Group A Lecture 1: The Self-Reducibility Technique

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Theorem 1.2

Theorem 1.2

Tally Set

A set T is a tally set exactly if $T \subseteq 1^*$

Theorem 1.2

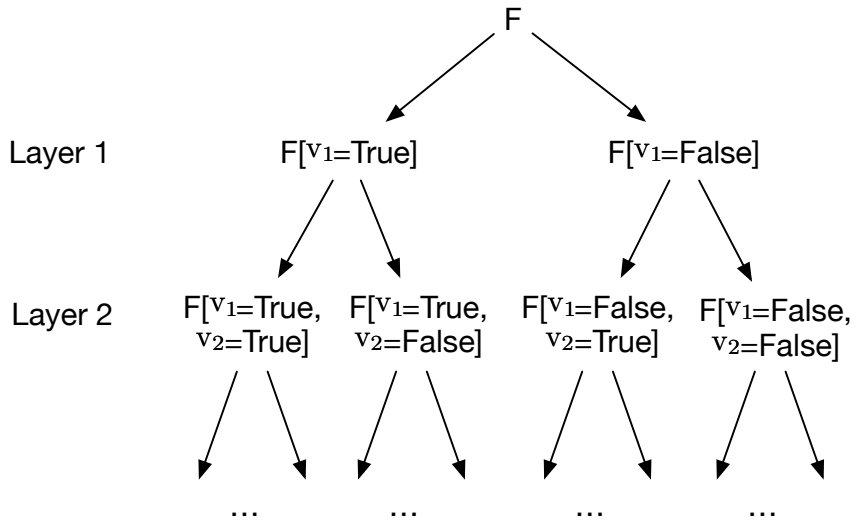
If there is a tally set that is $\leq_m^p$-hard for NP, then P=NP.

Corollary 1.3

If there is a tally set that is NP-complete, then P = NP.

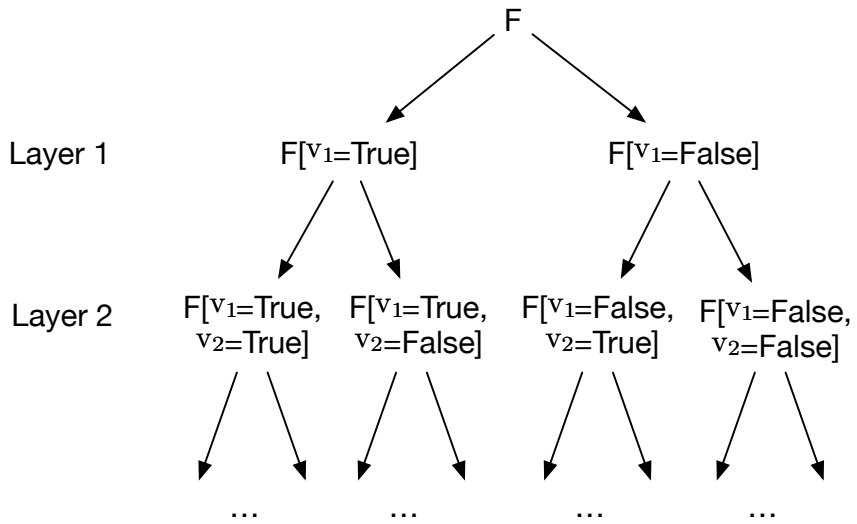
- Let T be a tally set that is $\leq_m^p$-hard. Then the NP-complete set $SAT \leq_m^p T$.
- Goal: We want to use $SAT \leq_m^p T$ to proof that SAT can be decided in polynomial time. Thus, $SAT \in$ P, then P = NP

Tree Pruning For *SAT* Problem



- F[v_i=True] denotes the resulting boolean formula when we assign True to variable v_i
- Boolean formula F is satisfiable if and only if F[v_1=True] is satisfiable or F[v_1=False] is satisfiable.
- Find the satisfiable assignment by traversing the tree. If the traverse can be done in polynomial time, then $SAT \in P$.

Tree Pruning For *SAT* Problem

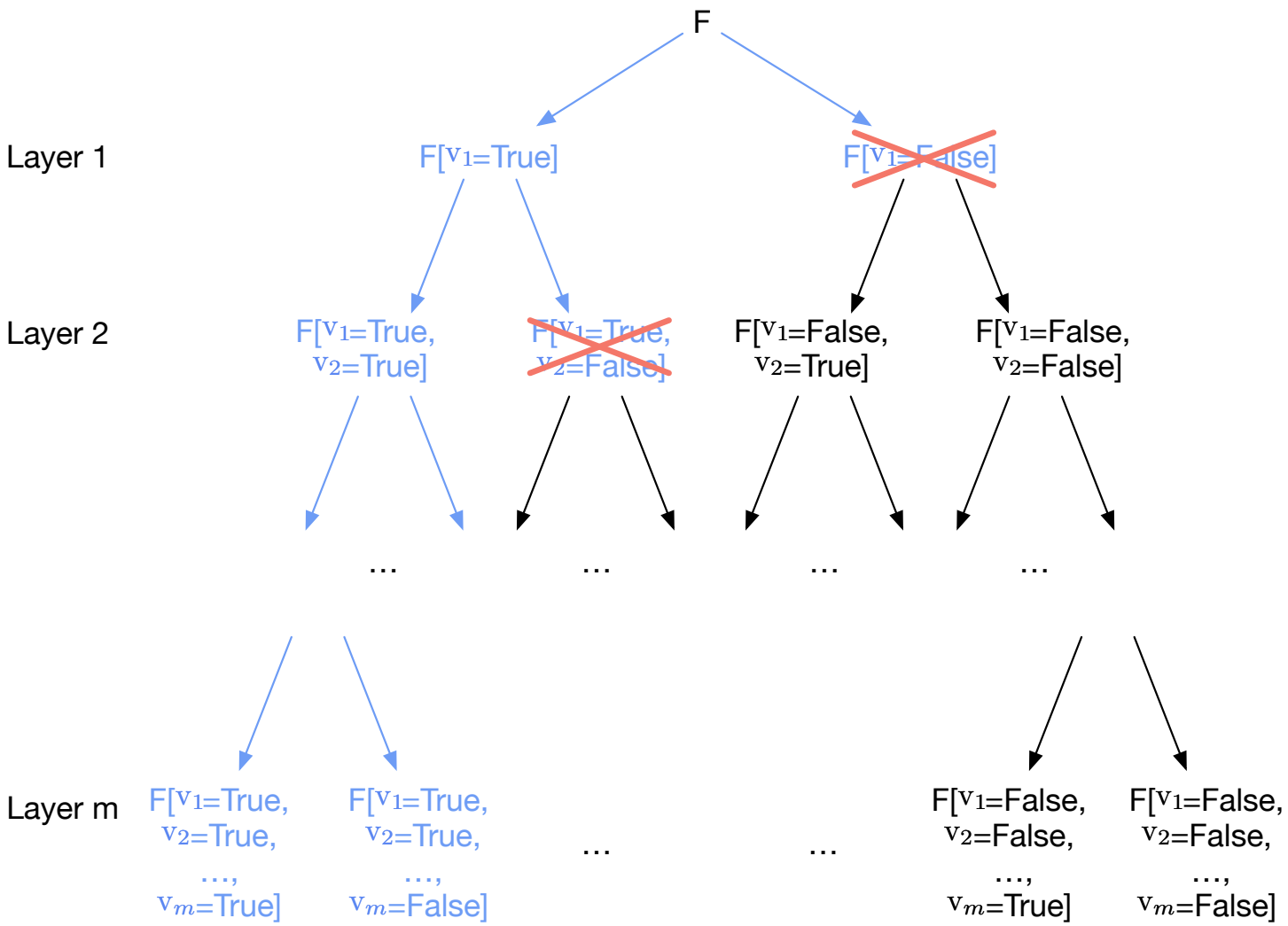


- Traverse is done layer by layer. The number of nodes in i^{th} layer is 2^i.
- If during the traverse we can ignore some redundant nodes (tree pruning) so that for each layer we only traverse polynomial number of nodes, then the entire traverse is polynomial.

Example: Tree Pruning For *SAT* Problem

- (Rabbit says) What nodes/formulas are redundant?
- If a formula is not satisfiable, then all of its descendants are not satisfiable. Thus, this formula is redundant.
- If a formula is "identical" to another formula, then it is redundant.
- If f_1 is satisfiable if and only if f_2 is satisfiable, then f_1 and f_2 is identical.
- (Rabbit says) How do we identify the redundancy?

Tree Pruning For *SAT* Problem



Identify Redundancy

- Let g be the deterministic polynomial-time function such that $\forall f \in SAT$ if and only if $g(f) \in T$, where T is the $\leq_m^p$-hard Tally set.
- Recall that $T \subseteq 1^*$. If $g(f) \notin 1^*$, then f is not satisfiable.
- For any two boolean formula $f \neq h$, and $g(f) = g(h)$, $f \in SAT \iff h \in SAT$.

$$\begin{array}{ccc} f \in SAT & \iff & g(f) \in T \\ & & \| \\ h \in SAT & \iff & g(h) \in T \end{array}$$

- (Rabbit says) How do we make sure the number of remaining nodes/formulas in each layer is polynomial?

Polynomial Bound

- The length of the output of a polynomial-time function is bounded by some polynomial
- Let $g(x)$ be a a polynomial-time function, there exists a integer k such that $\forall x, |g(x)| \leq |x|^k + k$
- If $g(x) \in 1^*$, then the longest possible output is $1^{|x|^k+k}$. Thus, the total number of possible outputs of $g(x)$ is $|x|^k + k + 1$.

Example

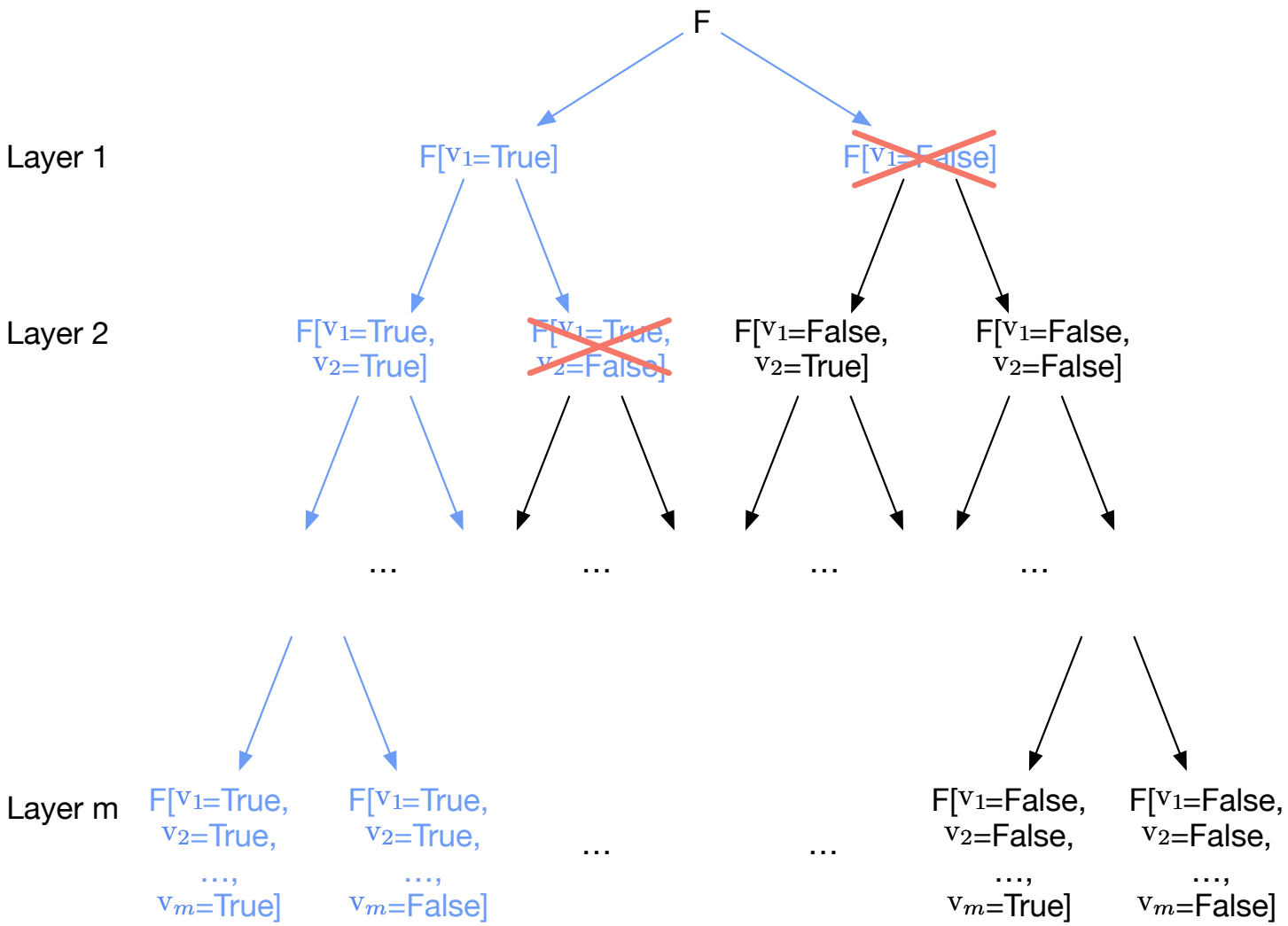
Given that $|g(x)| \leq |x|^k + k$ and $g(x) \in 1^*$, what are the possible outputs of $g(x)$?

$$\epsilon, 1, 11, 111, 1111, 11111, \ldots, \underbrace{11\ldots111}_{|x|^k+k}$$

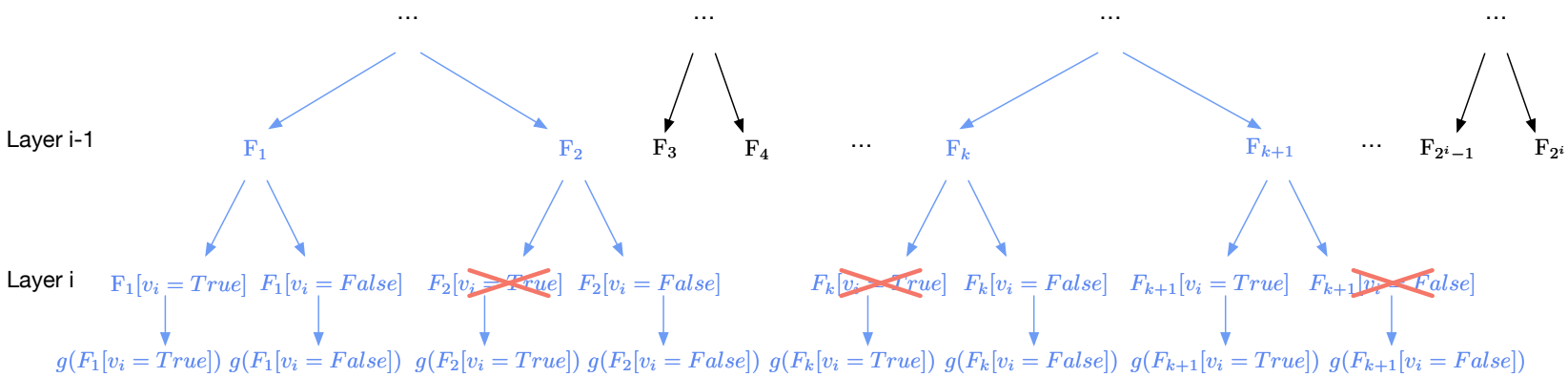
Polynomial Bound

- Recall that for any two boolean formula f, h, if $g(f) = g(h)$, then f and g are "identical". Similarly, if $g(f) \neq g(h)$, we say f and g are "distinct".
- Recall that the total number of possible outputs of $g(x)$ is $|x|^k + k + 1$.
- Let n be the size of formulas on the i^{th} layer. Thus, among the 2^i formulas in this layer, at most $n^k + k + 1$ of them are "distinct".

Proof Sketch



Proof Sketch



- The input of layer i are the output formulas from layer $i-1$.
- Expand each formula by assigning *True* and *False* value to v_i (Get the corresponding formulas in layer i).
- For each expanded formula f in layer i, calculate $g(f)$. If $g(f) \notin 1^*$, remove f. If $f \in 1^*$ but exists expanded formula $h \neq f$ such that $g(f) = g(h)$, remove f.
- Output the resulting formulas in layer i.

Proof

Stage 0

Outputs C=F where F is the original formula

Stage i

Input $C = \{F_1, \ldots, F_l\}$
Step 1: Replace v_i by True or False to get

$$C = \{F_1[v_i = True], F_2[v_i = True], \ldots, F_l[v_i = True],$$
$$F_1[v_i = False], F_2[v_i = False], \ldots, F_l[v_i = False]\}$$

Step 2: $C' = \emptyset$
Step 3: For each f in C do

1. Compute $g(f)$
2. If $g(f) \in 1^*$ and for no formula $h \in C'$ does $g(f) = g(h)$, then add f to C'.

Output of stage i : $C = C'$

Stage m+1

Input is C which is now a variable-free formula collection. F is satisfiable if an element in C is true.

Questions?

Theorem 1.4

Problem

Theorem 1.4

If there is a sparse set that is $\leq_m^p$-hard for coNP, then P=NP.

Corollary 1.5

If there is a sparse coNP-complete set, then P=NP.

Observation

Theorem 1.4

If there is **a sparse set** that is $\leq_m^p$**-hard for coNP**, then P=NP.

Definition

A set S is **sparse** if it contains at most polynomially many elements at each length, i.e.,

$$(\exists \text{ polynomial } p)(\forall n)[\|\{x|x \in S \wedge |x| = n\}\| \leq p(n)].$$

Definition

A language A is **coNP-hard**, if $\forall L \in coNP, L \leq_m^p A$.

Observation

Idea

Utilize **Tree-pruning** trick and the definition of **coNP-hard** to construct a polynomial-time algorithm for SAT. (SAT is NP-complete)

Explanation

- $\forall L \in NP, L \leq_m^p SAT$
- SAT solved in polynomial-time by deterministic Turing machine (DTM).
- $\Leftrightarrow$ All NP problems solved in polynomial-time by DTM.
- $\Leftrightarrow P = NP$.

Observation

Let S be a sparse set and also coNP-hard.

Definition

$\forall \ell$, $p_\ell(n)$ denotes the polynomial $n^\ell + \ell$.

Definition

$\|S^{\leq n}\|$ denotes the number of strings with length less than n in S.

Observation

Corollary

$\forall n,\ \|S^{\leq n}\| \leq p_d(n).$

Proof.

- S is sparse $\Rightarrow \|\{x|x \in S \wedge |x| = n\}\| \leq p(n)$
- We can obtain the upper bound $p_{max} = \max_n p(n)$, where p_{max} is bounded by polynomial.
- $\|S^{\leq n}\| = \sum_{i=0}^{n} p(i) \leq \sum_{i=0}^{n} p_{max} = (n+1)p_{max}$, which is bounded by polynomial.

Observation

Recall

$SAT \in NP \Rightarrow \overline{SAT} \in coNP$ and $\overline{SAT} \in coNP \Rightarrow \overline{SAT} \leq_m^p S$, since $S \in$ coNP-hard.

Let g denote the reduction function $\overline{SAT} \leq_m^p S$.

Corollary

$\forall x, |g(x)| \leq p_k(|x|)$.

Proof.

- Function g is computed by a DTM
- a DTM outputs at most 1 symbol in one step

$\Rightarrow |g(x)|$ is bounded by polynomial length, named $p_k(|x|)$. $\square$

Observation

Corollary

Since $\forall n,\ \|S^{\leq n}\| \leq p_d(n)$ *and* $\forall x, |g(x)| \leq p_k(|x|)$, *given* g *and* S,

$$\|S^{\leq |g(x)|}\| \leq p_d(p_k(|x|)).$$

Rabbit: Interesting! $S^{\leq |g(x)|}$ is a set with a polynomial number of elements.

Deterministic polynomial-time algorithm for SAT

Input

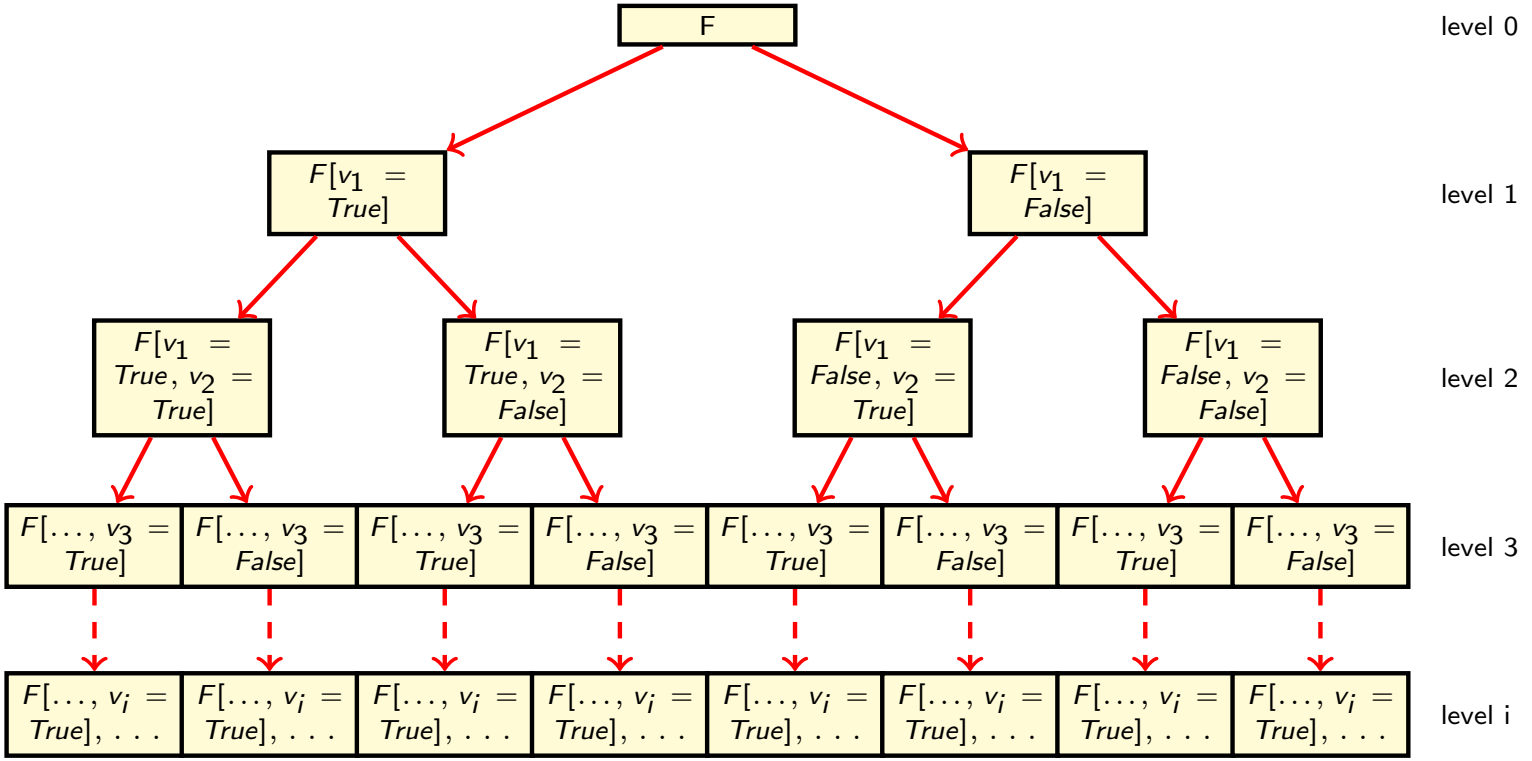
Boolean formula $F[v_1, v_2, ..., v_m]$, w.l.o.g, $m \geq 1$.

Stage 0

- Collection of boolean formulas, $C' = \{F\}$
- Pass C' to Stage 1.

Rabbit: That's pretty easy. I can do it.

Deterministic polynomial-time algorithm for SAT



Rabbit: If we keep this procedure to Stage m, the number of strings in each level will grow larger and larger!!!

Deterministic polynomial-time algorithm for SAT

A collection of formulas: Hi, we are from Stage i-1.

Stage i

Step 1: $\mathcal{C} = \{F_1[v_i = True], F_2[v_i = True], ..., F_\ell[v_i = True], F_1[v_i = False], F_2[v_i = False], ..., F_\ell[v_i = False]\}$.
Step 2: Set $\mathcal{C}' = \emptyset$.

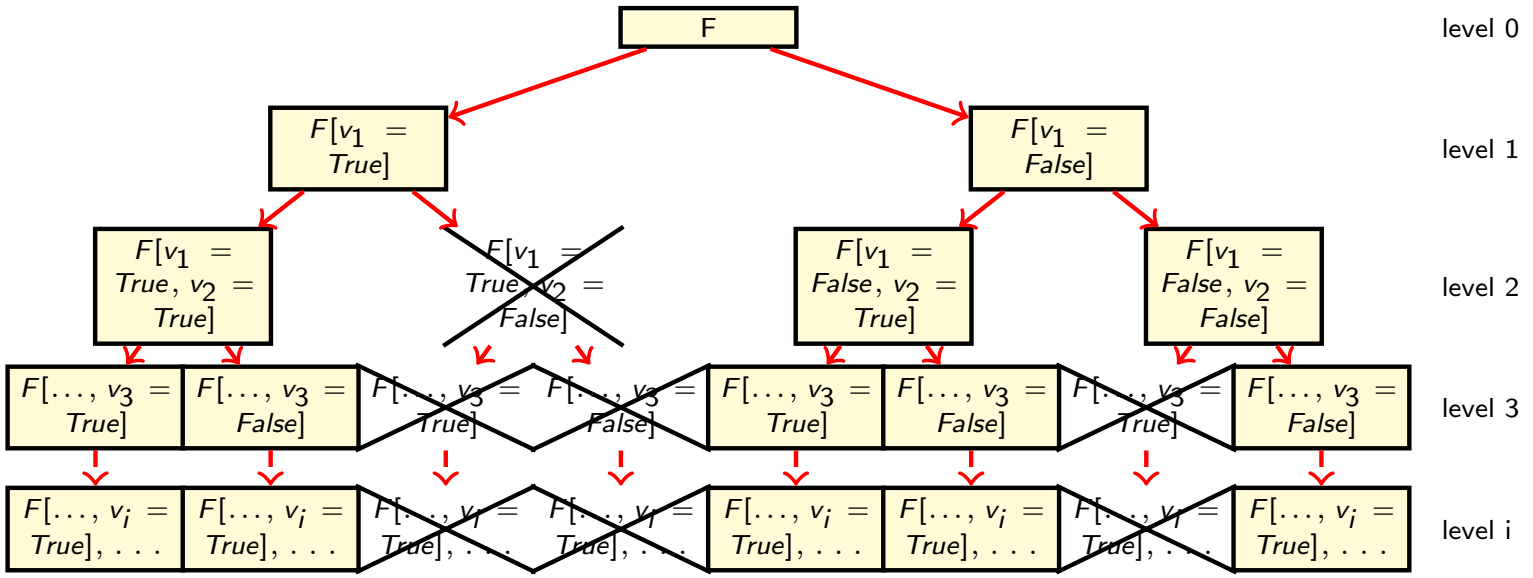
Rabbit: lol, I can do it but where is my carrot?

Deterministic polynomial-time algorithm for SAT

Stage i

Step 3: For each formula f in $\mathcal{C}$ do:

1. Compute $g(f)$.
2. If for no formula $h \in \mathcal{C}'$ does $g(f) = g(h)$, then add f to $\mathcal{C}'$



Deterministic polynomial-time algorithm for SAT

Stage i

Step 4: If C' contains at least $p_d(p_k(|F|)) + 1$ elements, stop and immediately declare that $F \in SAT$.

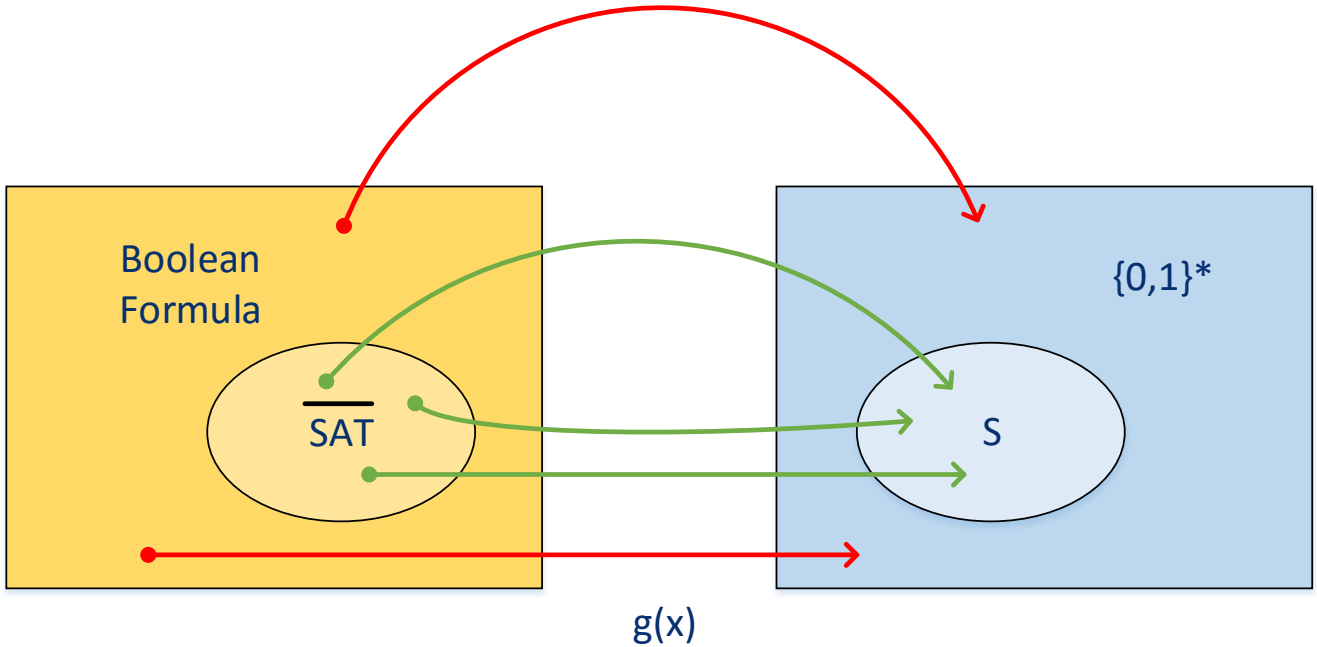


Figure: Reduction Mapping

Deterministic polynomial-time algorithm for SAT

Stage i

Step 4: If C' contains at least $p_d(p_k(|F|)) + 1$ elements, stop and immediately declare that $F \in SAT$.

Explanation

- Only $p_d(p_k(|F|))$ strings are in $S^{\leq p_k(|F|)}$.
- There is at least one formula named H maps to a string in $\overline{S}$, i.e., $g(H) \notin S$.
- Since g is the reduction function from $\overline{SAT}$ to S, H is satisfiable. It imply that F is satisfiable.

Deterministic polynomial-time algorithm for SAT

Stage i

End Stage i: $\mathcal{C}'$ is the collection that gets passed on to Stage $i+1$.

Deterministic polynomial-time algorithm for SAT

Stage m+1

If some member of the formula collection output by Stage m evaluates to being true, $F \in SAT$, and otherwise $F \notin SAT$.

Rabbit: Oh, my carrot! The proof is done here. Wait, rabbit!

Discuss

Comment

Obviously, this algorithm is computed by deterministic Turing machine.

- **Step 4 never met**
  Upper bound number of strings $p_{max} = \max p_d(p_k(|F|))$.
  $\Rightarrow$ time for whole algorithm $t \leq mp_{max}$
- **Step 4 invoked**
  This algorithm stops early before Stage m+1.
  $\Rightarrow$ The algorithm is polynomial-time.

We construct a deterministic polynomial-time algorithm for SAT.

Rabbit: If I find a carrot like this set S, I will buy a million carrots (plus 9 millions).

Thank You!

Group A Lecture 2: Mahaney's Theorem

Adam Scrivener, Haofu Liao, Nabil Hossain, Shupeng Gui, Thomas Lindstorm-Vautrin

Why Study Sparseness?

- Closely related to the p-isomorphism conjecture:

Lemma 5.1 Let L_1 and L_2 be two p-ismorphic languages. Then two polynomials p_1 and p_2 exist such that, for any n,

$$c_{L_1}(n) \leq c_{L_2}(p_1(n)) \qquad \text{and} \qquad c_{L_2}(n) \leq c_{L_1}(p_2(n)).$$

- In other words, the census functions are polynomially related.
- Since the census function for SAT is known to be exponential (Bov-Cre p. 83), if a sparse NP-complete language L is found, then the census function for L cannot be polynomially related to the census function for SAT. So, the p-isomorphism conjecture falls.

Difficulty of Finding a Sparse NP-C Language

- Due to a result by Mahaney:

Theorem 5.7 If a sparse NP-complete language exists, then P = NP.

- And thus, finding such an NP-complete language means proving that P=NP, which is widely believed to be false.
- But why does Theorem 5.7 hold?

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Theorem 5.6

Theorem 5.6 If an NP-complete sparse language exists such that its census function is computable in polynomial time, then P = NP. (A census function c_L is said to be computable in polynomial time if a Turing transducer T exists which reads 0^n as input and computes $c_L(n)$ in polynomial time.)

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Complement of S is in NP

```
begin {input: x}
  n := |x|;
  k := c_S(n);
  guess y_1, ..., y_k in set of k-tuples of distinct words
  each of which has length, at most, n;
  {check whether the guessed k-tuple coincides with S_{≤n}}
  for i = 1 to k do
    if NT(y_i) rejects then reject;
  {check if x ∈ S_{≤n}}
  for i = 1 to k do
    if y_i = x then reject;
  accept;
end.
```

Further, **the complement of SAT reduces to S.**

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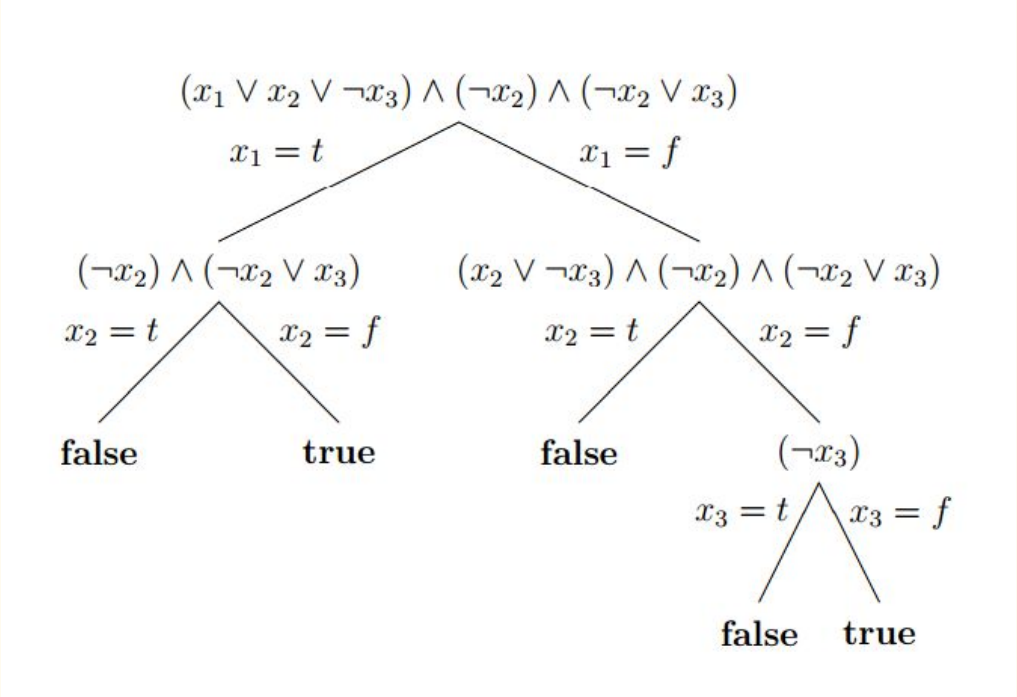
Satisfiability Trees and Tree Pruning



Satisfiability Trees and Tree Pruning

- Since the complement of SAT reduces to S by some function f, we can use f as a pruning function.
- If y is a node in the satisfiability tree and f(y) is in S, then we know that y is not satisfiable, and we can ignore y's children in the tree.
- It is difficult to determine if f(y) is in S directly since S is NP-C.
- To solve this, we build a list of elements in S as our tree pruning algorithm runs, which we can query.

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SAT is in P

```
begin {input: x} {main program}
  list := {f(false)};
  if sat(x) then accept else reject;
end.
```

```
function sat(y): Boolean;
begin
  if y = true then sat := true;
  if f(y) ∈ list then sat := false
  else
  begin
    derive y0 and y1 from y;
    if ¬sat(y0) ∧ ¬sat(y1) then
    begin
    {if both y0 and y1 are not satisfiable
    then y is not satisfiable}
      list := list ∪ {f(y)};
      sat := false;
    end
    else sat := true;
  end;
end;
```

Mahaney's Theorem

Theorem 5.7 If a sparse NP-complete language exists, then P = NP.

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Mahaney's Theorem

- We have shown that we can prove P = NP if the sparse NP-C set has a poly-time computable census function.
- Can we still do it if we don't have this assumption?
- Yes, and it will require "guessing" the correct value of the census function.

Pseudo-complement of S: (PC(S))

Define PC(S) as the set of triples <x,k,0^n> accepted by this machine:

begin {input: $x, k, 0^n$}
  if $|x| > n \vee k > p(n)$ **then** reject;
  guess $y_1, \ldots, y_k$ **in** set of k-tuples of distinct words
  each of which is of length, at most, n;
  for $i = 1$ **to** k **do**
    if $NT(y_i)$ rejects **then** reject;
  for $i = 1$ **to** k **do**
    if $y_i = x$ **then** reject;
  accept;
end.

Note that if |x| <= n and k = c_s(n), then <x,k,0^n> is in the set if and only if x is in S complement.

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Constructing a good pruning function

- Before, we used a reduction from the complement of SAT to S. This time we do the exact same thing, with a new reduction.
- First, let h be a reduction from SAT to S, and g be a reduction from PC(S) to S.
- Let p_h and p_g limit the length of h and g.
- Now, suppose that x, the string we want to determine is satisfiable or not, is of length n.
- Define F*(y) = $g(h(y), c_s(p_h(n)), 0^{p_h(n)})$
- Claim: F*(y) is a reduction from the complement of SAT to S.

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A set of pruning functions

- For each n and for each k <= p(p_h(n)), where p is a polynomial bounding the census function for S:
- Define $f_{n,k}(y) = g(h(y), k, 0^{p_h(n)})$.
- These are all poly-time computable when |y| <= n
- Note that F*(y) = $g(h(y), c_s(p_h(n)), 0^{p_h(n)}) = f_{n,c_s(p_h(n))}(y)$
- So, F* is among this set of functions, and it is a poly-time reduction from the complement of SAT to S, thus it is a valid pruning function.
- The problem: we do not know the census function, so we cannot compute F*(y).

A set of pruning functions

- Observation: $c_s(p_h(n)) \leq p(p_h(n))$
- Solution idea: On input x of size n, for all values of k from 0 to $p(p_h(n))$, try the tree-pruning algorithm with pruning function $f_{n,k}$
- One of these k's will work.
- Problem: we are not guaranteed that the tree-pruning algorithm will run in poly time with every choice of pruning function.

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Properties of the correct pruning function F*

- Observation: if k = c_s(p_h(n)), we have that

$$|\left\langle h(y), k, 0^{p_h(n)}\right\rangle| = 2p_h(n) + O[log(p(p_h(n)))]$$

- Thus, there is an n_0 such that for all n >= n_0,

$$|\left\langle h(y), k, 0^{p_h(n)}\right\rangle| \leq 2p_h(n) + p(p_h(n))$$

- This means that the list created by the tree-pruning algorithm will be at most

$$p(p_g\left[2p_h(n) + p(p_h(n))\right])$$

Properties of the correct pruning function F*

- So, we know from the poly-bound proof of the tree-pruning function, that the algorithm visits at most

$$2[|x|p(p_g\left[2p_h(n) + p(p_h(n))\right]) + |x| - 1]$$

  Nodes, which is polynomial in |x|.

- Further, if the amount of nodes explored exceeds this value, we know we have chosen the wrong k.

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SAT is in P

begin {input: x}
    for $k = 0$ to $p(p_h(|x|))$ do
    begin
        execute the tree-visiting algorithm described in the
        proof of Theorem 5.6 using $f_{|x|,k}$ as a pruning function
        and visiting, at most, $2[|x|p(p_g[2p_h(n) + p(p_h(n))]) + |x| - 1]$ inner nodes;
        if the algorithm accepts then accept;
    end;
    reject;
end.

Group A Lecture 3: Sparse Sets and Turing Reductions

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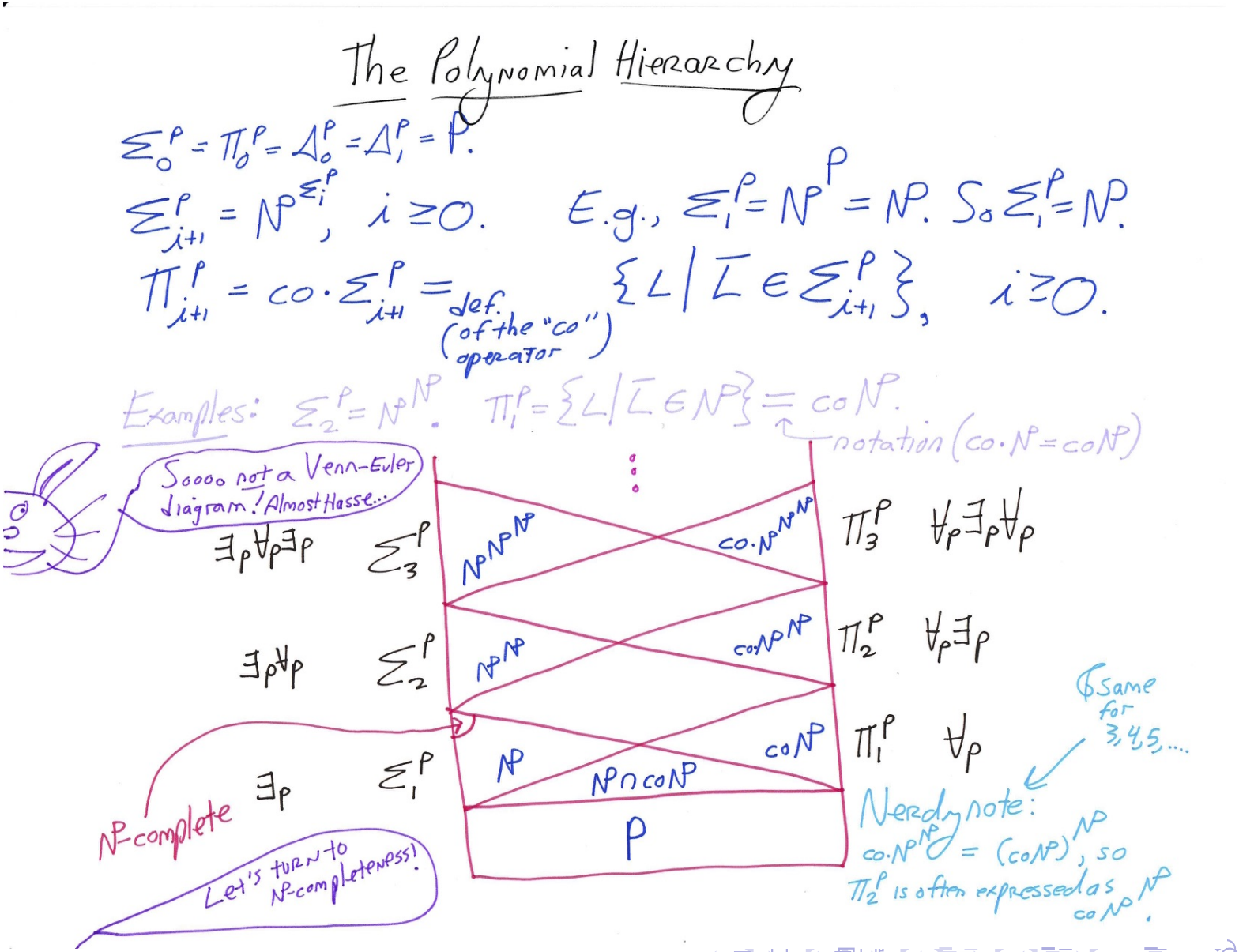
November 9, 2015

Introduction

- We've seen: if a sparse set is NP-hard or NP-complete w.r.t many-one reductions, then P = NP
- Today we investigate whether any sparse set can be NP-hard or NP-complete w.r.t. **Turing reductions**
  - weaker assumption (compared to many-one reductions)
- **Open Question:** do these Turing reduction based hypotheses imply that P = NP?
- We can show that the Polynomial Hierarchy **collapses**, given these assumptions hold
- Intuitively, PH **collapses** when all polynomial classes above a certain order are shown to be equal

The Polynomial Hierarchy



Polynomial Hierarchy Collapse

$$PH = \bigcup_i \Sigma_i^p$$

$$= P \cup NP \cup NP^{NP} \cup NP^{NP^{NP}} \cup NP^{NP^{NP^{NP}}} \cup \ldots$$

Theorem (Gold) [1]

If $\Sigma_i^p = \Pi_i^p$, then $PH = \Sigma_i^p$

Example

if NP = coNP, then PH = NP

- An implication: any problem, that can be solved using an NP machine with access to an NP oracle, can also be solved with a non-det poly-time TM with no access to oracle

[1] http://www.cs.cornell.edu/courses/cs6810/2009sp/scribe/lecture5.pdf

Topics

- Can sparse sets be NP-complete or NP-hard w.r.t Turing reductions? What are the implications?

Theorem 1.14 [Hemaspaandra-Ogihara]

If NP has sparse Turing-complete sets, then the Polynomial Hierarchy collapses to $P^{NP}[\log n]$

Theorem 1.15 [Hem-Ogi] (also called Karp-Lipton Thm)

If NP has sparse Turing-hard sets, then the Polynomial Hierarchy collapses to NP^{NP}

Bounded/restricted query classes[2]

- **Motivation:** add new classes to PH to capture problems that can be solved by restricting the number of queries made to oracle to $O(\log n)$ (instead of polynomial)

E.g. Odd colorability in graphs

- it is in P^{NP} and is NP-hard, but not known whether it is in NP
- but it can be solved in $P^{NP}[\log n]$ by using a P^{NP} machine making only $O(\log n)$ (rather than polynomially many) queries to an NP oracle.

[2] Wagner, Klaus W. "Bounded query classes." SIAM Journal on Computing 19.5 (1990): 833-846.

Theorem 1.14

Theorem 1.14

If NP has sparse NP $\leq_T^p$-complete sets, then

$$PH = P^{NP}[\log n]$$

Proof Strategy

- proof uses "census" approach
- For a sparse set, the census approach is to first obtain the exact number of elements in the set up to some given length, and then exploit that information.

Proof of Theorem 1.14

1) Let S be a sparse set s.t. S is $\leq_T^p$ $-complete$ for NP
2) For any ℓ, let $p_\ell(n) = n^\ell + \ell$
3) Let j be s.t. $(\forall n)[|S^{\leq n}| \leq p_j(n)]$
4) Let M be deterministic poly-time TM s.t. SAT $= \mathcal{L}(M^S)$
   - M exists since S is Turing-hard for NP
5) Let k be s.t. $p_k(n)$ bounds runtime of M regardless of M's oracle
6) Let L be an arbitrary set in Σ_2^p (i.e. NP^{NP})
   - Since SAT is NP-complete, we have $\Sigma_2^p = NP^{SAT}$
7) $\exists$ non-det poly-time TM N s.t. $L = \mathcal{L}(N^{SAT})$
8) Let ℓ be s.t. $p_\ell(n)$ bounds non-det runtime of N for all oracles
9) Note that $L = \mathcal{L}(N^{\mathcal{L}(M^S)})$

Proof of Theorem 1.14 (Continued)

$$V = \{0\#1^n\#1^q \mid |S^{\leq n}| \geq q\}$$
$$\bigcup$$
$$\{1\#x\#1^n\#1^q \mid (\exists Z \subseteq S^{\leq n})\ [|Z| = q; \wedge\ x \in \mathcal{L}(N^{\mathcal{L}(M^Z)})]\}$$

- $V \in NP$ since $S \in NP$
- 1st set (census): will be exploited to compute $|S^{\leq n}|$
- 2nd set: will be used to check whether $x \in L$

Proof of Theorem 1.14 (Continued)

- The following $P^{NP}[\log n]$ algorithm accepts L by making $O(\log n)$ calls to the NP oracle V, for each input string y

Step 1

- In $O(\log |y|)$ sequential queries to V, compute $|S^{\leq p_k(p_\ell(|y|))}|$
  - queries of the form $0\#1^{p_k(p_\ell(|y|))}\#1^z$
  - vary z as in binary search until we find $|S^{\leq p_k(p_\ell(|y|))}|$
- Since $|S^{\leq p_k(p_\ell(|y|))}|$ is bounded by $p_j(p_k(p_\ell(|y|)))$, thus $O(\log |y|)$ queries are sufficient to find census value
- Let the census value obtained be r

Proof of Theorem 1.14 (Continued)

Step 2

Ask V the query $1\#y\#1^{p_k(p_\ell(|y|))}\#1^r$, and accept if and only if this query $\in V$

- Clearly this is a $P^{NP}[\log n]$ algorithm
- Algorithm accepts L
- Since $L \in \Sigma_2^p$ $(= NP^{NP})$, we have $\Sigma_2^p = P^{NP}[\log n]$.
- Since $P^{NP}[\log n]$ is closed under complementation, we have $\Sigma_2^p = \Pi_2^p$
- Therefore by Theorem (Gold), $PH = \Sigma_2^p = P^{NP}[\log n]$ □

THE KARP-LIPTON THEOREM

If there is a sparse NP-Turing-Hard set ($\exists S \in SPARSE \; \forall T \in NP \; T \leq_T^P S$)

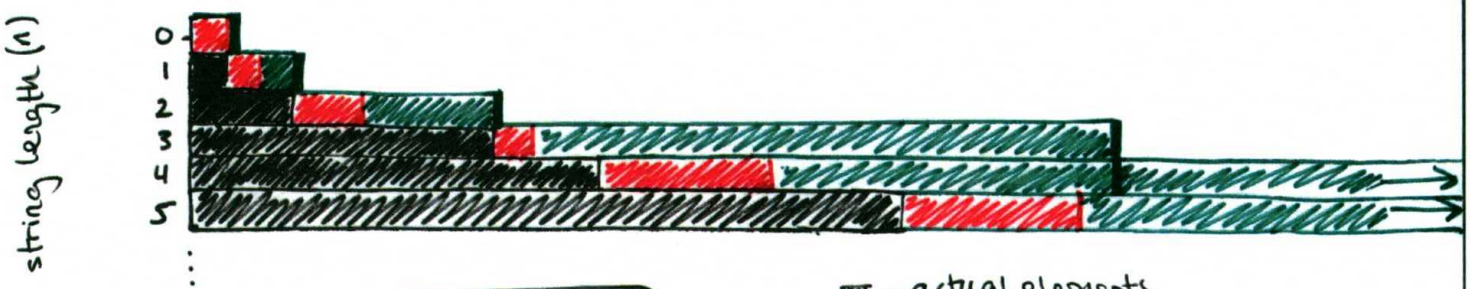
the polynomial hierarchy collapses to NP^{NP} ($PH = NP^{NP}$)

(also Congratulations if you did the reading, it is almost impossible)

A LITTLE REVIEW

What the heck does this mean?

SPARSE SETS:



$p(n) = n^2 + 1$

alphabet size $|\Sigma| = 3$

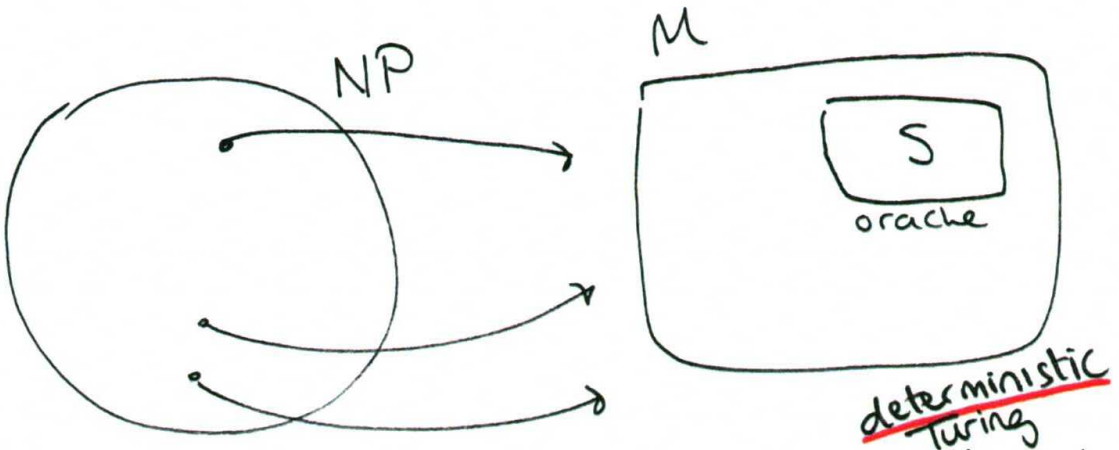
It's easy to see why they're called sparse.
As a proportion of possible elements, sparse sets contain almost nothing.

$$\left(|\Sigma^{*=n}| = |\Sigma|^n = 3^n\right)$$

A LITTLE REVIEW

A sparse NP-Turing-Hard set:

We have a sparse set S



every set in NP can be solved by a machine in polynomial time with an oracle for S.

(not necessarilly the same machine M)

A LITTLE REVIEW

The polynomial hierarchy (PH):

$$PH = P \cup NP \cup NP^{NP} \cup NP^{NP^{NP}} \cup NP^{NP^{NP^{NP}}} \cup \dots$$

Note: $NP^P = NP$

and

$$P \subseteq NP \subseteq NP^{NP} \subseteq NP^{NP^{NP}} \subseteq NP^{NP^{NP^{NP}}} \subseteq \dots$$

A LITTLE REVIEW

Polynomial time hierarchy collapse:

essentially when all subset-equals signs become equals signs above a certain order.

OR in other words

ALL polynomial classes above a certain order are shown to be equal

$$P \subseteq NP \subseteq NP^{NP} \overset{=}{\subseteq} NP^{NP^{NP}} \overset{=}{\subseteq} NP^{NP^{NP^{NP}}} \overset{=}{\subseteq} \dots$$

(a collapse of PH to NP^{NP})

$$PH = P \cup NP \cup NP^{NP} \cup NP^{NP^{NP}} \cup NP^{NP^{NP^{NP}}} \cup \dots$$

NP^{NP}

LINES OF INQUIRY

How do we show a polynomial hierarchy collapse to NP^{NP}?

Notice if we show $NP^{NP} = NP^{NP^{NP}}$, all other higher order polynomial classes will collapse ~~recursively~~ inductively:

$$\ldots \subseteq NP^{NP} = NP^{NP^{NP}} \stackrel{?}{=} NP^{NP^{NP^{NP}}} \stackrel{?}{=} NP^{NP^{NP^{NP^{NP}}}} \stackrel{?}{=} \ldots$$

so $NP^{NP} = NP^{NP^{NP}}$ is all we need to show

AVAILABLE TOOLS

Remember we need to use our sparse set S and the fact that any language in NP Turing reduces to S.

Since SAT is in NP,

$$SAT \leq_T^P S$$

So we have a machine M such that

$$SAT = L(M^S)$$

IMPORTANT: M is a deterministic Turing machine that runs in polynomial time

In other words this machine M given an oracle for S is a machine for SAT.

LINES OF INQUIRY

We want to show that an arbitrary element of $NP^{NP^{NP}}$ can be solved by an element of NP^{NP}.

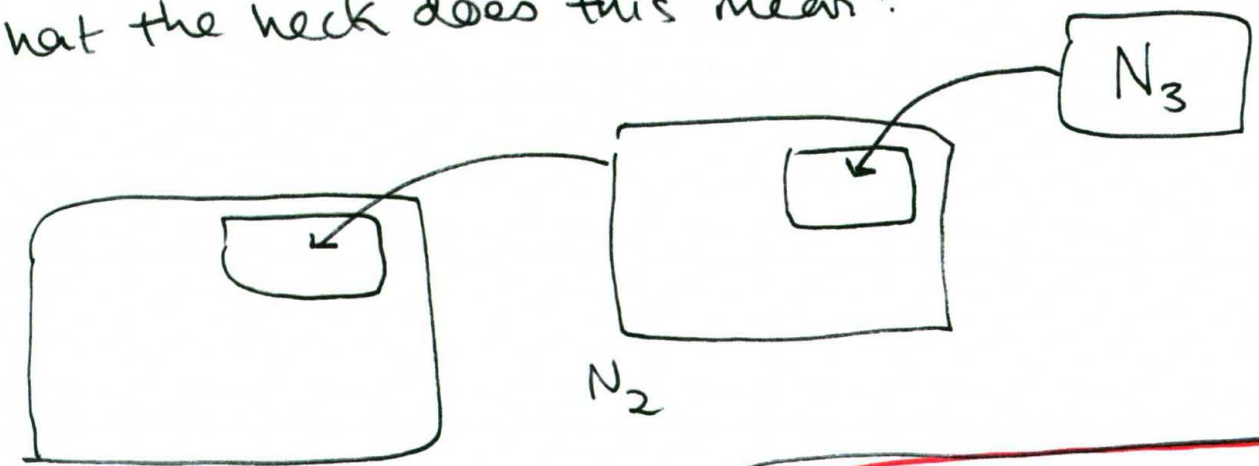
First we must see how to represent these:

$$L \in NP^{NP^{NP}}$$

$$L = L\left(N_1^{L(N_2^{L(N_3)})}\right)$$ for machines N_1, N_2, N_3 (nondeterministic polytime)

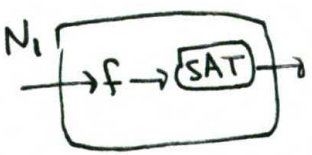
LINES OF INQUIRY

What the heck does this mean?



$$L = L\left(N_1^{L\left(N_2^{L(N_3)}\right)}\right)$$

Note: since SAT is NP complete,



we can make N_2 do the polynomial work of f and replace N_1 with a machine for SAT

POLYNOMIAL BOUNDS

Here are the polynomial bounds we will use and keep track of:

First, define $p_\ell(n)$ for natural numbers ℓ and n

$$p_\ell(n) = n^\ell + \ell.$$

size j: $\forall n \left[\|S^{\leq n}\| \leq p_j(n) \right]$ (we can polynomially bound any prefix of our sparse set)

runtime k: $\text{runtime}\left(M^A(x)\right) \leq p_k(|x|)$ For all oracles A
(M is our deterministic polynomial TM such that $L(M^S) = SAT$)

runtime ℓ: $\max\left(\text{runtime}\left(N_1^A(x)\right), \text{runtime}\left(N_2^A(x)\right)\right) \leq p_\ell(|x|)$ for all oracles A
(N_1 and N_2 are nondeterministic polynomial TMs we will be investigating in our proof)

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Pause to Ponder 1.15 *Show why this "without loss of generality claim" holds.*

(Answer sketch for Pause to Ponder 1.15: Given a machine M, let the machines $M_1, M_2, \ldots$, be as follows. $M_i^A(x)$ will simulate the action of exactly $p_i(|x|)$ steps of the action of $M^A(x)$, and then will halt in an accepting state if $M^A(x)$ halted and accepted within $p_i(|x|)$ steps, and otherwise will reject. Note that since the overhead involved in simulating one step of machine is at most polynomial, for each i, there will exist an $\widehat{i}$ such that for every A it holds that M_i^A runs in time at most $p_{\widehat{i}}(n)$. Furthermore, in each relativized world A in which M^A runs in time at most p_i, it will hold that $L(M^A) = L(M_i^A)$. Relatedly, in our proof, given the machine M such that SAT $= L(M^S)$, we will in light of whatever polynomial-time bound M^S obeys similarly replace M with an appropriate M_j from the list of machines just described.)

HOW DO POLYNOMIAL BOUNDS WORK?

Consider $M^A(|x|)$. Its runtime is bounded by $p_k(|x|)$.

At each step M^A can print at most one character/letter.
So the largest query M can make to A is one in which it uses all its steps to form the query and submits it on the last step.
So query size is bounded by $p_k(|x|)$.

MAIN INSIGHTS

Can we simulate N_2^{SAT} which works in NP^{NP} with a machine that works in NP?

$N_2^{SAT \leq |x|}$ works in "NP" since $SAT^{\leq |x|}$ is finite and in "P" and "NP^P" = NP

(x is the input to SAT, not to N_2. It is the query.)

(Since we can't actually easily find $SAT^{\leq n}$ this is assuming we can)

MAIN INSIGHTS

Simulating SAT:

Suppose you know the length of the input to M, $|x|$.

M can make a query of length at most $p_K(|x|)$.

On inputs of these lengths,

$M^{S \leq p_K(|x|)}$ simulates SAT

Takeaway:

A finite prefix of S is enough to simulate SAT

gives the input length.

Since M is deterministic and polynomical and $S^{\leq P_R(|x|)}$, being finite is in P

$$M^{S^{\leq P_R(|x|)}} \in P^P = P$$

Now that we have the main insights, I will walk you through the meat of the formal proof.

Define

$V_0 = \{0\#1^n\#S' \mid (\exists z \in (\Sigma^*)^{\leq n})[$(a) z is not a well-formed formula and $M^{S'}(z)$ accepts; or (b) z is a well-formed formula free variables and either (b1) $M^{S'}(z)$ accepts and $z \notin$ SAT or (b2) $M^{S'}(z)$ rejects and $z \in$ SAT; or (c) z is a well-formed formula variables $z_1, z_2, \ldots$ and it is *not* the case that: $M^{S'}(z)$ accepts if and only if

$(M^{S'}(z[z_1 = \text{True}])$ accepts $\vee\, M^{S'}(z[z_1 = \text{False}])$ accepts$)]\,\}$,

where, as defined earlier in this chapter, $z[...]$ denotes z with the indicated variables assigned as noted.

$V_1 = \{1\#S'\#z \mid z \in L(N_2^{L(M^{S'})})\}$.

$V = V_0 \bigcup V_1$.

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Step 1 Nondeterministically guess a set $S' \subseteq (\Sigma^*)^{\leq p_k(p_\ell(p_\ell(|y|)))}$ satisfying $||S'|| \leq p_j(p_k(p_\ell(p_\ell(|y|))))$. If $0\#1^{p_k(p_\ell(p_\ell(|y|)))}\#S' \in V$ then reject. Otherwise, go to Step 2.

Step 2 Simulate the action of $N_1(y)$ except that, each time $N_1(y)$ makes a query z to its $L(N_2^{\mathrm{SAT}})$ oracle, ask instead the query $1\#S'\#z$ to V.