25 students are sitting in professor Nutty’s MAT1313 class in a 5 X 5 square scheme as shown below. Professor Nutty asks all the students to stand up and change their position by moving to the seat to their front, left, right, or back (as marked by cross for Jack, Mary, and Sue).

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Prove that what professor Nutty wants is not possible.
A Solution to Quiz 0

Imagine the 5X5 square colored in alternating black and white.

To follow professor’s request everybody who is sitting in a black cell has to move to a white cell. But there 13 black cells while only 12 white cells. Hence it is impossible!
Intuitive vs. Rigorous Proofs

• It is great to be intuitive when dealing with science and engineering.
• Do listen to your intuition, but do not trust them all the time.
• The famous P vs. NP problem has claimed to be solved many times in the last 50 years!
• Theoretical mistakes usually occur as a result of trusting intuition and not being rigorous.
This course is about being **RIGOROUS** in defining concepts in stating theorems and in proving theorems
Welcome to
CSC 280
Computer Models and Limitations
Course Information

- Instructor: Mehdi Manshadi
- Course webpage: www.cs.rochester.edu/u/mehdih/CSC280.html
- Grad TAs
  - Curtis Menton
- Undergrad TA/Workshop leader:
  - Adina Rubinoff
Grading

- Workshop 5%
- Homework 15%
- Quiz 10%
- Short Exams 10%
- Midterm 20%
- Final: 40%
What else is this course about?

• What are the limitations of the computers? What kind of problems cannot ever be solved by a computer even if we let it run for arbitrarily long time.

• What kinds of problem can be solved by current computers within a reasonable amount of time, using a manageable amount of memory.
Course Topics

• Finite State Automata:
  – Simple models of computation
  – Study their limitations

• Turing Machine and Computability:
  – A model of a true computer
  – Study what kinds of problem are not computable.

• Computational Complexity:
  – Study the time and space complexity of problems.
  – Introducing classes P, NP, PSPACE, etc.
Some proof techniques

• The following techniques do not usually help to find a proof for a theorem.

• However, once you have a clear idea about why a proposition holds, sometimes they help you to present your ideas in a formal way.
The technique we used for solving quiz 0 is called coloring.

Another example: Knight’s tour.
  - Open vs. closed tour.

Using coloring technique, it is straightforward to prove that there is no closed tour for an $n \times n$ chessboard when $n$ is odd.
Constructive proofs

• Let’s play a game.
• Assume there is 2010 pins in a pot.
• Two players alternate taking 1 or 2 pins from the pot. The one who picks the last pin(s) wins the game.
• Prove that the second player has a winning strategy.
• A strategy is winning if the player following it is guaranteed to win independent of his opponent’s moves.
A Constructive proof

• The second player’s strategy is to take two pins if his opponent takes one, and to take one if his opponent takes two.

• Why he is guaranteed to win?

• What happens if there are 2011, or 2012 pins?

• Can you generalize this for the case where there is N pins and each player can take 1..K pins from the pot at each step?
Non-constructive proofs

• Let’s play another game: *Chomp*
Non-constructive proofs

• Using a strategy-stealing technique, we can prove that there exists a winning strategy for the first player, but unlike the pins game, we do not present the actual strategy.

• This kind of proof sometimes is called existence proof (not a great term though).
Proof using contradiction

• Sometimes it is much easier to prove “\(\sim P\) does not hold”, than to prove “\(P\) holds”.

• The assumption that \(\sim P\) holds results in a contradiction, proving that the assumption is false, concluding that \(P\) is true.

• Before giving an example, let’s define some graph theory concepts.
Graph Theory

• A **graph** is a pair $G=(V, E)$ where $V$ is a set of nodes and $E$ is a set of edges of the form $\{u, v\}$ where $u, v \in V$.

• A **path** between two nodes $u$ and $v$ is a sequence of edges of the form $P = \{u, x_1\}, \{x_1, x_2\}, \ldots, \{x_n, v\}$ s.t. all the nodes on the path except possibly $u$ & $v$ are distinct.

• If $u=v$ the path is called a **cycle**.

• A graph is **connected** iff every two nodes in the graph are connected by some path.

A connected graph with no cycle is called a **tree**.

**Theorem 1**: If $G$ is a tree, then every two nodes of $G$ are connected by exactly one path.
Implicit Multiple Parts

• Be aware of implicit multiple parts

• There is exactly one path means:
  a) There is at least one path
  b) There is no more than one path

• Other examples:
  – if and only if
  – equality of two sets
Proof of theorem 1

a) \(G\) is connected, therefore following the definition every two nodes in \(G\) are connected by some path.

b) Assume to the contrary that there are some pair of nodes in \(G\) connected with at least two distinct paths \(P_1\) and \(P_2\). Among all those pairs, let’s pick \(u\) and \(v\) with the smallest \(|P_1|+|P_2|\) (\(|P|\) is the length of the path). \(P_1+P_2\) is a cycle in \(G\). Contradiction!
Proof of theorem 1

• Be *concise* and *precise*!

• Why smallest $|P1| + |P2|$?

• Having figures would always help.
Pigeonhole Principle

• In this class, there are at least two persons born in the same month.

• There are 8 green and 11 red buttons in a box. How many buttons shall we pick from the box, to guarantee that there is at least two button of the same color?
Pigeonhole Principle

If $N+1$ pigeons live in $N$ pigeonholes, there are at least two pigeons live in the same hole.
Pigeonhole Principle

• In every party there are at least two people who have shaken hands with the same number of people.

• In every group of six or more, there are at least three people such that either every two of them know each other, or no two of them know each other.
Tiling problem

• Consider the following 5×5 square board in which one cell is covered with a 1×1 tile. We want to tile the rest of the board with the L-shaped tiles of size 3:

Is that possible?
**Tiling problem**

- Consider the following 5×5 square board in which one cell is covered with a 1×1 tile. We want to tile the rest of the board with the L-shaped tiles of size 3:

Is that possible? Yes
Tiling problem

• Now consider a $1024 \times 1024$ square board in which an arbitrary cell is covered with a $1\times1$ tile. Is it possible to cover the rest of the board with L-shaped tiles of size 3? Prove your answer.
Induction Proof

- Using induction on $n$, we prove that every board of size $2^n \times 2^n$ ($n \geq 0$) in which an arbitrary cell is already covered by a 1×1 tile can be tiled using the L-shaped tiles of size 3.

  a) Base case: $n=0$ Trivial!

  b) Assume that the proposition holds for $n=k \geq 0$. We prove that it holds for $n=k+1$. 
Tiling problem

$2^{k+1}$

$2^k \times 2^k$

$2^k \times 2^k$

$2^k \times 2^k$
Induction is tricky!

• Preposition: All people are bald!
  – Base case: $n = 0$, trivial!
  – Induction step: Assume that every person with $n \geq 0$ hairs is bald. Consider a person $A$ with $n+1$ hairs. After removing one of his/her hairs, $A$ will be bald. But 1 hair does not change the baldness of a person, therefore $A$ was originally bald.

• What is wrong with this argument?
Induction

• Proposition: In every group of people, everybody has the same number of hairs!
Induction

• Proposition: In every group of people, everybody has the same number of hairs!

• Proof by induction on the number of people.
  – Base case: n=1 trivially holds.
  – Induction step:
    • Remove a person from the group.
    • Induction assumption: the proposition holds for the rest.
    • Replace back that person and remove another person.
    • Again, the proposition holds for this group of n-1 people.
    • Therefore all the n people have the same number of hairs!
Always watch for border cases!