Chapter 5:
The Witness Reduction Technique

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Is #P closed under proper subtraction?

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Definition 8.2 (**#P**) **#P** is the set of all functions $f: \{0, 1\}^* \to \mathbb{N}$ such that there is a NPTM M such that for all $x \in \{0, 1\}^*$,

$f(x)$ = number of accepting branches in M's computation graph on x

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Proper Subtraction ($\ominus$) $a \ominus b = \max(\{0, a - b\})$

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Let σ be an operation from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ and let $\mathcal{F}$ be a class of functions from $\mathbb{N}$ to $\mathbb{N}$. We say that $\mathcal{F}$ is closed under (the operation) σ if

$$(\forall f_1 \in \mathcal{F})(\forall f_2 \in \mathcal{F})[h_{f_1,f_2} \in \mathcal{F}]$$

where $h_{f_1,f_2}(n) = \sigma[f_1(n), f_2(n)]$

Other #P Closures

Example: #**P** is closed under *addition*

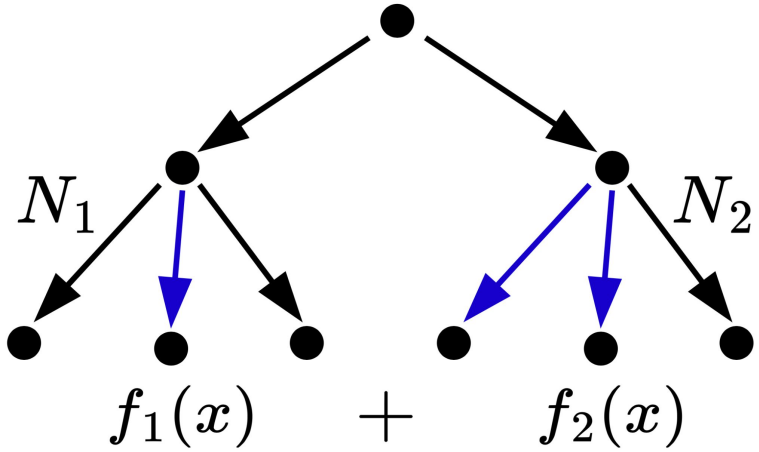
?

Example: #**P** is closed under *multiplication*

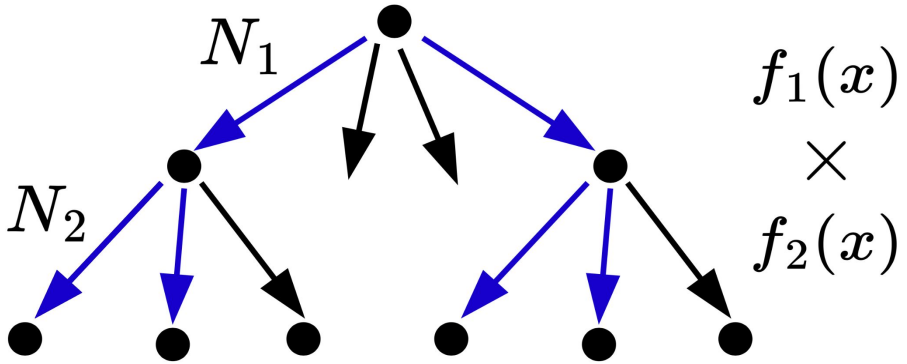
?

Other #P Closures

Example: #**P** is closed under *addition*



Example: #**P** is closed under *multiplication*



Preliminary Definitions

Predicate (does N accept x on y)

Polynomial length of path

Certificates (paths in N)

$$\{ y \mid |y| \leq q(|x|) \wedge R(x, y) \}$$

Preliminary Definitions

Definition of UP

A language L is in **UP** if there is a polynomial-time predicate R and a polynomial q such that for all x,

$$||\{ y \mid |y| \leq q(|x|) \wedge R(x, y)\}|| = \begin{cases} 0 \text{ if } x \notin L \\ 1 \text{ if } x \in L \end{cases}$$

Preliminary Definitions

Definition of PP

A language L is in **PP** if there exists a polynomial q and a polynomial-time predicate R such that for all x,

$$||\{\, y \mid |y| = q(|x|) \wedge R(x, y)\}|| = \begin{cases} < 2^{q(|x|)-1} \text{ if } x \notin L \\ \geq 2^{q(|x|)-1} \text{ if } x \in L \end{cases}$$

Preliminary Definitions

Definition of $\oplus$P

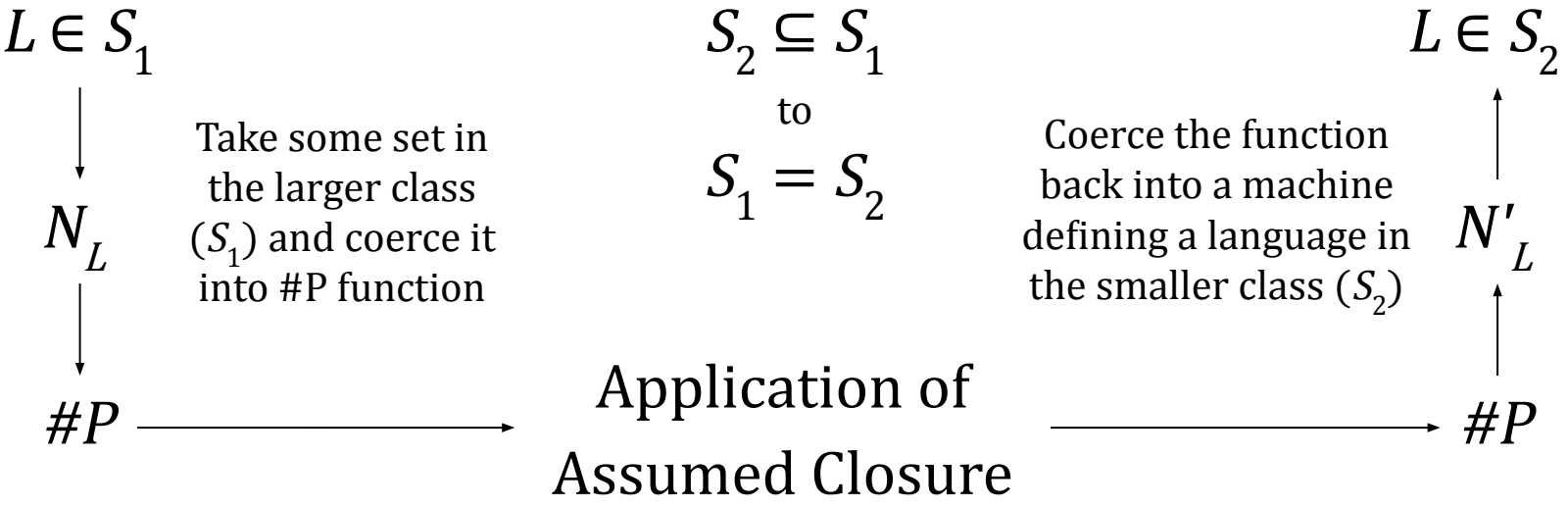
A language L is in $\oplus$**P** if there exists a polynomial q and a polynomial-time predicate R such that for all x,

$$||\{\, y \mid |y| \leq q(|x|) \wedge R(x, y)\}|| = \begin{cases} even \text{ if } x \notin L \\ odd \text{ if } x \in L \end{cases}$$

Witness Reduction Technique

- Witness?
- Adding is easy, removing is difficult
- Consequences of the possibility of removing witnesses
- Complexity class collapse
- Idea of the technique

Witness Reduction Technique



Is #P closed under proper subtraction?

- Find a collapse that characterizes the closure

- Operator ‘completeness’ for #**P**

Is #P closed under proper subtraction?

Theorem 5.6 *The following statements are equivalent*

1. **#P** is closed under proper subtraction.
2. **#P** is closed under every polynomial-time computable operation
3. **UP = PP**

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1. Closed (**#P**,⊖)
2. Closed (**#P**,*op*) (*op* is any arbitrary polynomial-time computable operation)
3. UP = PP

Is #P closed under proper subtraction?

$$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{UP} = \mathbf{PP}$$

$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \mathbf{UP} \wedge \mathbf{UP} \subseteq \mathbf{PP}$

$\mathbf{UP} \subseteq \mathbf{PP}$ by direct proof

Is #P closed under proper subtraction?

UP $\subseteq$ **PP** by direct proof

1. Let L be a UP language and let N be the NPTM that accepts L
2. Let N' be a NPTM with the same depth, $q(|x|)$, as N and accepts on all paths but one
3. Let N_{PP} be a NPTM that chooses between simulating N and N'

Computation Paths : $2^{q(|x|)+1}$

Accepting Paths : $2^{q(|x|)}$ $(N: 1, N': 2^{q(|x|)} - 1)$

Is #P closed under proper subtraction?

$$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{UP} = \mathbf{PP}$$

$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \mathbf{UP} \wedge \mathbf{UP} \subseteq \mathbf{PP}$

$\mathbf{UP} \subseteq \mathbf{PP}$ by direct proof

$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \mathbf{UP}$

$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \text{Co}\mathbf{NP} \wedge \text{Co}\mathbf{NP} \subseteq \mathbf{UP}$

Is #P closed under proper subtraction?

$$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \text{Co}\mathbf{NP}$$

$L \in \text{PP}$

$L = \{ x \mid ||\{ y \mid |y| = q(|x|) \wedge \text{R}(x, y)\}|| \geq 2^{q(|x|)-1} \}$

Let $q'(|x|) = q(|x|) + 1$ and for $b \in \{0, 1\}$, $\text{R}'(x, yb) = \text{R}(x, y)$ and, for all n, $q(n) \geq 1$ (avoid $q(|x|) = 0$)

Is #P closed under proper subtraction?

$$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \text{Co}\mathbf{NP}$$

Let N be an NPTM that on input x guesses each y such that $|y| = q(|x|)$ then tests $R(x, y)$

The $\#\mathbf{P}$ function f defined by this NPTM has that

$x \in L \rightarrow f(x) \geq 2^{q(|x|)\,-\,1}$

$x \notin L \rightarrow f(x) < 2^{q(|x|)\,-\,1}$

Is #P closed under proper subtraction?

$$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \text{Co}\mathbf{NP}$$

$g(x) = 2^{q(|x|)-1} - 1$ is a #**P** function

By assumption of closure under proper subtraction

$$\hbar(x) = f(x) \ominus g(x)$$

is a #**P** function, and by substituting yields

$\hbar(x) \geq 1$ if $x \in L$

$\hbar(x) = 0$ if $x \notin L$ Clearly **NP**, but …

Is #P closed under proper subtraction?

$$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \text{Co}\mathbf{NP}$$

There exists a NPTM N for which $\hbar$ computes the number of accepting paths

The values of $\hbar$ are such that N is an NP machine, so the arbitrary **PP** language is in **NP**

It follows that $\mathbf{PP} \subseteq \mathbf{NP}$

$\mathbf{PP} = \text{Co}\mathbf{PP} \rightarrow \mathbf{PP} \subseteq \text{Co}\mathbf{NP}$

Is #P closed under proper subtraction?

$$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \text{Co}\mathbf{NP} \subseteq \mathbf{UP}$$

Let L be an arbitrary Co**NP** language

There exists a NPTM N that accepts $\overline{L}$

N defines a #**P** function f such that

$x \in L \rightarrow f(x) = 0$

$x \notin L \rightarrow f(x) \geq 1$

Is #P closed under proper subtraction?

$$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \text{Co}\mathbf{NP} \subseteq \mathbf{UP}$$

The constant function $g(x) = 1$ is a #**P** function

By assumption of closure under proper subtraction

$$h(x) = g(x) \ominus f(x)$$

is a #**P** function, and by substituting yields

$h(x) = 1$ if $x \in L$

$h(x) = 0$ if $x \notin L$

h corresponds to a **UP** machine

Is #P closed under proper subtraction?

Theorem 5.6 *The following statements are equivalent*

1. **#P** is closed under proper subtraction.
2. **#P** is closed under every polynomial-time computable operation
3. **UP = PP**

?

Is #P closed under proper subtraction?

$$\mathbf{UP} = \mathbf{PP} \rightarrow \text{Closed}(\#\mathbf{P}, op)$$

1. Let op be an arbitrary polynomial-time computable operation
2. Let f and g be arbitrary #**P** functions

$$B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \in \mathbf{PP}$$
$$B_g = \{ \langle x, n \rangle \mid g(x) \geq n \} \in \mathbf{PP}$$

Is #P closed under proper subtraction?

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$$B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \in \mathbf{PP}$$
$$B_g = \{ \langle x, n \rangle \mid g(x) \geq n \} \in \mathbf{PP}$$

$$V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \wedge \langle x, n_1 + 1 \rangle \notin B_f \wedge \langle x, n_2 \rangle \in B_g \wedge \langle x, n_2 + 1 \rangle \notin B_g \}$$

Is #P closed under proper subtraction?

$$\mathbf{UP} = \mathbf{PP} \rightarrow \text{Closed}(\#\mathbf{P}, op)$$

V gives precise values of $f(x)$ and $g(x)$ by testing adjacent ns to find transition points in B_f and B_g

V 4-truth-table reduces to the language $B_f \oplus B_g$

$\oplus$ denotes disjoin Union: $Y \oplus Z = \{0x \mid x \in Y\} \cup \{1x \mid x \in Z\}$

Is #P closed under proper subtraction?

$$\mathbf{UP} = \mathbf{PP} \rightarrow \text{Closed}(\#\mathbf{P}, op)$$

V gives precise values of $f(x)$ and $g(x)$ by testing adjacent ns to find transition points in B_f and B_g

Corollary 9.17 PP is closed under polynomial time bounded-truth-table reductions and disjoint union $\rightarrow V \in \mathbf{PP}$

$V \in \mathbf{UP}$ by assumption that $\mathbf{UP} = \mathbf{PP}$

Is #P closed under proper subtraction?

$$\mathbf{UP} = \mathbf{PP} \rightarrow \text{Closed}(\#\mathbf{P}, op)$$

f and g are #**P** functions, so for some polynomial q and for all x
$\max\{f(x), g(x)\} \leq 2^{q(|x|)}$

Consider the **NP** machine N that on input x

1. Nondeterministically choose an integer i, $0 \leq i \leq 2^{q(|x|)}$
2. Nondeterministically choose an integer j, $0 \leq j \leq 2^{q(|x|)}$
3. Guesses a computation path of V on input $\langle x, i, j \rangle$. If the path rejects, reject. Otherwise, nondeterministically guess an integer k, $1 \leq k \leq op(i, j)$ and accept.

Is #P closed under proper subtraction?

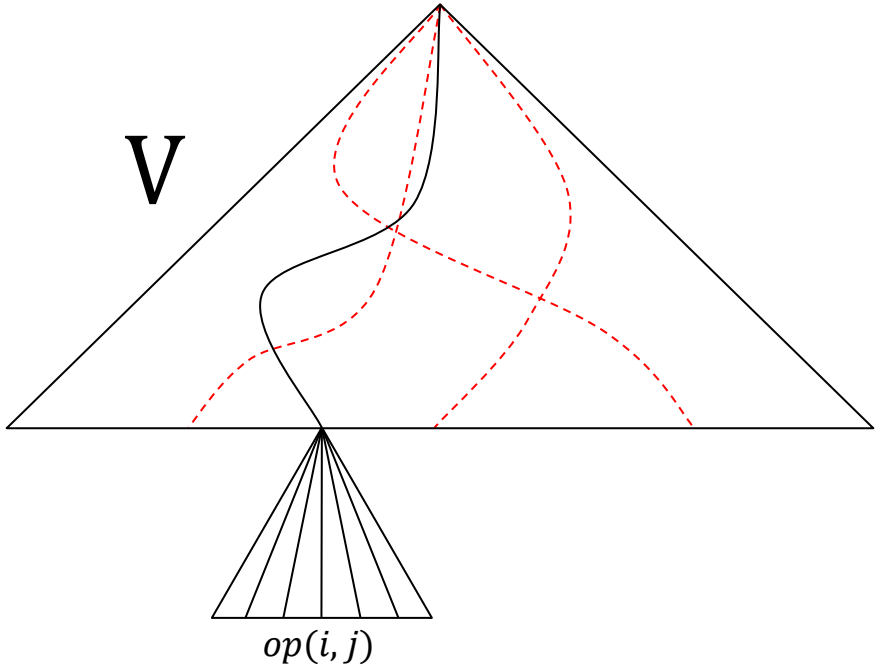
$$\mathbf{UP} = \mathbf{PP} \rightarrow \text{Closed}(\#\mathbf{P}, op)$$

For all $i \neq f(x)$ and $j \neq g(x)$, $V(\langle x, i, j \rangle)$ rejects

For the correct i and j, $N(x)$ accepts along precisely $op(i, j)$ paths

The $\#\mathbf{P}$ function defined by this machine is $h(x) = op(f(x), g(x))$

Is #P closed under proper subtraction?



How Significant is the Collapse?

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Theorem 5.7 *The following statements are equivalent*

1. $\mathbf{UP} = \mathbf{PP}$
2. $\mathbf{UP} = \mathbf{NP} = \mathrm{co}\mathbf{NP} = \mathbf{PH} = \oplus\mathbf{P} = \mathbf{PP} = \mathbf{PP} \cup \mathbf{PP}^{\mathbf{PP}} \cup \ldots$

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Theorem 5.7 *The following statements are equivalent*

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Prove each of the above in sequence

How Significant is the Collapse?

$$\mathbf{UP} = \mathbf{NP} = \mathbf{coNP} = \mathbf{PH} = \oplus\mathbf{P} = \mathbf{PP} = \mathbf{PP} \cup \mathbf{PP}^{\mathbf{PP}} \cup \ldots$$

NP ⊆ PP

1. Let $L \in$ **NP** and let N be the NPTM that accepts L
2. Let N' be a NPTM with the same depth, $q(|x|)$, as N and accepts on all paths but one
3. Let N_{PP} be a NPTM that chooses between simulating N and N'

Computation Paths : $2^{q(|x|)+1}$

Accepting Paths : $\geq 2^{q(|x|)}$ (N : ≥ 1, N' : $2^{q(|x|)} - 1$)

How Significant is the Collapse?

$$\mathbf{UP} = \mathbf{NP} = \mathbf{coNP} = \mathbf{PH} = \oplus\mathbf{P} = \mathbf{PP} = \mathbf{PP} \cup \mathbf{PP}^{\mathbf{PP}} \cup \ldots$$

$$\mathbf{UP} \subseteq \mathbf{NP} \subseteq \mathbf{PP}, \mathbf{UP} = \mathbf{PP} \rightarrow \mathbf{UP} = \mathbf{NP}$$

1. Let $L \in \mathbf{NP}$ and let N be the NPTM that accepts L
2. Let N' be a NPTM with the same depth, $q(|x|)$, as N and accepts on all paths but one
3. Let N_{PP} be a NPTM that chooses between simulating N and N'

Computation Paths : $2^{q(|x|)+1}$

Accepting Paths : $\geq 2^{q(|x|)}$ (N : ≥ 1, N' : $2^{q(|x|)} - 1$)

How Significant is the Collapse?

$$\mathbf{UP} = \mathbf{NP} = \mathrm{Co}\mathbf{NP} = \mathbf{PH} = \oplus\mathbf{P} = \mathbf{PP} = \mathbf{PP} \cup \mathbf{PP}^{\mathbf{PP}} \cup \ldots$$

$$\mathbf{PP} = \mathrm{Co}\mathbf{PP}$$

1. Let $L \in \mathbf{PP}$ and let N be the NPTM that accepts L
2. Let N' be equivalent to N, but ensuring the rightmost path rejects
3. Let $N_{\mathrm{Co}\mathbf{PP}}$ a NPTM that takes N' and expands down one level

For the rightmost leaf node of N', one child accepts and one rejects

For accepting leaf nodes of N', both children reject

For rejecting leaf nodes of N', both children accept

How Significant is the Collapse?

$$\mathbf{UP} = \mathbf{NP} = \mathrm{co}\mathbf{NP} = \mathbf{PH} = \oplus\mathbf{P} = \mathbf{PP} = \mathbf{PP} \cup \mathbf{PP}^{\mathbf{PP}} \cup \ldots$$

$$\mathbf{PP} = \mathrm{co}\mathbf{PP}$$

1. Let $L \in \mathbf{PP}$ and let N be the NPTM that accepts L
2. Let N' be equivalent to N, but ensuring the rightmost path rejects
3. Let y be the number of accepting paths in N'
4. Let $h - 1$ be the depth of N

Rejecting Paths : $2y + 1$

Accepting Paths : $2^{h+1} - 2y - 1$

How Significant is the Collapse?

$$\mathbf{UP} = \mathbf{NP} = \mathrm{co}\mathbf{NP} = \mathbf{PH} = \oplus\mathbf{P} = \mathbf{PP} = \mathbf{PP} \cup \mathbf{PP}^{\mathbf{PP}} \cup \ldots$$

$$\mathbf{PP} = \mathrm{co}\mathbf{PP}$$

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3. Let y be the number of accepting paths in N'
4. Let $h - 1$ be the depth of N

$$\mathbf{PP} = \mathrm{co}\mathbf{PP}, \mathbf{NP} = \mathbf{PP} \rightarrow \mathbf{NP} = \mathrm{co}\mathbf{NP}$$

How Significant is the Collapse?

$$\mathbf{UP} = \mathbf{NP} = \mathrm{co}\mathbf{NP} = \mathbf{PH} = \oplus\mathbf{P} = \mathbf{PP} = \mathbf{PP} \cup \mathbf{PP}^{\mathbf{PP}} \cup \ldots$$

$$\mathbf{NP} = \mathrm{co}\mathbf{NP} \rightarrow \mathbf{NP}^{\mathbf{NP} \cap \mathrm{co}\mathbf{NP}} = \mathbf{NP}^{\mathbf{NP}} = \mathbf{NP} \rightarrow \mathbf{PH} = \mathbf{NP}$$

How Significant is the Collapse?

$$\mathbf{UP} = \mathbf{NP} = \mathrm{co}\mathbf{NP} = \mathbf{PH} = \oplus\mathbf{P} = \mathbf{PP} = \mathbf{PP} \cup \mathbf{PP}^{\mathbf{PP}} \cup \ldots$$

$$\mathbf{P}^{\mathbf{NP}} \subseteq \mathbf{PH},\ \mathbf{NP} = \mathbf{UP} \rightarrow \mathbf{P}^{\mathbf{UP}} \subseteq \mathbf{PH},\ \mathbf{UP} = \mathbf{PH} \rightarrow \mathbf{P}^{\mathbf{UP}} = \mathbf{UP}$$

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$$\mathbf{Lemma\ 4.14} \rightarrow \mathbf{PP}^{\oplus\mathbf{P}} \subseteq \mathbf{P}^{\mathbf{PP}},\ \mathbf{P}^{\mathbf{PP}} = \mathbf{UP},\ \mathbf{UP} \subseteq \oplus\mathbf{P} \subseteq \mathbf{PP}^{\oplus\mathbf{P}}$$

$$\rightarrow \mathbf{UP} = \oplus\mathbf{P}$$

How Significant is the Collapse?

$$\mathbf{UP} = \mathbf{NP} = \mathrm{co}\mathbf{NP} = \mathbf{PH} = \oplus\mathbf{P} = \mathbf{PP} = \mathbf{PP} \cup \mathbf{PP}^{\mathbf{PP}} \cup \ldots$$

$$\mathbf{P}^{\mathbf{NP}} \subseteq \mathbf{PH},\ \mathbf{NP} = \mathbf{UP} \rightarrow \mathbf{P}^{\mathbf{UP}} \subseteq \mathbf{PH},\ \mathbf{UP} = \mathbf{PH} \rightarrow \mathbf{P}^{\mathbf{UP}} = \mathbf{UP}$$

$$\textbf{Lemma 4.14} \rightarrow \mathbf{PP}^{\oplus\mathbf{P}} \subseteq \mathbf{P}^{\mathbf{PP}},\ \mathbf{PP}^{\mathbf{PP}} = \mathbf{UP},\ \mathbf{UP} \subseteq \oplus\mathbf{P} \subseteq \mathbf{PP}^{\oplus\mathbf{P}}$$

$$\rightarrow \mathbf{UP} = \oplus\mathbf{P}$$

$$\oplus\mathbf{P} = \mathbf{PP},\ \mathbf{PP}^{\oplus\mathbf{P}} = \mathbf{PP} \rightarrow \mathbf{PP}^{\mathbf{PP}} = \mathbf{PP}^{\oplus\mathbf{P}} = \mathbf{PP}$$

$\rightarrow$ any stack of **PP**s of arbitrary height can be reduced to **PP**

More Witness Reduction

Theorem 5.9 *The following statements are equivalent*

1. **#P** is closed under integer division
2. **#P** is closed under every polynomial-time computable operation
3. **UP** = **PP**

Definition 5.8 Let $\mathcal{F}$ be a class of functions from $\mathbb{N}$ to $\mathbb{N}$. We say that $\mathcal{F}$ is closed under integer division ($\oslash$) if

$$(\forall f_1 \in \mathcal{F})(\forall f_2 \in \mathcal{F} : (\forall n)[f_2(n) > 0])[f_1 \oslash f_2 \in \mathcal{F}]$$

More Witness Reduction

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More Witness Reduction

$$\text{Closed}(\#\mathbf{P}, \oslash) \rightarrow \mathbf{UP} = \mathbf{PP}$$

Let L be any **PP** set, there is a NPTM N and integer $k \geq 1$ such that

1. On each input x, $N(x)$ has exactly $2^{|x|^k}$ computation paths, each containing $|x|^k$ binary choices,
2. On each input x, $x \in L$ iff $N(x)$ has at least $2^{|x|^k - 1}$ accepting paths, and
3. On each input x, $N(x)$ has at least one rejecting path.

More Witness Reduction

$$\text{Closed}(\#\mathbf{P}, \oslash) \rightarrow \mathbf{UP} = \mathbf{PP}$$

Let f be the $\#\mathbf{P}$ function for N

Let g be the $\#\mathbf{P}$ function $g(x) = 2^{|x|^k} - 1$

By assumption of closure under integer division

$$h(x) = f(x) \oslash g(x)$$

is a $\#\mathbf{P}$ function

More Witness Reduction

$$\text{Closed}(\#\mathbf{P},\varnothing) \rightarrow \mathbf{UP} = \mathbf{PP}$$

$$\text{If } x \in L,\ \mathcal{h}(x) = \left\lfloor \frac{2^{|x|^k-1} \leq f(x) \leq 2^{|x|^k} - 1}{2^{|x|^k-1}} \right\rfloor = 1$$

$$\text{If } x \notin L,\ \mathcal{h}(x) = \left\lfloor \frac{0 \leq f(x) < 2^{|x|^k-1}}{2^{|x|^k-1}} \right\rfloor = 0$$

The NPTM corresponding to $\mathcal{h}$ is a **UP** machine for L

Intermediate Potential Closure Properties

P vs NP Analogues

2CNF-SAT ∈ P	Closed(#P, +)
SAT is NP-complete	Closed(#P, $\ominus$) ⇔ $\forall_{\text{poly-time}}$ *op*, Closed(#P, *op*)
SAT ∈ P ⇔ P = NP	Closed(#P, $\ominus$) ⇔ UP = PP
PRIME ∈ NP, not known to be NP-complete or in P	????

Is #P closed under proper decrement?

- Decrement a restrictive form of subtraction
  - Intuitively more likely that #P could be closed under proper decrement than proper subtraction
- Ideal result:
  - Two classes C and D s.t.:

    #P is closed under proper decrement if and only if $C = D$
- How to approach this? Witness reduction technique!

Is #P closed under proper decrement?

Theorem 5.11

1. If #P is closed under proper decrement, then NP $\subseteq$ SPP.
2. If UP = NP, then #P is closed under proper decrement.

SPP - Generalized UP

Polynomial-time computable function f

NPTM N

$$x \notin L \implies \#\mathrm{acc}_N(x) = f(x) - 1$$
$$x \in L \implies \#\mathrm{acc}_N(x) = f(x)$$

Is #P closed under proper decrement?

Theorem 5.11

1. If #P is closed under proper decrement, then NP $\subseteq$ SPP.
2. If UP = NP, then #P is closed under proper decrement.

$$\text{Closed}(\#\text{P}, \ominus 1) \Longrightarrow \text{NP} \subseteq \text{SPP}$$

$$L \in \text{NP} \longrightarrow \text{NPTM } N \longrightarrow f(x) = \#\text{acc}_N(x) \in \#\text{P}$$

$$\text{Closed}(\#\text{P}, \ominus 1) \Longrightarrow \text{NP} \subseteq \text{SPP}$$

$$L \in \text{NP} \longrightarrow \text{NPTM } N \longrightarrow f(x) = \#\text{acc}_N(x) \in \#\text{P}$$

assume closure under proper decrement

$$g(x) = f(x) \ominus 1 \in \#\text{P}$$

$$\text{Closed}(\#\text{P}, \ominus 1) \Longrightarrow \text{NP} \subseteq \text{SPP}$$

$L \in \text{NP} \longrightarrow \text{NPTM } N \longrightarrow f(x) = \#\text{acc}_N(x) \in \#\text{P}$

assume closure under proper decrement

$$g(x) = f(x) \ominus 1 \in \#\text{P}$$

reverse acceptance behavior of all paths

$$2^{p(|x|)} - g(x) \in \#\text{P}$$

$\text{Closed}(\#\text{P}, \ominus 1) \Longrightarrow \text{NP} \subseteq \text{SPP}$

$L \in \text{NP} \longrightarrow \text{NPTM } N \longrightarrow f(x) = \#\text{acc}_N(x) \in \#\text{P}$

assume closure under proper decrement

$$g(x) = f(x) \ominus 1 \in \#\text{P}$$

reverse acceptance behavior of all paths

$$2^{p(|x|)} - g(x) \in \#\text{P}$$

#P is closed under addition

$$f(x) + 2^{p(|x|)} - g(x) \in \#\text{P}$$

$\text{Closed}(\#\text{P}, \ominus 1) \Longrightarrow \text{NP} \subseteq \text{SPP}$

$$x \notin L \;\rightarrow\; f(x) = 0 \;\rightarrow\; 2^{p(|x|)} + \;\; 0 \;\; - \;\; (0 \ominus 1) \;\; = 2^{p(|x|)}$$

$$x \in L \;\rightarrow\; f(x) > 0 \;\rightarrow\; 2^{p(|x|)} + f(x) - (f(x) - 1) = 2^{p(|x|)} + 1$$

$$f(x) + 2^{p(|x|)} - g(x) \in \#\text{P}$$

$\text{Closed}(\#\text{P}, \ominus 1) \Longrightarrow \text{NP} \subseteq \text{SPP}$

$L \in \text{NP} \longrightarrow \text{NPTM } N \longrightarrow f(x) = \#\text{acc}_N(x) \in \#\text{P}$

assume closure under proper decrement

$g(x) = f(x) \ominus 1 \in \#\text{P}$

reverse acceptance behavior of all paths

$2^{p(|x|)} - g(x) \in \#\text{P}$

#P is closed under addition

$L \in \text{SPP} \longleftarrow \text{NPTM } N' \longleftarrow f(x) + 2^{p(|x|)} - g(x) \in \#\text{P}$

Is #P closed under proper decrement?

Theorem 5.11

1. If #P is closed under proper decrement, then NP $\subseteq$ SPP.
2. If UP = NP, then #P is closed under proper decrement.

$\mathrm{UP} = \mathrm{NP} \Longrightarrow \mathrm{Closed}(\#\mathrm{P}, \ominus 1)$

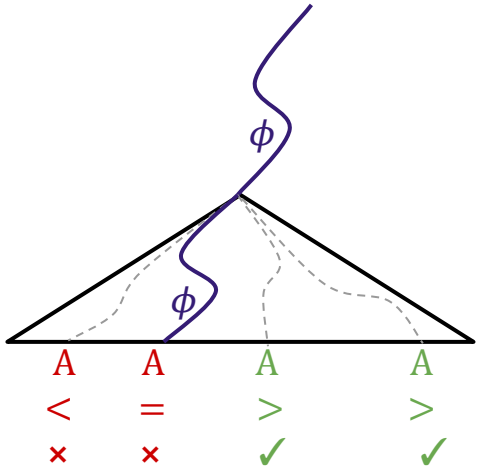
$f \in \#\mathrm{P} \longrightarrow \exists\ \mathrm{NPTM}\ N\ \text{s.t.}\ \#\mathrm{acc}_N(x) = f(x)$

$B = \{\langle x, \phi \rangle : \phi \text{ is an accepting path of } N(x) \wedge$
$\exists \phi' \text{ s.t } \phi < \phi' \wedge\ \phi' \text{ is acc. path of } N(x)\}$

$\mathrm{UP} = \mathrm{NP} \Longrightarrow \mathrm{Closed}(\#\mathrm{P}, \ominus 1)$

$f \in \#\mathrm{P} \longrightarrow \exists$ NPTM N s.t. $\#\mathrm{acc}_N(x) = f(x)$

$B = \{\langle x, \phi \rangle : \phi$ is an accepting path of $N(x) \wedge$
$\exists \phi'$ s.t $\phi < \phi' \wedge \phi'$ is acc. path of $N(x)\}$



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$f \in \#\mathrm{P} \longrightarrow \exists\ \mathrm{NPTM}\ N\ \text{s.t.}\ \#\mathrm{acc}_N(x) = f(x)$

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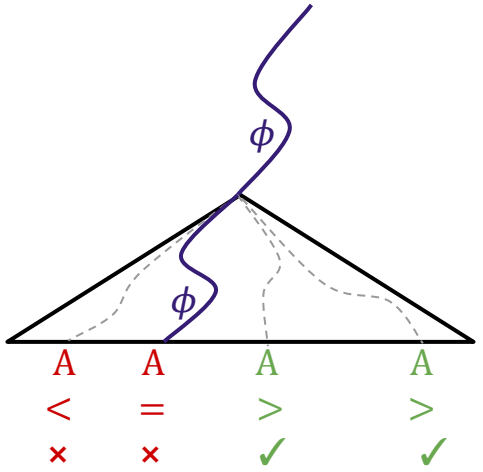
by assumption that UP = NP

$B \in \mathrm{NP} \longrightarrow B \in \mathrm{UP}$



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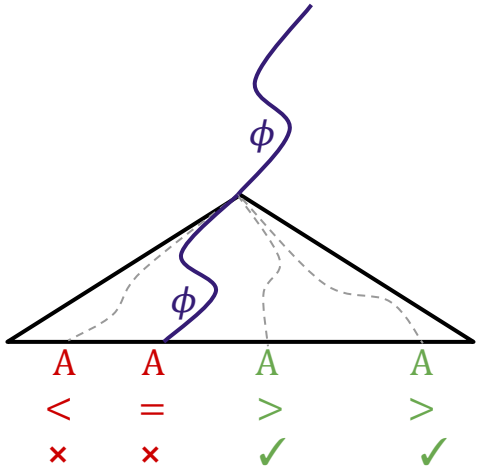
by assumption that UP = NP

$B \in \mathrm{NP} \longrightarrow B \in \mathrm{UP}$

$\exists\ \mathrm{NPTM}\ N'\ \text{s.t.}\ \mathrm{L}(N') = B\ \wedge\ \#\mathrm{acc}_{N'}(x) \leq 1$



$$\mathrm{UP} = \mathrm{NP} \Longrightarrow \mathrm{Closed}(\#\mathrm{P}, \ominus 1)$$

$$\mathrm{L}(N') = B \ \wedge \ \#\mathrm{acc}_{N'}(x) \leq 1$$

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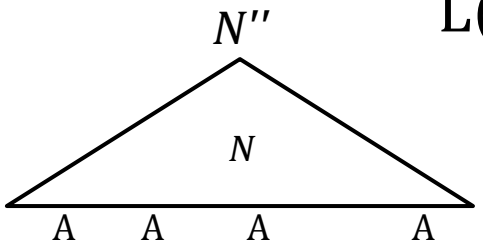
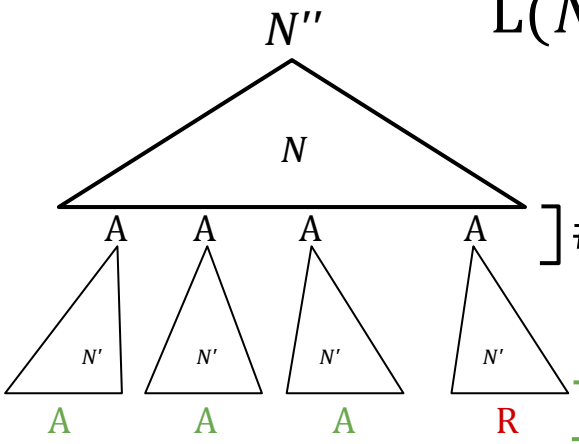


$\mathrm{L}(N') = B \ \wedge \ \#\mathrm{acc}_{N'}(x) \leq 1$

$\#\mathrm{acc}_{N}(x) = f(x)$

$\mathrm{UP} = \mathrm{NP} \Longrightarrow \mathrm{Closed}(\#\mathrm{P}, \ominus 1)$

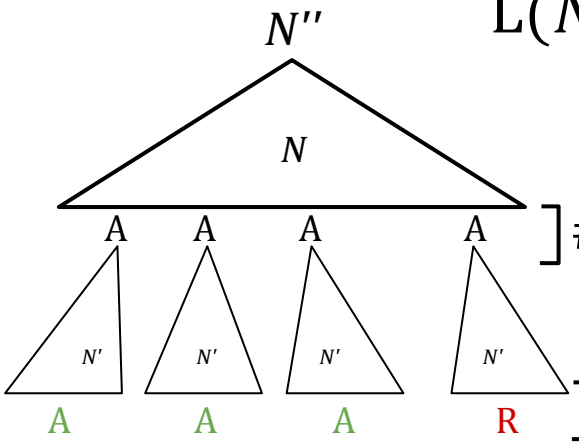


$\mathrm{L}(N') = B \ \wedge \ \#\mathrm{acc}_{N'}(x) \leq 1$

$\#\mathrm{acc}_{N}(x) = f(x)$

$\#\mathrm{acc}_{N'}(x) = \#\mathrm{acc}_{N}(x) \ominus 1$

$\mathrm{UP} = \mathrm{NP} \Longrightarrow \mathrm{Closed}(\#\mathrm{P}, \ominus 1)$



$\mathrm{L}(N') = B \ \wedge \ \#\mathrm{acc}_{N'}(x) \leq 1$

$\#\mathrm{acc}_N(x) = f(x)$

$\#\mathrm{acc}_{N'}(x) = \#\mathrm{acc}_N(x) \ominus 1 = f(x) \ominus 1$

$f(x) \ominus 1 \in \#\mathrm{P}$

Is #P closed under integer division by 2?

- Restricting subtraction to decrement was successful
- Restricting division to division by 2 might be similar

Is #P closed under integer division by 2?

- Restricting subtraction to decrement was successful
- Restricting division to division by 2 might be similar

Theorem 5.12

If #P is closed under integer division by two, then $\oplus P = SPP$ (and thus $PH \subseteq PP$)

Note: $SPP \subseteq \oplus P$ holds unconditionally, we need only show that $\oplus P \subseteq SPP$.

$$\text{Closed}(\#\text{P}, \oslash 2) \Longrightarrow \bigoplus \text{P} = \text{SPP}$$

$$L \in \bigoplus \text{P} \longrightarrow \text{NPTM } N \longrightarrow f(x) = \#\text{acc}_N(x) \in \#\text{P}$$

$$\text{Closed}(\#\text{P}, \oslash 2) \Longrightarrow \oplus\text{P} = \text{SPP}$$

$$L \in \oplus\text{P} \longrightarrow \text{NPTM } N \longrightarrow f(x) = \#\text{acc}_N(x) \in \#\text{P}$$

assume closure under integer division by 2

$$g(x) = 2(f(x) \oslash 2) \in \#\text{P}$$

$$\text{Closed}(\#\text{P}, \oslash 2) \Longrightarrow \oplus\text{P} = \text{SPP}$$

$$L \in \oplus\text{P} \longrightarrow \text{NPTM } N \longrightarrow f(x) = \#\text{acc}_N(x) \in \#\text{P}$$

assume closure under integer division by 2

$$g(x) = 2(f(x) \oslash 2) \in \#\text{P}$$

reverse acceptance behavior of all paths

$$2^{p(|x|)} - f(x) \in \#\text{P}$$

$$\text{Closed}(\#\text{P}, \oslash 2) \Longrightarrow \oplus\text{P} = \text{SPP}$$

$$L \in \oplus\text{P} \longrightarrow \text{NPTM } N \longrightarrow f(x) = \#\text{acc}_N(x) \in \#\text{P}$$

assume closure under integer division by 2

$$g(x) = 2(f(x) \oslash 2) \in \#\text{P}$$

reverse acceptance behavior of all paths

$$2^{p(|x|)} - f(x) \in \#\text{P}$$

#P is closed under addition

$$2^{p(|x|)} - f(x) + 2(f(x) \oslash 2) \in \#\text{P}$$

$\text{Closed}(\#\text{P}, \oslash 2) \Longrightarrow \oplus\text{P} = \text{SPP}$

$$x \notin L \rightarrow f(x) \text{ even} \rightarrow 2^{p(|x|)} - f(x) + 2(\ \ f(x)\ \ )/2 = 2^{p(|x|)}$$

$$x \in L \rightarrow f(x) \text{ odd} \ \rightarrow 2^{p(|x|)} - f(x) + 2(f(x)-1)/2 = 2^{p(|x|)} - 1$$

$$2^{p(|x|)} - f(x) + 2(f(x) \oslash 2) \in \#\text{P}$$

$$\text{Closed}(\#\text{P}, \oslash 2) \Longrightarrow \oplus\text{P} = \text{SPP}$$

$L \in \oplus\text{P} \longrightarrow$ NPTM $N \longrightarrow f(x) = \#\text{acc}_N(x) \in \#\text{P}$

assume closure under integer division by 2

$$g(x) = 2(f(x) \oslash 2) \in \#\text{P}$$

reverse acceptance behavior of all paths

$$2^{p(|x|)} - f(x) \in \#\text{P}$$

#P is closed under addition

$$2^{p(|x|)} - f(x) + 2(f(x) \oslash 2) \in \#\text{P}$$

NPTM N'

$L \in \text{SPP}$

Other operators?

We know UP = NP $\Longrightarrow$ Closed(#P, $\ominus$1), is there any operator that can reverse the direction?

Closed(#P, ???) $\Longrightarrow$ UP = NP

UP is like NP with *at most* 1 accepting path

For NPTM N where $f(x) = \#\text{acc}_N(x)$, what function could we use to coerce this into a UP machine?

$\text{Closed}(\#\text{P}, \min) \Longrightarrow \text{UP} = \text{NP}$

UP is like NP with *at most* 1 accepting path

For NPTM N where $f(x) = \#\text{acc}_N(x)$, what function could we use to coerce this into a UP machine?

$$\min(f(x), 1)$$

(Theorem 5.13.1)

$$\text{Closed}(\#\text{P}, \min) \Longrightarrow \text{UP} = \text{NP}$$

$$L \in \text{NP} \longrightarrow f(x) \in \#\text{P}$$

$$\downarrow$$

$$L \in \text{UP} \longleftarrow \min(f(x), 1) \in \#\text{P}$$

$\text{Closed}(\#\text{P}, \text{min}) \Longrightarrow \text{Closed}(\#\text{P}, \ominus 1)$

Recall that $\text{UP} = \text{NP} \Longrightarrow \text{Closed}(\#\text{P}, \ominus 1)$

We just proved $\text{Closed}(\#\text{P}, \text{min}) \Longrightarrow \text{UP} = \text{NP}$

So as a direct corollary we get

$$\text{Closed}(\#\text{P}, \text{min}) \Longrightarrow \text{Closed}(\#\text{P}, \ominus 1)$$

We can continue chaining prior results to get

$$\text{Closed}(\#\text{P}, \text{min}) \Longrightarrow \text{NP} \subseteq \text{SPP}$$

Is #P closed under min/max?

Theorem 5.13

1. If #P is closed under minimum then UP = NP ✓
2. If #P is closed under maximum or under minimum then $C_{=}P = SPP$

SPP - Generalized UP

Polynomial-time computable function f

NPTM N

$$x \notin L \implies \#\mathrm{acc}_N(x) = f(x) - 1$$
$$x \in L \implies \#\mathrm{acc}_N(x) = f(x)$$

C_P - Exact Counting

Polynomial-time computable function f

NPTM N

$$x \in L \Leftrightarrow \#\mathrm{acc}_N(x) = f(x)$$

C_P - Alternate Definition

Polynomial q

Polynomial-time predicate R

Polynomial-time function f

$$x \in L \Leftrightarrow \|\{y : |y| = q(|x|) \wedge R(x, y)\}\| = f(x)$$

C_P - Alternate Definition

Polynomial q

Polynomial-time predicate R

Polynomial-time function f

$$x \in L \Leftrightarrow \|\{y : |y| = q(|x|) \wedge R(x, y)\}\| = f(x)$$

Certificates (paths in N) Polynomial length of path

C$_=$P - Alternate Definition

Polynomial q

Polynomial-time predicate R

Polynomial-time function f

$$x \in L \Leftrightarrow \|\{y : |y| = q(|x|) \wedge R(x, y)\}\| = f(x)$$

Does N accept x along path y?

C_P - Alternate Definition

Polynomial q

Polynomial-time predicate R

Polynomial-time function f

$$x \in L \Leftrightarrow \|\{y : |y| = q(|x|) \wedge R(x, y)\}\| = f(x)$$

Exact number of accepting paths

C_P - Alternate Definition, Fixed f

Polynomial q

Polynomial-time predicate R

~~Polynomial-time function f~~

$$x \in L \Leftrightarrow \|\{y : |y| = q(|x|) \wedge R(x, y)\}\| = 2^{q(|x|)-1}$$

Fix f to be exactly half of all possibilities

C_=P - Single Sided Definition (Lemma 5.14)

Polynomial r

Polynomial-time predicate S

$$x \in L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| = 2^{r(|x|)-2}$$

$$x \notin L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| < 2^{r(|x|)-2}$$

C_P - Single Sided Definition (Lemma 5.14)

$$r(n) = 2q(n)$$

z is two paths in N rather than just one

$$S(x, z) = (\exists w_1, w_2 : |w_1| = |w_2| = q(|x|))$$
$$[w_1 \cdot w_2 = z \wedge R(x, w_1) \wedge \neg R(x, w_2)]$$

1^{st} path accepts

2^{nd} path rejects

$$x \in L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| = 2^{r(|x|)-2}$$
$$x \notin L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| < 2^{r(|x|)-2}$$

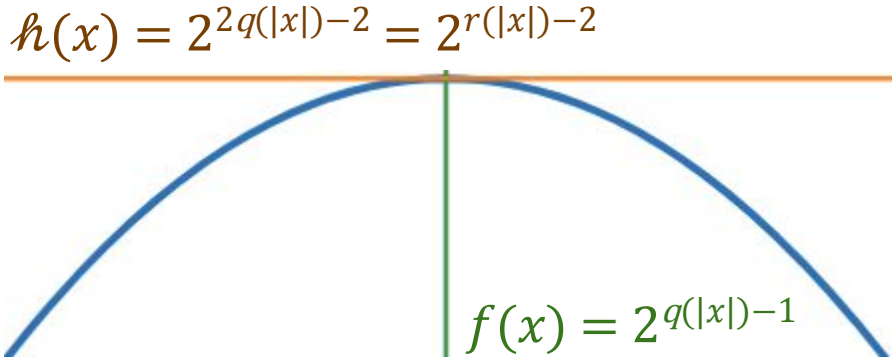
C_P - Single Sided Definition (Lemma 5.14)

$$r(n) = 2q(n)$$

$$S(x, z) = (\exists w_1, w_2 : |w_1| = |w_2| = q(|x|))$$
$$[w_1 \cdot w_2 = z \wedge R(x, w_1) \wedge \neg R(x, w_2)]$$

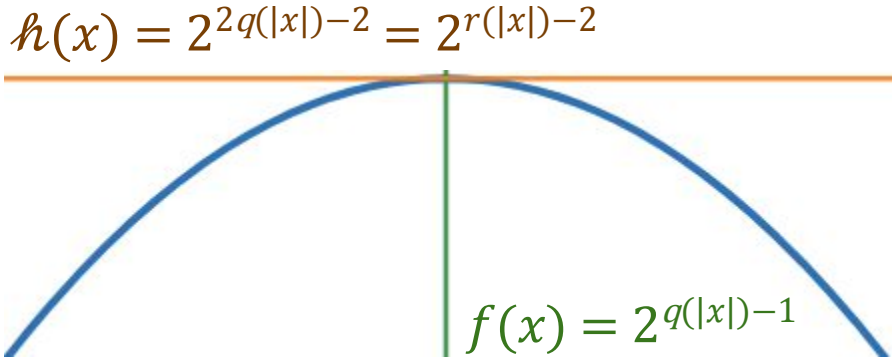
$$h(x) = f(x)(2^{q(|x|)} - f(x))$$

C_P - Single Sided Definition (Lemma 5.14)



$$h(x) = f(x)(2^{q(|x|)} - f(x))$$

C_P - Single Sided Definition (Lemma 5.14)



$$\hbar(x) = f(x)(2^{q(|x|)} - f(x))$$

$$\max(\hbar(x)) = 2^{r(|x|)-2}$$

C_P - Single Sided Definition (Lemma 5.14)

$$x \in L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x,y)\}\| = 2^{r(|x|)-2}$$
$$x \notin L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x,y)\}\| < 2^{r(|x|)-2}$$

$$h(x) = f(x)(2^{q(|x|)} - f(x))$$
$$\max(h(x)) = 2^{r(|x|)-2}$$

C_P - Single Sided Definition (Lemma 5.14)

$$x \in L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x,y)\}\| = 2^{r(|x|)-2}$$
$$x \notin L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x,y)\}\| < 2^{r(|x|)-2}$$

$$\downarrow$$

$$q = r \qquad R = S \qquad f(x) = 2^{r(|x|)-2}$$
$$x \in L \Leftrightarrow \|\{y : |y| = q(|x|) \wedge R(x,y)\}\| = f(x)$$

Is #P closed under min/max?

Theorem 5.13

1. If #P is closed under minimum then UP = NP ✓
2. If #P is closed under maximum or under minimum then $C_{=}P = SPP$

Note: $SPP \subseteq C_{=}P$ holds unconditionally, proof is trivial.

We need only show $C_{=}P \subseteq SPP$.

$\text{Closed}(\#\text{P}, \max) \Longrightarrow \text{C}_{=}\text{P} = \text{SPP}$

$$x \in L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| = 2^{r(|x|)-2}$$

$$x \notin L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| < 2^{r(|x|)-2}$$

$\text{Closed}(\#\text{P}, \max) \Longrightarrow \text{C}_{=}\text{P} = \text{SPP}$

$$x \in L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| = 2^{r(|x|)-2}$$
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$$\downarrow$$

$$\exists f \in \#\text{P}$$
$$x \in L \Longrightarrow f(x) = 2^{r(|x|)-2}$$
$$x \notin L \Longrightarrow f(x) < 2^{r(|x|)-2}$$

$\text{Closed}(\#\text{P}, \max) \Longrightarrow \text{C}_{=}\text{P} = \text{SPP}$

$$x \in L \Longrightarrow f(x) = 2^{r(|x|)-2}$$
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$$g(x) = 2^{r(|x|)-2} - 1 \in \#\text{P}$$

$\text{Closed}(\#\text{P}, \max) \Longrightarrow \text{C\_P} = \text{SPP}$

$$x \in L \Longrightarrow f(x) = 2^{r(|x|)-2}$$

$$x \notin L \Longrightarrow f(x) < 2^{r(|x|)-2}$$

$$g(x) = 2^{r(|x|)-2} - 1 \in \#\text{P}$$

$$h(x) = \max(f(x), g(x))$$

$\text{Closed}(\#\text{P}, \max) \Longrightarrow \text{C}_{=}\text{P} = \text{SPP}$

$$x \in L \Longrightarrow f(x) = 2^{r(|x|)-2}$$
$$x \notin L \Longrightarrow f(x) < 2^{r(|x|)-2}$$

$$g(x) = 2^{r(|x|)-2} - 1 \in \#\text{P}$$

$$h(x) = \max(f(x), g(x))$$

$$x \in L \Longrightarrow \max(2^{r(|x|)-2}, \; 2^{r(|x|)-2} - 1) = 2^{r(|x|)-2}$$
$$x \notin L \Longrightarrow \max(<2^{r(|x|)-2}, \; 2^{r(|x|)-2} - 1) = 2^{r(|x|)-2} - 1$$

$\text{Closed}(\#\text{P}, \max) \Longrightarrow \text{C}_{=}\text{P} = \text{SPP}$

$x \in L \Longrightarrow f(x) = 2^{r(|x|)-2}$

$x \notin L \Longrightarrow f(x) < 2^{r(|x|)-2}$

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$h(x) = \max(f(x), g(x))$

NPTM N

$L \in \text{SPP}$

Is #P closed under min/max?

Theorem 5.13

1. If #P is closed under minimum then UP = NP ✓
2. If #P is closed under maximum or under minimum then $C_{=}P$ = SPP ✓

$\mathrm{Closed}(\#\mathrm{P}, \min) \Longrightarrow \mathrm{C}_{=}\mathrm{P} = \mathrm{SPP}$

$$x \in L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| = 2^{r(|x|)-2}$$
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Closed(#P, min) $\Longrightarrow$ C_P = SPP

$$x \in L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| = 2^{r(|x|)-2}$$

$$x \notin L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| < 2^{r(|x|)-2}$$

↓ negate *S*, essentially flipping accepting/rejecting paths

$$x \in L \Longrightarrow \|\{y : |y| = r(|x|) \wedge \neg S(x, y)\}\| = 2^{r(|x|)} - 2^{r(|x|)-2}$$

$$x \notin L \Longrightarrow \|\{y : |y| = r(|x|) \wedge \neg S(x, y)\}\| > 2^{r(|x|)} - 2^{r(|x|)-2}$$

Closed(#P, min) $\Longrightarrow$ C_P = SPP

$$x \in L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| = 2^{r(|x|)-2}$$

$$x \notin L \Longrightarrow \|\{y : |y| = r(|x|) \wedge S(x, y)\}\| < 2^{r(|x|)-2}$$

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$$x \notin L \Longrightarrow \|\{y : |y| = r(|x|) \wedge \neg S(x, y)\}\| > 2^{r(|x|)} - 2^{r(|x|)-2}$$

↓

$$x \in L \Longrightarrow f(x) = (¾)2^{r(|x|)}$$

$$x \notin L \Longrightarrow f(x) > (¾)2^{r(|x|)}$$

Closed(#P, min) $\Longrightarrow$ C_P = SPP

$$x \in L \Longrightarrow f(x) = (¾)2^{r(|x|)}$$
$$x \notin L \Longrightarrow f(x) > (¾)2^{r(|x|)}$$

$$g(x) = (¾)2^{r(|x|)} + 1 \in \#\mathrm{P}$$

Closed(#P, min) $\Longrightarrow$ C_P = SPP

$x \in L \Longrightarrow f(x) = (¾)2^{r(|x|)}$

$x \notin L \Longrightarrow f(x) > (¾)2^{r(|x|)}$

$g(x) = (¾)2^{r(|x|)} + 1 \in \#\mathrm{P}$

$h(x) = \min(f(x), g(x))$

Closed(#P, min) $\Longrightarrow$ C_P = SPP

$x \in L \Longrightarrow f(x) = (¾)2^{r(|x|)}$

$x \notin L \Longrightarrow f(x) > (¾)2^{r(|x|)}$

$g(x) = (¾)2^{r(|x|)} + 1 \in \#\mathrm{P}$

$h(x) = \min(f(x), g(x))$

$x \in L \Longrightarrow \min((¾)2^{r(|x|)}, (¾)2^{r(|x|)} + 1) = (¾)2^{r(|x|)}$

$x \notin L \Longrightarrow \min(>(¾)2^{r(|x|)}, (¾)2^{r(|x|)} + 1) = (¾)2^{r(|x|)} + 1$

Closed(#P, min) $\Longrightarrow$ C$_=$P = SPP

$x \in L \Longrightarrow f(x) = (¾)2^{r(|x|)}$

$x \notin L \Longrightarrow f(x) > (¾)2^{r(|x|)}$

$g(x) = (¾)2^{r(|x|)} + 1 \in \#\mathrm{P}$

$h(x) = \min(f(x), g(x))$

NPTM N

$L \in \mathrm{SPP}$

More to consider

- Classes other than #P
  - OptP (max output)
  - SpanP (cardinality of set of outputs)
- Complete characterizations
  - Open issue for #P under proper decrement

References

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Thanks!

Bonus Content: OptP

OptP - Optimization Functions

$f \in$ OptP iff, for some NPTM N:

$$f(x) = \max\{i \in \mathbb{N} : \text{a path of } N(x) \text{ outputs } i\}$$

Is OptP closed under proper subtraction?

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Theorem 5.15 *The following are equivalent:*

1. OptP is closed under proper subtraction
2. OptP is closed under every polynomial-time operation
3. NP = coNP

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$$\mathrm{NP} = \mathrm{coNP} \Longrightarrow \mathrm{Closed}(\mathrm{OptP}, op)$$

$$f \in \mathrm{OptP} \longrightarrow \mathrm{NPTM}\ N_f$$
$$g \in \mathrm{OptP} \longrightarrow \mathrm{NPTM}\ N_g$$

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$$f \in \text{OptP} \longrightarrow \text{NPTM } N_f$$
$$g \in \text{OptP} \longrightarrow \text{NPTM } N_g$$

$$L_f = \{\langle x, i\rangle : f(x) > i\} \in \text{NP}$$
$$L_g = \{\langle x, i\rangle : g(x) > i\} \in \text{NP}$$

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by assumption

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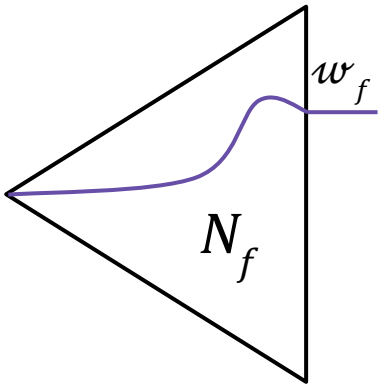
by assumption

$\bar{L}_f \in \mathrm{NP} \longrightarrow \mathrm{NPTM}\ N_{\geq f}$

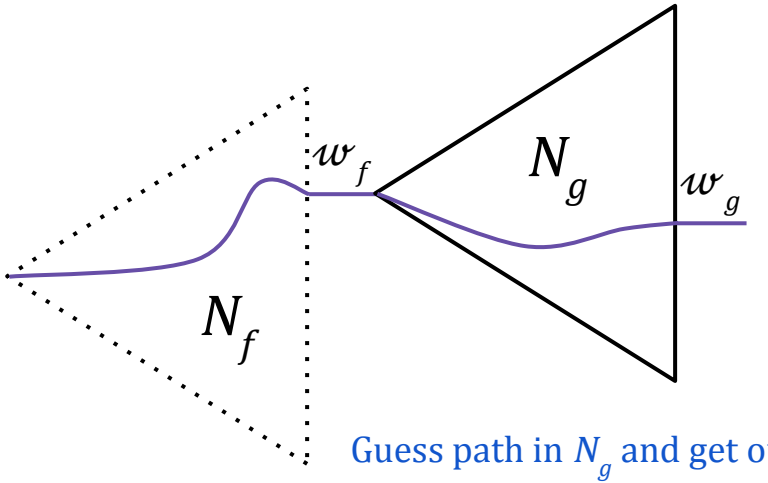
$\bar{L}_g \in \mathrm{NP} \longrightarrow \mathrm{NPTM}\ N_{\geq g}$

$$\mathrm{NP} = \mathrm{coNP} \Longrightarrow \mathrm{Closed}(\mathrm{OptP}, op)$$



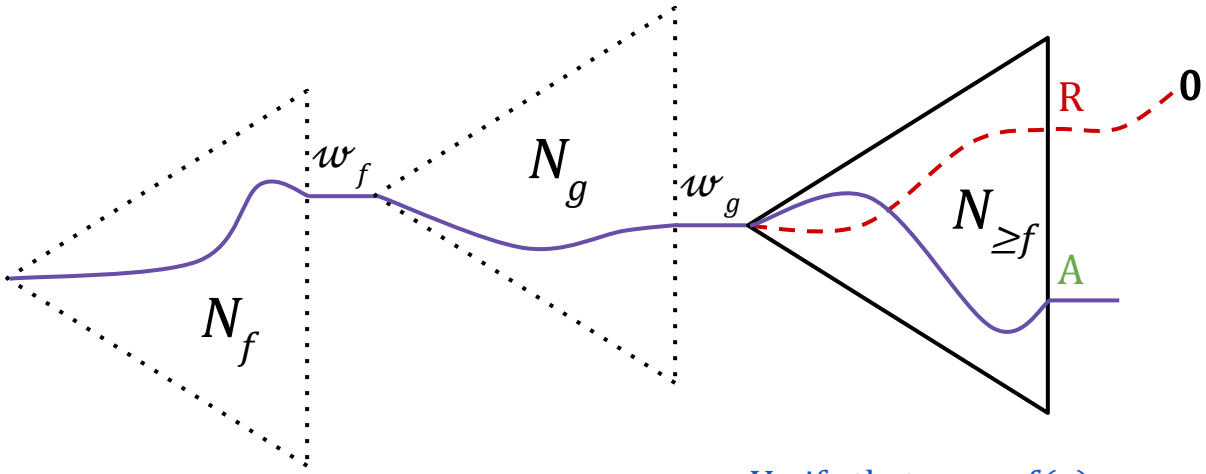
Guess path in N_f and get output

$\mathrm{NP} = \mathrm{coNP} \Longrightarrow \mathrm{Closed}(\mathrm{OptP}, op)$



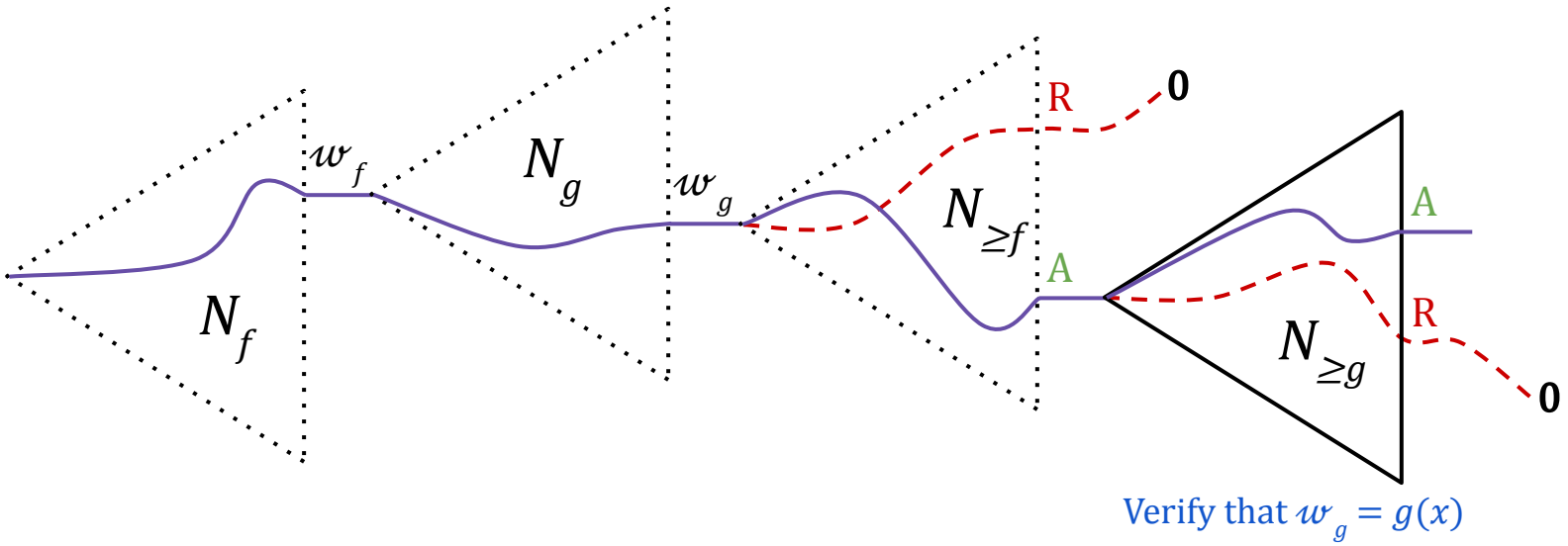
Guess path in N_g and get output

NP = coNP ⟹ Closed(OptP, *op*)



Verify that $w_f = f(x)$

NP = coNP ⟹ Closed(OptP, *op*)



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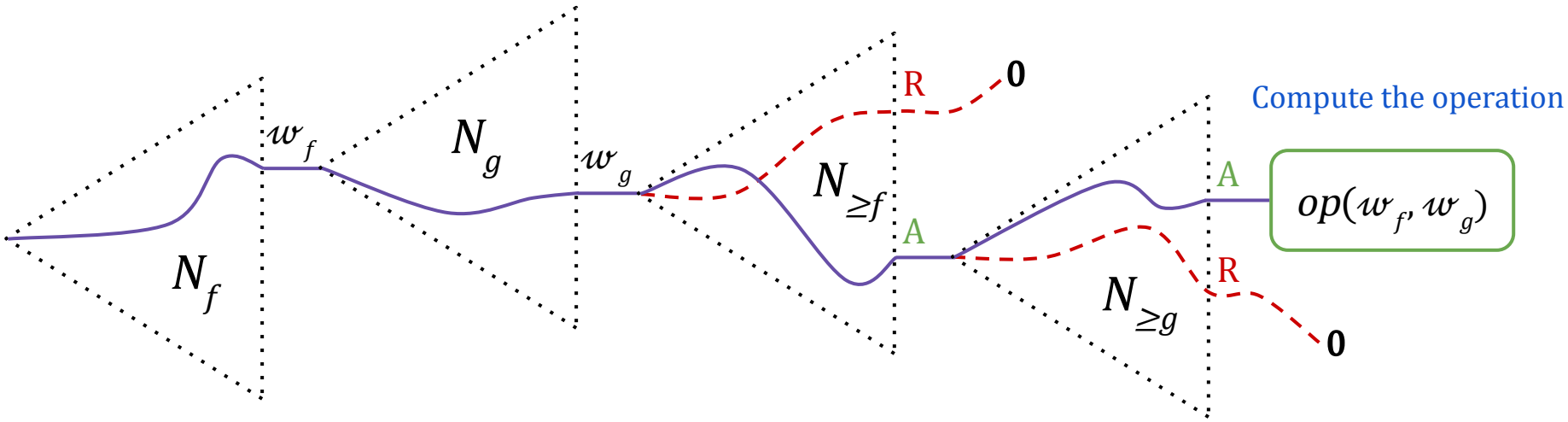

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$$\text{Closed}(\text{OptP}, \ominus) \Longrightarrow \text{NP} = \text{coNP}$$

output 1 on accepting paths, 0 on rejecting paths

$$L \in \text{NP} \longrightarrow \text{NPTM } N \longrightarrow f(x) = \begin{cases} 1 \text{ if } x \in L \\ 0 \text{ otherwise} \end{cases} \in \text{OptP}$$

$$\text{Closed}(\text{OptP}, \ominus) \Longrightarrow \text{NP} = \text{coNP}$$

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assume closure under proper subtraction

$$g(x) = 1 \ominus f(x) \in \text{OptP}$$

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assume closure under proper subtraction

$$g(x) = 1 \ominus f(x) \in \text{OptP}$$

accept only on paths that output 1

$$\text{NPTM } N' \longleftarrow g(x) = \begin{cases} 0 \text{ if } x \in L \\ 1 \text{ otherwise} \end{cases} \in \text{OptP}$$

$$\text{Closed}(\text{OptP}, \ominus) \Longrightarrow \text{NP} = \text{coNP}$$

output 1 on accepting paths, 0 on rejecting paths

$$L \in \text{NP} \longrightarrow \text{NPTM } N \longrightarrow f(x) = \begin{cases} 1 \text{ if } x \in L \\ 0 \text{ otherwise} \end{cases} \in \text{OptP}$$

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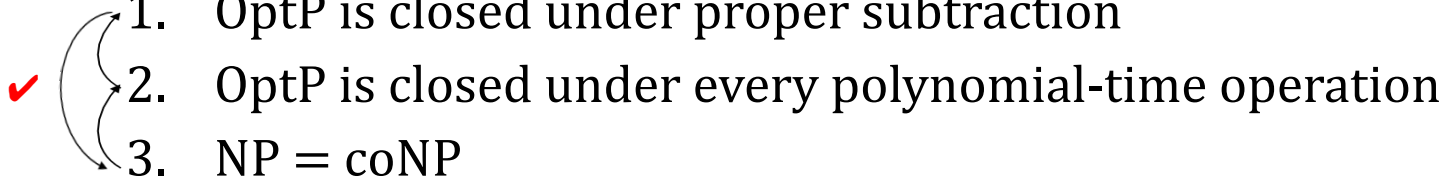

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SpanP - Output Cardinality

$f \in$ SpanP iff, for some NPTM N:

$$f(x) = \|\{w \in \Sigma^* : \text{an accepting path of } N(x) \text{ outputs } w\}\|$$

Is SpanP closed under proper subtraction?

Theorem 5.16 *The following are equivalent:*

1. SpanP is closed under proper subtraction
2. SpanP is closed under every polynomial-time operation
3. $\mathrm{PP}^{\mathrm{NP}} = \mathrm{PH} = \mathrm{NP}$

Proof is left as an exercise for the reader.