Problem Set 11  
CSC 280, 2006  
Due before class, Wednesday Apr. 26, 2006

Do one of problems 1 & 2, one of problems 3 & 4, and two of problems 5–7. Half of the marks received for any extra problems done will count as extra credit.

**Problem 1**  Show that if $P = NP$, a polynomial time algorithm exists that, given a boolean formula $\phi$, actually produces a satisfying assignment for $\phi$ if it is satisfiable. (*Hint:* This is a simple problem, if you consider that if $P = NP$, then there is a polynomial time “subroutine” for the satisfiability problem, that you can use to actually find a satisfying assignment. Expect to use the subroutine more than once...)

**Problem 2**  Show that if $P = NP$, we can find all the factors of any integer $x$ in polynomial time.  
*Note:* An NTM that merely verifies in polynomial time that $x$ does have some nontrivial factor (as in the COMPOSITES problem) won’t help as a subroutine for actually finding the factors of $x$!

**Problem 3**  Consider the following problem. Given a graph $G$, with positive integer distances on the edges, and two integers $k$ and $d$, is there a way to select $k$ vertices of $G$ on which to locate “firehouses”, so that no vertex is at distance more than $d$ from a firehouse? Show that this problem is NP-complete.  
*Suggestion:* Show this is in NP, and then do a reduction from a known NP-complete problem. One possibility for the latter is as follows. Start with a 3CNF formula, and derive from it the same graph as was used in reducing 3SAT to VERTEX-COVER, but giving each edge a length of 1. This graph consists of “triangles” for clauses and “linked pairs” for literals $x_i, \overline{x}_i$, $i = 1, \ldots, n$ (and connections between the triangles and pairs). Now add a further node $y_i$ for each pair, connecting it to $x_i$ and $\overline{x}_i$ with 2 edges, each of length 2. Thus we have $n$ new nodes and $2n$ new edges. Let the corresponding “firehouse” problem have parameters $k = n$ (i.e., $n$ firehouses) and $d = 2$ (maximum distance to nearest firehouse). Show that this “works”, with proofs in both of the required directions. (The trickier direction is showing that a “firehouse solution” implies satisfiability of the 3CNF formula. Argue that for each triple of nodes $x_i, \overline{x}_i, y_i$ – i.e., the linked pairs that were expanded to triples – one of the 3 nodes must be a firehouse; and if it is $y_i$ that is a firehouse (as may happen), then either of $x_i, \overline{x}_i$ can be chosen to be 1, i.e., true; whereas if one of $x_i, \overline{x}_i$ is a firehouse, that is the one chosen to be true, to obtain a satisfying assignment.)

**Problem 4**  Let $U = \{(M,x,1^t) \mid M$ is an NTM that accepts input $x$ within $t$ steps\}. Show that $U$ is NP-complete. (*Hint:* Show that $U$ is in NP, and then show that for any arbitrary problem $A$ in NP, $A$ can be reduced to $U$.)

**Problem 5**  Show that PSPACE is closed under $\cup, \cap$, and Kleene *. 

**Problem 6**  Show that $\text{SPACE}(f(n))$ is the same for $f(n) \geq n$, whether we use a 1-tape TM model or a 2-tape model with a read-only input tape (with complexity measured in terms of the number of tape cells used on the second tape).

**Problem 7**

(a) Show that any PSPACE-hard language (one such that every language in PSPACE is polynomial-time reducible to it) is also NP-hard.

(b) Show that if every NP-hard language is also PSPACE-hard, then PSPACE = NP.