Question 1. Define $A = \{⟨i, q⟩| (∃x ∈ Σ^*) [M_i on input x halts after a finite number of steps AND (either $M_q$ on input x runs forever or $M_q$ on input x halts after a strictly greater number of steps than $M_i$ on input x does)] \}$. (Loosely put, there is some string on which $M_i$ is faster than $M_q$.)

Define $B = \{k|L(M_k) \neq \emptyset\}$. (a) Prove, by giving a recursive reduction and proving it correct, that $A \leq_m B$. (b) Prove, by giving a recursive reduction and proving it correct, that $B \leq_m A$.

Answer: An answer is on the next page!
**Answer:** (a) We will give a recursive function $\sigma$ such that for all pairs $\langle i, q \rangle$, $\langle i, q \rangle \in A \iff \sigma(\langle i, q \rangle) \in B$. We define $\sigma(\langle i, q \rangle) = k$ which is the index of a TM that on arbitrary input $y$ does the following. $M_k$ simulates $M_i$ on $y$ step by step, and at the end of each step a counter is incremented by one. (The counter is initialized to be zero.) If $M_i$ halts, let $t$ be the value of the counter when $M_i$ halts, then $M_k$ simulates $M_q$ on $y$ for $t$ steps. $M_k$ accepts if and only if $M_q$ doesn’t halt within $t$ steps.

If $\langle i, q \rangle \in A$, then there is a string $x$ on which $M_i$ is faster than $M_q$. By the definition of $M_k$, $M_k$ will accept $x$, which implies $k \in B$.

If $\langle i, q \rangle \notin A$, then for all $x$, either $M_i$ won’t halt on $x$ or $M_i$ on $x$ halts after a greater or equal number of steps than $M_q$ on input $x$ does. By the definition of $M_k$, $M_k$ won’t accept any string, which implies that $k \notin B$.

(b) We will give a recursive function $\sigma$ such that for all $k$, $k \in B \iff \sigma(k) \in A$. We define $\sigma(k) = \langle i, q \rangle$ such that $M_q$ is a TM which runs forever on every input and $M_i$ is a TM that on arbitrary input $y$ does the following. $M_i$ ignores $y$ and simulates the enumerator for $L(M_i)$, and then halts and accepts if and only if the enumerator enumerates something.

If $k \in B$, then $L(M_k) \neq \emptyset$. By the definition of $M_i$, $M_i$ on every input will halt and accept. Since $M_q$ runs forever on every input, we have $\langle i, q \rangle \in A$.

If $k \notin B$, then $L(M_k) = \emptyset$. By the definition of $M_i$, $M_i$ will run forever on every input, which implies $\langle i, q \rangle \notin A$. 

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