Question 1. [10 points] Prove that if $A \leq^p_m B$ and $B \leq^p_m C$, then $A \leq^p_m C$.
That is, prove that many-one polynomial-time reductions are transitive.
Answer:

Let $A, B, C$ be sets such that $A \leq^p_m B$ through a function $f$ and $B \leq^p_m C$
through $g$. $g \circ f$ is a function that provides the reduction $A \leq^p_m C$ (that is, the composition of $g$ and $f$, or $g(f(x)))$.

First, $g \circ f$ runs in polynomial time. Both $g$ and $f$ are polynomial time
functions, and $g \circ f$ will run in time bounded by the product of the polyno-
mials that bound $g$ and $f$.

Second, $x \in A \iff g \circ f(x) \in C$. Since $f$ first maps exactly the elements in
$A$ to elements of $B$, and then $g$ maps exactly the elements of $B$ to elements
of $C$, the composition of these two functions serves as a reduction from $A$ to
$C$.

Question 2. [10 points] Prove or disprove and state explicitly which you
are doing: $\text{SAT} \leq^p_m \emptyset$. SAT is the famous NP-complete problem, satisfiability
of Boolean formulas.
Answer:

Disprove. If $\text{SAT} \leq^p_m \emptyset$, there would need to be a function $\sigma$ such that $x \in \text{SAT} \rightarrow \sigma(x) \in \emptyset$. However, there are not any elements in $\emptyset$, and so there
is nothing that $\sigma$ can map elements of $\text{SAT}$ to. Therefore there can be no
such reduction. In fact, the only set that $\leq^p_m$-reduces to the empty set is the
empty set itself.