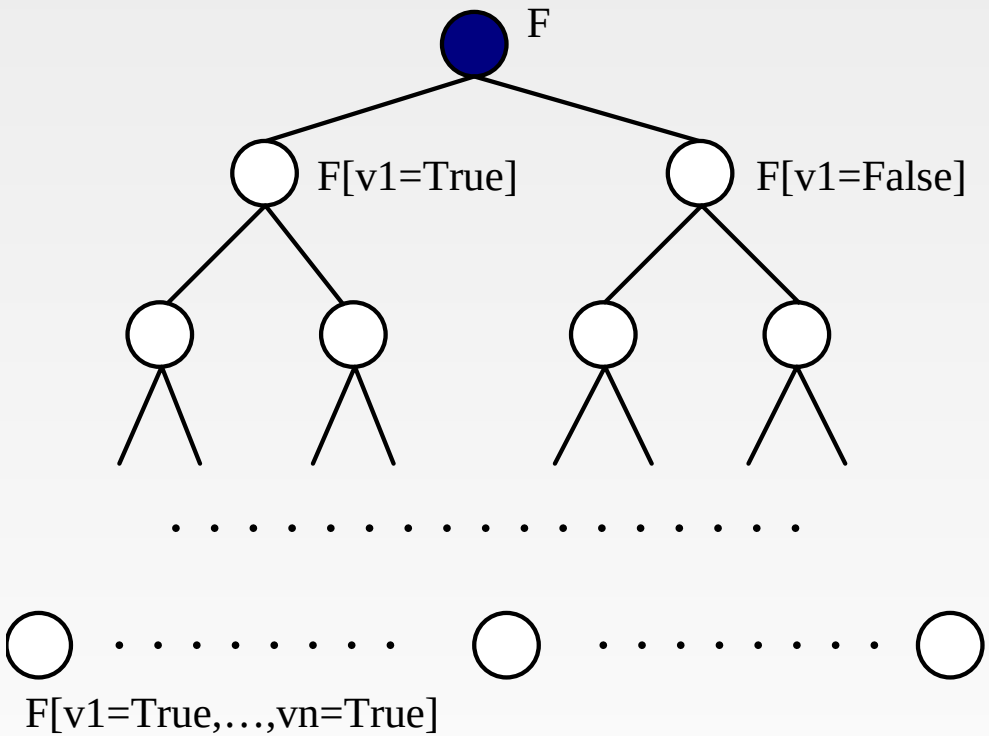
NP Algorithm for S^c

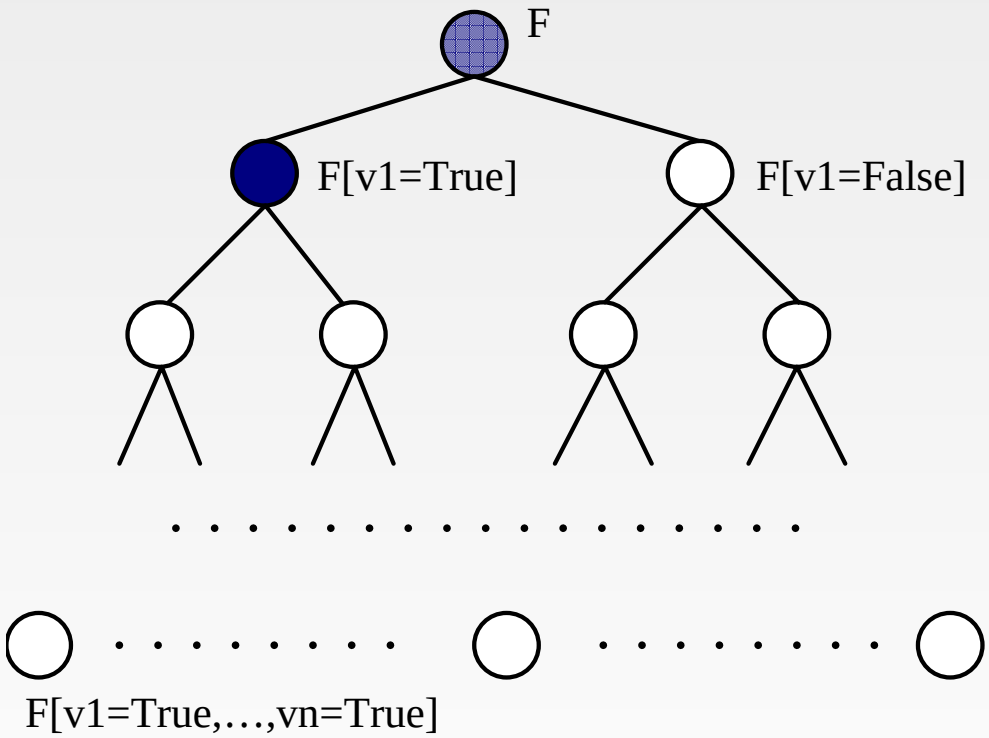
Input: x

1. Let $n = |x|, k = c_S(n)$.
2. Guess k distinct strings of length at most n.
3. Nondeterministicly test to make sure that all these k strings are in S, and if they all are then accept iff x is not one of these strings.

Sparse NP-complete Language (Cont.)

Next, we will prove the theorem that if $SAT^c \leq_m^p S$ for some sparse set S then P=NP via depth-first search and this approach will be useful in the later proof.


F
F[v1=True]
F[v1=False]
F[v1=True,…,vn=True]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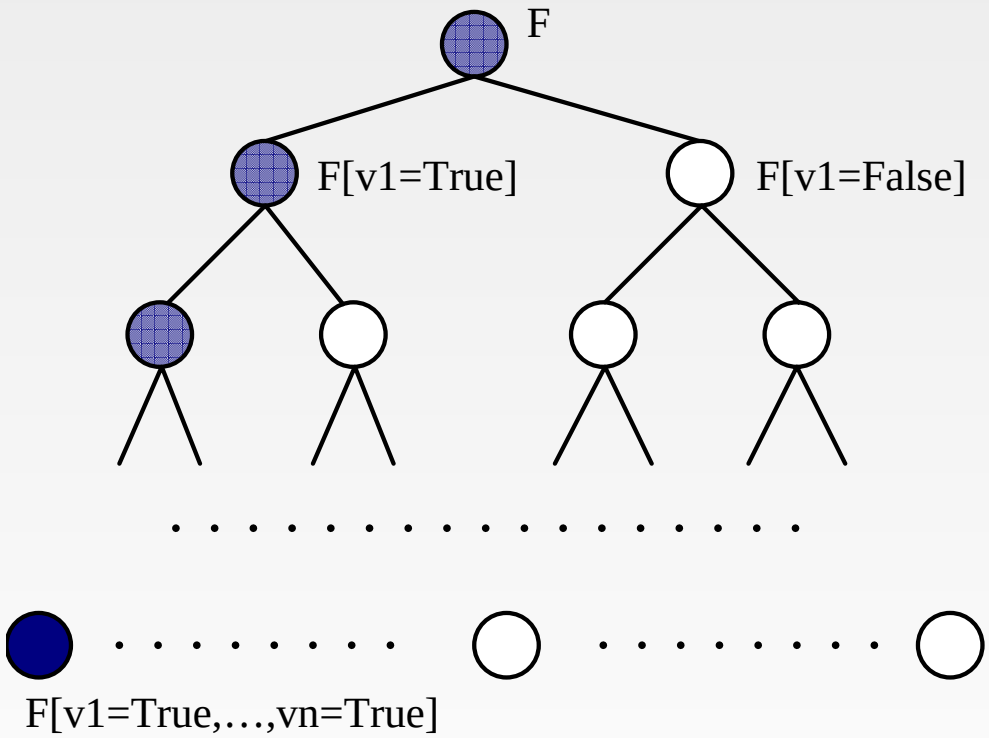
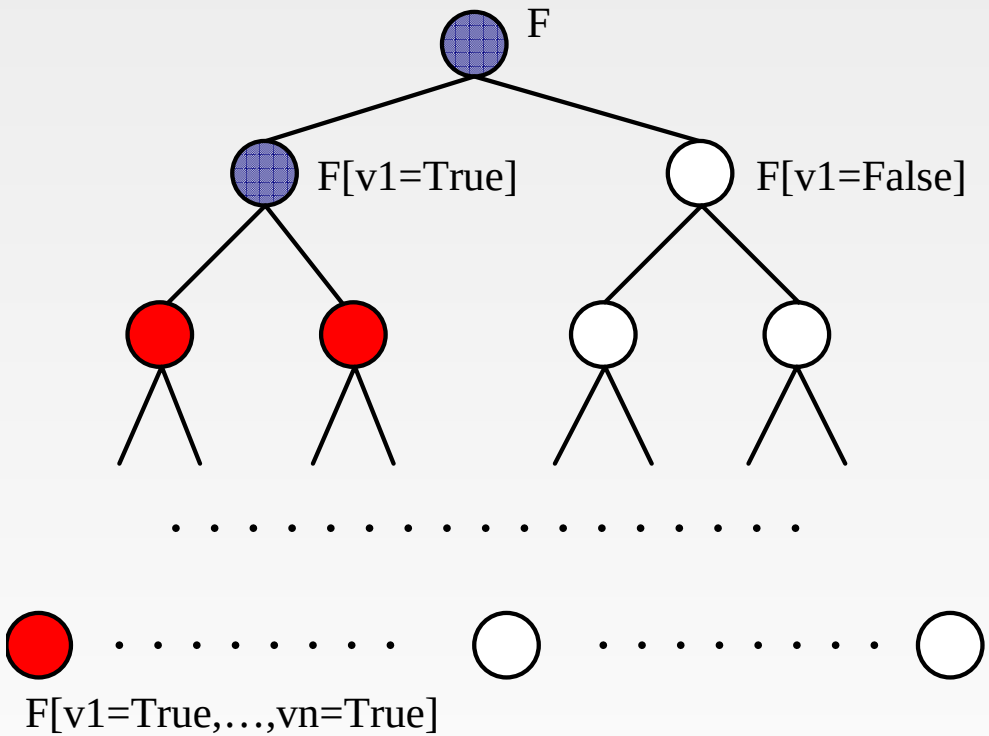
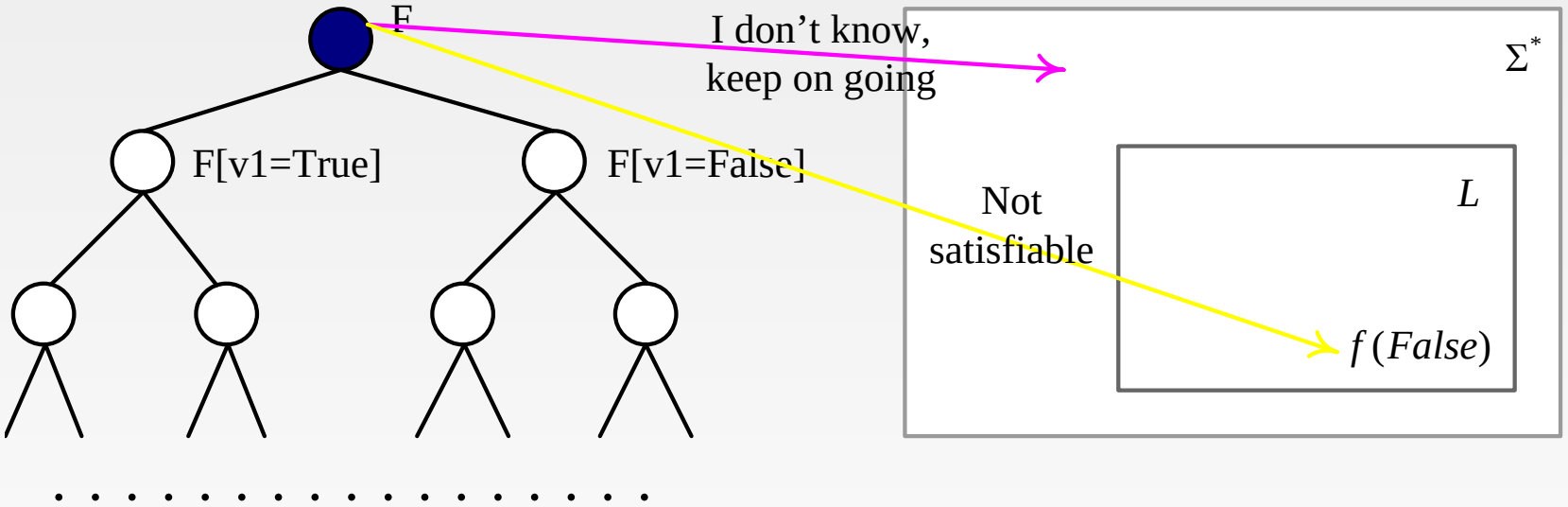
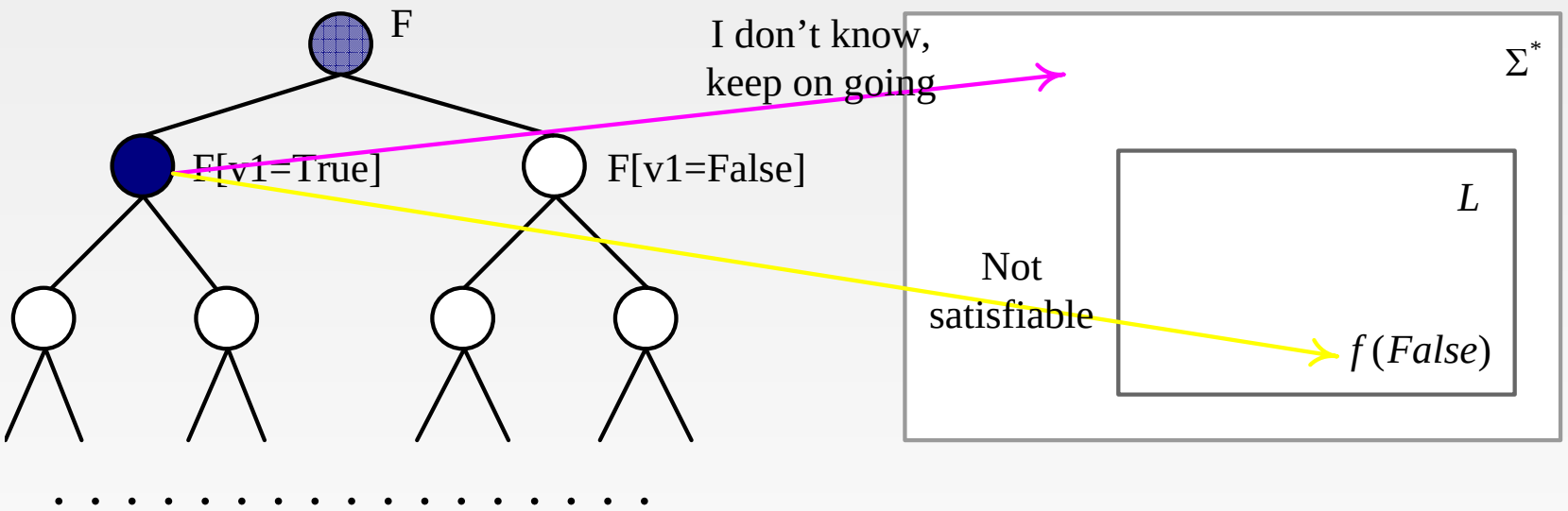
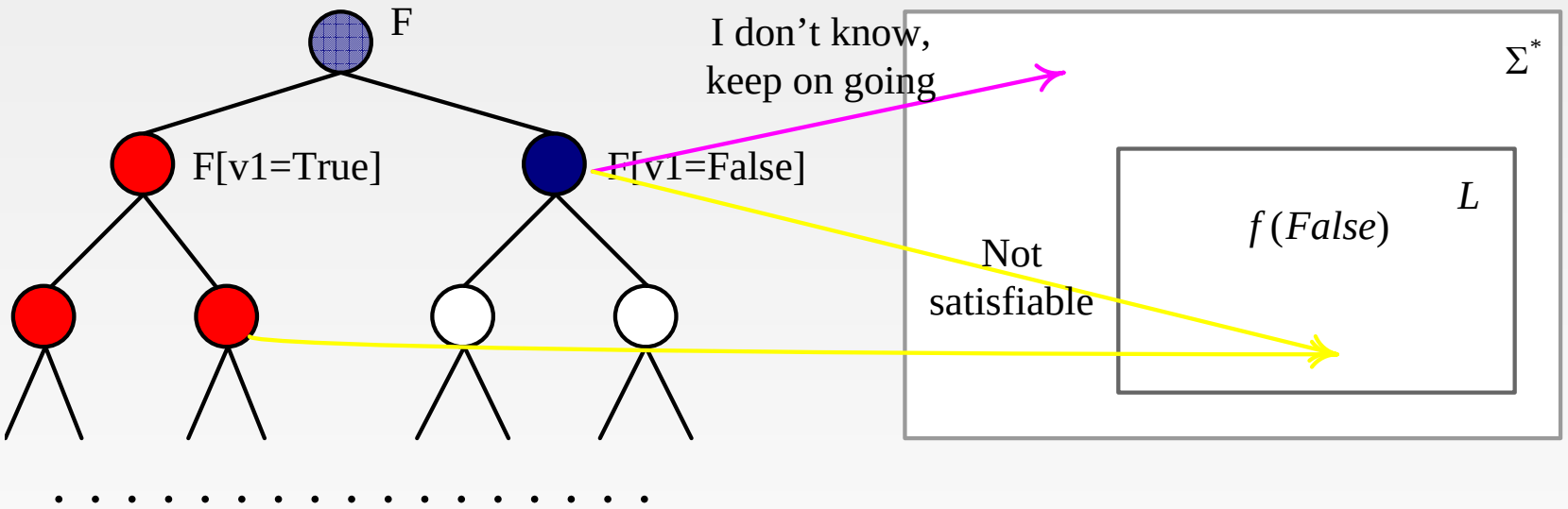
F[v1=True]
F[v1=False]
F[v1=True,…,vn=True]


F
F[v1=True]
F[v1=False]
F[v1=True,…,vn=True]


F
F[v1=True]
F[v1=False]
F[v1=True,…,vn=True]


F
I don’t know,
keep on going
F[v1=True]
F[v1=False]
Not
satisfiable
Σ^*
L
$f(False)$


F
F[v1=True]
F[v1=False]
I don't know,
keep on going
Not
satisfiable
Σ^*
L
$f(False)$


F
F[v1=True]
F[v1=False]
I don't know,
keep on going
Not
satisfiable
Σ^*
L
$f(False)$

Sparse NP-complete Language (Cont.)

Algorithm for SAT

Input: formula F

1. Use a set L to keep the images of unsatisfiable formula under f, initialize L to be $\{\text{f(False)}\}$;
2. If F has no variables and evaluates to True, then F is satisfiable;
3. If $f(F)$ is in L, then F is unsatisfiable;
4. Otherwise recursively test $F[v_i = True]$ and $F[v_i = False]$;
5. If neither of the two formula is satisfiable then F is unsatisfiable and add $f(F)$ to L.
6. Otherwise F is satisfiable.

Sparse NP-complete Language (Cont.)

Correctness and time complexity of the algorithm:

- Tree pruning.
- The size of L is at most $p(q(|F|))$.
- Visit at most $np(q(|F|)) + n - 1$ inner nodes.

Sparse NP-complete Language (Cont.)

- What is the situation if the census function cannot be computed in polynomial time?
- Just “guess” a value for census function.

Sparse NP-complete Language (Cont.)

- Define pseudo-complement of S to be $PC(S) = \{\langle x, k, 0^n\rangle \mid |x| \leq n \wedge k \leq p(n) \wedge (k < c_S(n) \vee (k = c_S(n) \wedge x \in S^c))\}$.
- If S is a sparse NP language, then $PC(S)$ is in NP.

Sparse NP-complete Language (Cont.)

NP Algorithm for $PC(S)$

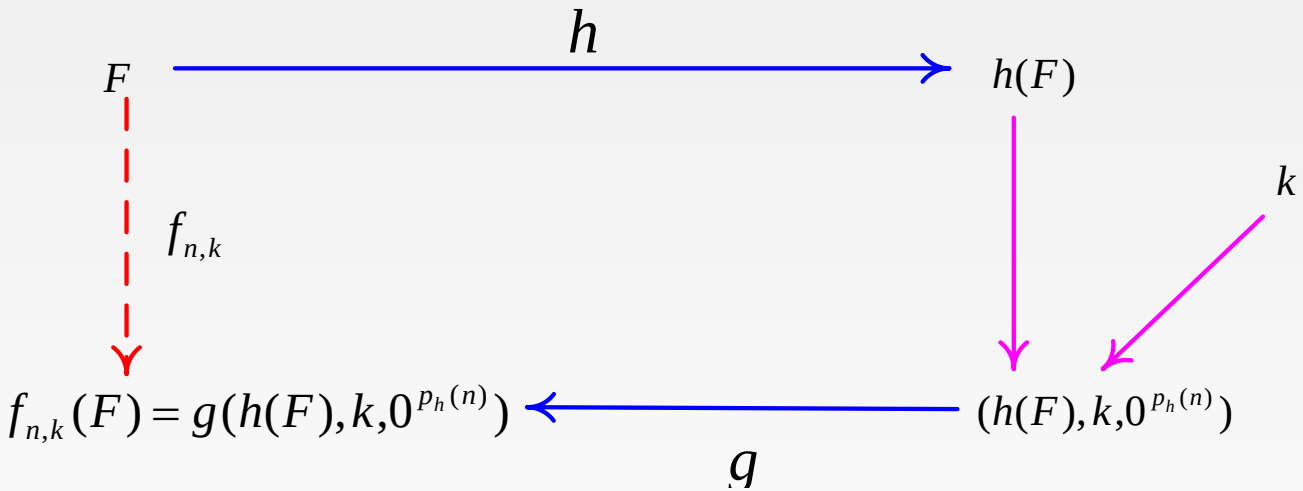
Input: $\langle x, k, 0^n \rangle$

1. Check if $|x| \leq n \wedge k \leq p(n)$, if not, reject.
2. Guess k distinct strings of length at most n.
3. Nondeterministicly test to make sure that all these k strings are in S, and if they all are then accept iff x is not one of these strings.

Sparse NP-complete Language (Cont.)

What do we have now?

- $SAT^c \leq_m^p S^c$
- $PC(S) \leq_m^p S$
- S is sparse.
- SAT is 2-disjunctive-self-reducible.


h
F
h(F)
k
$f_{n,k}$
$f_{n,k}(F) = g(h(F), k, 0^{p_h(n)})$
$(h(F), k, 0^{p_h(n)})$
g

Sparse NP-complete Language (Cont.)

- If $k = c_S(p_h(|F|))$, $|\langle h(F), k, 0^{p_h(|F|)}\rangle|$ is bounded by some polynomial.
- If $k = c_S(p_h(|F|))$, in the tree-traversal algorithm, at most some polynomial number of inner nodes will be visited.

Sparse NP-complete Language (Cont.)

Algorithm for SAT

Input: formula F

- For $k = 0$ to $p(p_h(|F|))$
- Use $f_{|F|,k}$ as pruning function to execute the tree-traversal algorithm and visit at most $|F|p(p_g(3|F|)) + |F| - 1$ inner nodes;
- If the algorithm accepts then accepts;
- reject;

Sparse NP-complete Languages (Cont.)

Correctness and time complexity of the algorithm:

- The algorithm accepts a formula F iff for some k, there is a tree-traversal path to reach a leaf node which evaluates to True.
- The algorithm never rejects a formula which is satisfiable.
- The number of visited tree nodes is bounded by some polynomial.

E and NE

- $E=\bigcup_{c>0}DTIME[2^{cn}]$
- $NE=\bigcup_{c>0}NTIME[2^{cn}]$
- $E \subseteq NE$

Theorem

Theorem

The following statements are equivalent

1. E=NE
2. NP-P contains no sparse sets
3. NP-P contains no tally sets

If NP-P contains no tally sets then E=NE.

Sub-Theorem(3⇒1)

If NP-P contains no tally sets then E=NE.

What we have:

- A. NP-P contains no tally sets
- B. P⊆NP
- C. E⊆NE

If NP-P contains no tally sets then E=NE.

Proof for NE$\subseteq$E

Start Point

$L \in$ NE $\Rightarrow$ there exists a nondeterministic exponential-time TM N_L and $L(N_L) = L$

Construct a tally set

$T = \{1^k | (\exists x \in L)[k = (1x)_{bin}]\}$

If NP-P contains no tally sets then E=NE.

$T \in$ NP? –YES.

Algorithm for T

- Input string y
- Reject if y is not of the form 1^k $(k > 0)$
- Simulate $N_L(x)$ $(k = (1x)_{bin})$ if y is of the form 1^k $(k > 0)$

The algorithm is nondeterministic and polynomial-time.
$T \in$ NP $\Rightarrow$ $T \in$ P $\Rightarrow$ there exists a deterministic polynomial-time TM M_T and $L(M_T) = T$

If NP-P contains no tally sets then E=NE.

Algorithm for L

- Input string y
- Get string $x = 1^{(1y)_{bin}}$
- Simulate $M_T(x)$ and accept if and only if $M_T(x)$ accepts

The algorithm is deterministic and exponential-time. So $L \in$ E and NE $\subseteq$ E.
The claim is proved.

If E=NE then NP-P contains no tally sets.

Warm Up for Sub-Theorem(1⇒2)

If E=NE then NP-P contains no sparse sets.

Sub-Theorem(1⇒3)

If E=NE then NP-P contains no tally sets.

What we have:

- A. E=NE
- B. P⊆NP

If E=NE then NP-P contains no tally sets.

Proof for tally set $T \in$ P if $T \in$ NP

Start Point

Tally set $T \in$ NP $\Rightarrow$ there exists a nondeterministic polynomial-time TM N_T and $L(N_T) = T$

Construct an NE language

$L = \{x | (x = 0 \vee$ x is a binary string of nonzero length with no leading zeros$) \wedge 1^{(x)_{bin}} \in T\}$

If E=NE then NP-P contains no tally sets.

$L \in$ NE? –YES.

Algorithm for L

- Input string y
- Reject if $x \neq 0$ and x is a binary string with leading zeros.
- Otherwise,
  - Get string $x = 1^{(y)_{bin}}$.
  - Simulate $N_T(x)$ and accept if and only if $N_T(x)$ accepts.

The algorithm is nondeterministic and exponential-time.
$L \in$ NE $\Rightarrow L \in$ E $\Rightarrow$ there exists a deterministic exponential-time TM M_L and $L(M_L) = L$

If E=NE then NP-P contains no tally sets.

Algorithm for T

- Input string y
- Reject if y is not of the form 1^k $(k \geq 0)$
- Otherwise, simulate $M_L(x)$ $(k = (x)_{bin})$ if y is of the form 1^k $(k \geq 0)$

The algorithm is deterministic and polynomial-time. So $T \in$ P and no tally sets in NP-P.
The claim is proved.

If E=NE then NP-P contains no sparse sets.

Sub-Theorem(1⇒2)

If E=NE then NP-P contains no sparse sets.

What we have:

- A. NE=E
- B. P⊆NP

If E=NE then NP-P contains no sparse sets.

Proof for sparse sets $S \in$ P if $S \in$ NP

Start Point

Sparse set $S \in$ NP $(\forall n)[||S^{=n}|| \leq q(n)] \Rightarrow$ there exists a nondeterministic polynomial-time TM N_S and $L(N_S) = S$

Difference from Warm Up

How to construct an NE language L from S that a string in S can be checked by L easily?

If E=NE then NP-P contains no sparse sets.

Construct an NE language (HIS encoding set)

$L = \{0\#n\#k \mid ||S^{=n}|| \geq k\} \bigcup$
$\{1\#n\#c\#i\#j \mid (\exists z_1, z_2, ..., z_c \in S^{=n})[z_1 <_{lex} z_2 <_{lex} .... <_{lex} z_c \wedge$ the jth bit of z_i is 1$]\}$

If E=NE then NP-P contains no sparse sets.

$L \in$ NE? –YES.

Algorithm for L

- Input string y
- Reject if y is syntactically illegal
- Check the first bit of y
  - The first bit of y is 0.
    - Get binary value of n and k.
    - Guess k distinct strings that are n bits long. If there is such a strings set that each string can be accepted by N_S then accept. Otherwise, reject.
  - The first bit of y is 1.
    - ...

If E=NE then NP-P contains no sparse sets.

Algorithm for L (Cont.)

- Input string y
- Reject if the string is not either form of L
- Check the first bit of y
  - The first bit of y is 0.
    - ...
  - The first bit of y is 1.
    - Get binary value of n, c, i and j.
    - Guess c distinct strings that are n bits long. Then make them in lexical order. If there is such a set that each string is accepted by N_S and the jth bit of ith string is 1 then accept. Otherwise, reject.

If E=NE then NP-P contains no sparse sets.

The algorithm is nondeterministic and exponential-time.
$L \in \text{NE} \Rightarrow L \in \text{E} \Rightarrow$ there exists a deterministic exponential-time TM M_L and $L(M_L) = L$

If E=NE then NP-P contains no sparse sets.

Algorithm for S

- Input string y
- Let $n = |y|$ and simulate $M_L(0\#n\#0)$, $M_L(0\#n\#1)$, $M_L(0\#n\#2)$, ..., $M_L(0\#n\#q(n))$. Then get $c = Max\{k|0 \leq k \leq q(n) \wedge 0\#n\#k \in L\} = ||S^{=n}||$.

If E=NE then NP-P contains no sparse sets.

Algorithm for S (Cont.)

- Simulate
  $M_L(1\#n\#c\#1\#1)$, ..., $M_L(1\#n\#c\#1\#n)$,
  $M_L(1\#n\#c\#2\#1)$, ..., $M_L(1\#n\#c\#2\#n)$,
  ....,
  $M_L(1\#n\#c\#c\#1)$, ..., $M_L(1\#n\#c\#c\#n)$.
  Then we can get all strings in L that are n bits long by checking each bit of them. So y is accepted if and only if y belongs to this set.

The algorithm is deterministic and polynomial-time. So $S \in$ P and no sparse sets in NP-P.
The claim is proved.

Summary

- Two classes: E and NE.
- Theorem 1.18 of Hem-Ogi: The three statements are equivalent.
  - E=NE
  - NP-P contains no sparse set
  - NP-P contains no tally sets

Thanks!