The Tournament Divide and Conquer Technique

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K-tournaments

Consider the set of k people playing a game with the following properties:

1) Game consists of multiple matches, with only two players in each match
2) Each player play against each other exactly once.
3) Every match has a winner

Such a setting can be represented by a directed graph.
Example: 4-tournament
Curious fact about tournaments

The interesting fact about tournaments is that we can select a very small subset of players such that each player not from that subset defeats someone from that set.

How small?

A logarithmically small!
Theorem 3.1:

Let $G = (V, E)$ be a $k$-tournament, where $V = \{1, \ldots, k\}$ is the set of nodes or players, and $E$ is the set of edges or matches.

Then there exists a subset $H$ of $V$, such that:

1) $|H| \leq \lceil \log(k + 1) \rceil$

2) $\forall v (v \in (V - H) \implies \exists g (g \in H \land (g, v) \in E))$
Example: 7-tournament
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Proof of Theorem 3.1:

- In a k-tournament, each player plays exactly $k - 1$ games.
- There must be at least one player who lost to at least $\left\lfloor \frac{k - 1}{2} \right\rfloor$ other players.
- Add this player to $H$.
- Remove the nodes corresponding to that player and to all the players who defeated him from the graph. The resulting graph has at most $\left\lfloor \frac{k}{2} \right\rfloor - 1$ vertices.
- Apply the same procedure to the new graph.
- Since at each step we decrease the size of graph by factor of 2 (at least), we end up with at most $\left\lfloor \log(k + 1) \right\rfloor$ steps, so $|H| \leq \left\lfloor \log(k + 1) \right\rfloor$.
Definitions

● **P-sel**
  ○ A language $L$ is P-selective if there is a polynomial-time function $f$ such that:
  ○ $f(x,y)$ is either $x$ or $y$
  ○ If $x \in L$ or $y \in L$, $f(x,y) \in L$
  ○ Notice that $f(x,y)$ can do anything if $x$ and $y$ are not in $L$, so in some sense, $f(x,y)$ selects which string is “more likely” to be in $L$

● **Example:**
  ○ For a fixed real number $r$, $\{<a,b> | a/b < r\}$
Definitions

- **P/poly**
  - Can be thought of as having “small circuits” that decide each length
  - Easier to think of it as polynomial amount of advice
  - More generally, for a class of languages C and class of functions F, let C/F denote the class of languages L such that:
    - There exists a language $A \in C$, and $h(n)$ such that $|h(n)| \in F$, and $L = \{ x \mid <x,h(|x|)> \in A \}$
  - So P/poly is equivalently:
    - There is a language $A \in P$ and $h(n)$ of polynomial length such that $x \in L$ iff $<x,h(|x|)> \in A$
Connection Between Tournaments and P-sel

- If $L$ is P-selective, let $f$ be a P-selector for $L$
- Then for a given length $n$, $f$ gives a tournament on $L^{=n}$
  - We have an edge from $a$ to $b$ if $f(a,b) = b$
  - Note: This requires $f(a,b) = f(b,a)$
- Then Theorem 3.1 gives a set $H$ with at most $\log(2^n+1) \leq n+1$ strings
- Every string in $L^{=n}$ is either in $H$, or beats a string in $H$
- Furthermore, any string in $H$ or that beats a string in $H$ is in $L^{=n}$
P-sel \subseteq P/poly

- Recall our definitions. To show a language L is in P/poly, we need a function g and a language A \in P such that:
  - x \in L iff <x, g(|x|)> \in A
- So let g(n) encode the, at most, n+1 strings in H
  - |g(n)| is polynomial in n
- Then A is accepted by the following deterministic Turing machine:
  - On input <x, y> do the following:
    - For each h in y, accept if x=h or f(x, h)=x
    - If we fail all the above, reject
- Then x \in L iff <x, g(|x|)> \in A, so L is in P/poly
- In fact, since g(n) has at most n+1 strings of length n, it has quadratic length
- So we can strengthen this result to P-sel \subseteq P/quadratic
3.2 Optimal Advice for the Semi-feasible Sets
Theorem 3.9

If $G$ is a $k$-tournament, then there is a $v \in V_G$ such that $V_G = R_{2,G}(v)$. 
Required Notations

Given a directed graph $G$, and a node $v \in V_G$, let
\[ R_{0,G}(v) = \{v\} \]
And for each $i > 0$, let
\[ R_{i,G}(v) = R_{i-1,G}(v) \cup \{z \in V_G| (\exists w \in R_{i-1,G}(v))[(w,z) \in E_G]\}. \]

For any $i$, $G$, and $S \subseteq V_G$, define
\[ R_{i,G}(S) = \{w \in V_G|(\exists v \in S)[w \in R_{i,G}(v)]\}. \]
If $G$ is a $k$-tournament, then there is a $v \in V_G$ such that $V_G = R_{2,G}(v)$.

Proof: By Induction

1-tournament and 2-tournament holds trivially.

Assume: $k'$-tournament holds in $G'$ graph.

Consider: $k'+1$ -tournament in $G$ graph where $a$ is the new vertex.
If $G$ is a $k$-tournament, then there is a $v \in V_G$ such that $V_G = R_{2,G}(v)$.

There is a node $b$ in $G'$ such that $R_{2,G'}(b) = V_{G'}$.

Case 1: If the edge between $a$ and $b$ points to $a$.
Case 2: If the edge between $a$ and $b$ points to $b$. 
If $G$ is a k-tournament, then there is a $v \in V_G$ such that $V_G = R_{2,G}(v)$.

- **Case 1:** If the edge between $a$ and $b$ points to $a$.
We are done. As $R_{2,G}(b) = V_G$
If $G$ is a $k$-tournament, then there is a $v \in V_G$ such that $V_G = R_{2,G}(v)$.

- **Case 2:** If the edge between $a$ and $b$ points to $b$.
  - If $a \in R_{2,G}(b)$ then we are done.
  - If $a \notin R_{2,G}(b)$ then
    - For each node $c \in R_{1,G}(b)$ edge between $a$ and $c$ points to $c$.
    - So $R_{2,G}(a) = V_G$
Theorem 3.10 \( \text{P-sel} \subseteq \text{NP/linear} \)

What does it mean?
Theorem 3.10 \( \text{P-sel} \subseteq \text{NP/linear} \)

What does it mean?

If \( L \) is a P-sel set, then \( L \) is also a NP/linear set

\( L \) is a P-sel set \( \Rightarrow \) there exists a selector function \( f \) for \( L \) (which selects the one that is “more likely” in \( L \))

\( L \) is a NP/linear set \( \Rightarrow \) there is an advice function \( g \) and set \( A \) such that if \( x \in L \) then \( (x, g(|x|)) \in A \)

How to prove this?

**Hint Hint:** We’ll use Theorem 3.9 (if \( G \) is a k-tournament, we can find a “core” node which has a relatively short distance (\( \leq 2 \)) to any other nodes)
Theorem 3.10  \( P\text{-sel} \subseteq NP/linear \)

What we have: a selector function \( f \) for set \( L \)

What we want: a linear advice function and an “interpreter” set \( A \)

Consider this advice function \( g \):

\[
g(n) = \begin{cases} 
1^{n+1} & \text{if } L^{=n} = \emptyset \\
0\omega_n & \text{otherwise}
\end{cases}
\]

\( \omega_n \) is a length \( n \) string in \( L^{=n} \) such that, by Theorem 3.9, each node in the tournament induced on \( L^{=n} \) by \( f \) can be reached from \( \omega_n \) via paths of length at most two. (Basically \( \omega_n \) is the core of the tournament graph!)

This is great, now we have a linear advice (=n+1) and it seems to be useful, but how to construct the set \( A \) (the interpreter) based on the advice function?
Theorem 3.10  \( \text{P-sel} \subseteq \text{NP/linear} \)

The interpreter is defined as:

\[
A = \{ \langle x, 0\omega \rangle \mid \text{there is a path of length at most two, in the tournament induced on } L^n \text{ by } f, \text{ from } \omega \text{ to } x \} 
\]

So if \( x \) is in \( L \), by construction of the advice function we know \( \langle x, g(|x|) \rangle \) is in \( A \)

What if \( x \) is not in \( L \)?
If \( x \notin L \), then \( \langle x, g(|x|) \rangle \notin A \), as if \( \langle x, g(|x|) \rangle \in A \), then we have a directed path from \( \omega \) to \( x \) within 2 steps (0, 1, 2):

\[
\omega \to ... \to x
\]

Remember an edge from \( a \to b \) means that \( f(a, b) = b \), or ”b beats a in the tournament”. So b is ”more likely” in L than a. If a is in already in L, then b must be in L.

In our this case a is \( \omega \), b is x. So based on our assumption (\( \langle x, g(|x|) \rangle \in A \)), x turns out to be in L!!! We get a contradiction, so our assumption is not correct.

So now we know if \( x \notin L \), then \( \langle x, g(|x|) \rangle \notin A \) ! And thus by our construction of g and A, L is a NP/linear set.
This is the end of lecture 1.