UR Honor Pledge for Exams  “I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.” (UR now requires you to by hand copy the above sentence onto the blank lines below and then sign it.)

Signature: ________________________________

Instructions: There are six problems here. Do four. If you choose to answer more than four problems (you may if you want to), we will grade all that you answer, and will count the highest (that is not a typo; it really says and means “highest”) four scores.

The highest possible score on this exam is 100%, i.e., 100/100 (see, at your leisure, the 150928 “Midterm I Tidbit,” which is related to this.)

Throughout this exam, $HP$ denotes $\{i|M_i(i)\text{ accepts}\}$. Note carefully that the word there is not “halts” but is “accepts.”

Don’t just jot down random stuff (doing so may LOSE points)—think before answering (we’ll hand out scrap paper so you have scrap paper on which to doodle/plan/try-things).

Note that not all questions are necessarily identical in difficulty, so don’t, for example, make the mistake of spending all your time on one hard problem and leaving the others blank.

Don’t confuse “from” and “to” on reductions (if when trying to prove $A \leq_m B$ you accidentally just prove $B \leq_m A$, you’ll probably get no points). Note, for example, that there is a many-one reduction from $\emptyset$ to $HP$, but there exists no many-one reduction from $HP$ to $\emptyset$. (On this exam, “many-one” of course refers to recursive many-one reductions.)

Your handwriting must be clear and readable. We will not guess that some unclear character is what is needed to make your answer correct; your written characters must be clearly readable on their own.

Rules: Closed book, closed notes, no computers or calculators, use pen not pencil. An exception to the “closed notes” rule is that each person can have one sheet of self-prepared, hand-written notes.

Make sure to put all your answers that you want graded onto (not the scrap paper but rather) the stapled test sheets.

Scoring: To repeat: There are six problems here. Do four. If you choose to answer more than four problems (you may if you want to), we will grade all that you answer, and will count the highest four scores. Since your grade will be based on four 25-point problems, there are 100 points available here.

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Question 1 [25 points]
(a) [5 points] Give the definition of what it means for $A$ to be a recursive set.
(b) [20 points] Professor Foo claims that the following result holds: A set $B$ is recursive if and only if every Turing machine accepting $B$ is a total Turing machine. Prove or disprove this result.
**Question 2** [25 points] For each part, please place each of the sets given into the lowest level of the Kleene hierarchy (among the ones listed in the next sentence) that contains it. Choose from the levels: \( \Sigma_0, \Sigma_1, \Pi_1, \Sigma_2, \Pi_2 \), etc.

Hints: Recall that \( \Sigma_0 = \Pi_0 = \) the recursive sets; \( \Sigma_1 = \) the recursively enumerable sets; \( \Pi_1 = \) coRE sets. Recall the quantifier structures of the levels (informally put, \( \Sigma_1 \) is \( \exists \), \( \Sigma_2 \) is \( \exists \forall \), \( \Pi_1 \) is \( \forall \), \( \Pi_2 \) is \( \forall \exists \), \( \Pi_3 \) is \( \forall \exists \forall \), \( \Sigma_3 \) is \( \exists \forall \exists \), etc.). Rice’s Theorems might (or might not) be helpful on some of these. Your knowledge of complete sets, many-one reductions, and Turing reductions might (or might not) be helpful. Your ability to see quantifier structure might (or might not) be helpful.

There is no penalty for guessing on this problem so you might want to answer everything. Please use the 1-inch answer lines for the answers. Do all scrap work on scrap paper; the scrap work will not be collected.

The \( M_i \)'s below are a standard enumeration of (one-head, one-tape) Turing machines.

1. [Place in the best level from the choices mentioned above]
   \( \{ i \mid L(M_i) \text{ is finite} \} \).

   Answer: ____________.

2. [Place in the best level from the choices mentioned above]
   \( \{ (x, i) \mid x \notin L(M_i) \} \).

   Answer: ____________.

3. [Place in the best level from the choices mentioned above]
   \( \{ i \mid L(M_i) \text{ is an r.e. set} \} \).

   Answer: ____________.

4. [Place in the best level from the choices mentioned above]
   \( \{ (i, j) \mid L(M_i) = \emptyset \text{ and } L(M_j) = \Sigma^* \} \).

   Answer: ____________.

5. [Place in the best level from the choices mentioned above]
   \( \{ i \mid L(M_i) \in \Pi_{2009} - \Sigma_0 \} \).

   Answer: ____________.
Question 3 [25 points] (a) [12.5 points] Prove by a clear Tarski-Kuratowski quantifier analysis that \( \{i \# j \mid L(M_i) = L(M_j)\} \) is in \( \Pi_2 \).
(b) [12.5 points] Let $R = \{ i \mid M_i \text{ is a total Turing machine} \}$. State Rice’s Theorem (the easy 1-part theorem, not the 3-part theorem). Then either use it to prove that $R$ is not recursive, or explain clearly why Rice’s Theorem is not appropriate for this problem.
Question 4 [25 points] Let $A = \{ j \# k \mid L(M_j) \not\subseteq L(M_k) \}$. Recall that $HP = \{ i \mid i \in L(M_i) \}$, i.e., those $i$ such that $M_i$ accepts $i$ (note it says “accepts,” not “halts”). Give an explicit, direct, many-one reduction (and show that it is indeed a reduction between those) from $HP$ to $A$ (i.e., $HP \leq_m A$).
Question 5 [25 points] Prove that there exists an infinite set $A \subseteq \{0, 1\}^*$ such that (a) $A$ has no infinite r.e. subset, and (b) $(\forall n \in \{0, 1, 2, \ldots\}) |A \cap \{0, 1\}^n| = n$. (Part (b) says that at each length $n$ the set $A$ contains exactly $n$ of the strings of length $n$.)
Question 6 [25 points] Let \( B = \{ i \mid L(M_i) \text{ is infinite} \} \). Give an explicit, direct, many-one reduction (and show that it is indeed a reduction between those) from \( \overline{HP} \) to \( B \) (i.e., \( \overline{HP} \leq_m B \)).