

General instructions:

- You will receive email saying which parts you need to solve (based on the results of the 0-th quiz).
- The solutions need to be typeset on a computer, using a program which allows you to use mathematical notation (open office, tex/latex, word, etc are OK, plain text file is NOT OK).
- Turn in 2 print-outs (one for the instructor, one for the grader).
- The exercises are from your MTH 150 book (Discrete Mathematics and Its Applications, Kenneth H. Rosen, 4th Edition), part C (294-295), part A (90-91), part I (200-201).

## 1 PART C (basic counting)

Solve the following problems in part C: 12, 15, 16(a,b,d), 21, 22, 27, 28, 37, 39.

For part “C” the your goal should be to get the results right. The following solved exercises may give you an idea on how much you should write.

**14)** How many solutions are there to the equation  $x_1 + x_2 + x_3 + x_4 = 17$ , where  $x_1, x_2, x_3, x_4$  are nonnegative integers.

There is a 1-1 correspondence between the solutions and reorderings of 17 ones and 3 zeros ( $x_1$  is the number of ones before the first zero,  $x_2$  the number of ones between the first and the second zero,  $x_3$  the number of ones between the second and the third zero, and  $x_4$  is the number of ones after the third zero). Hence the answer is

$$\binom{20}{3} = 1140.$$

□

**16c)** How many solutions are there to the equation  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$ , where  $x_1, x_2, x_3, x_4, x_5, x_6$  are nonnegative integers such that  $x_1 \leq 5$ .

If  $x_1$  is fixed then the number of solutions for the other variables is

$$\binom{29 - x_1 + 4}{4},$$

since as in 14), the solutions are in 1-1 correspondence with reorderings of  $29 - x_1$  ones and 4 zeros.

The possible values for  $x_1$  are 0, 1, 2, 3, 4, 5 and hence the answer is

$$\binom{33}{4} + \binom{32}{4} + \binom{31}{4} + \binom{30}{4} + \binom{29}{4} + \binom{28}{4} = 179976.$$

□

**26)** How many different strings can be made from the letters in MISSISSIPPI, using all the letters?

We have one  $M$ , four  $I$ , four  $S$ , and two  $P$ .

$$\binom{11}{1, 4, 4, 2} = \frac{11!}{1! 4! 4! 2!} = 34650.$$

□

## 2 Part A (asymptotic notation)

Solve the following problems in part A (excluding the parts solved below):

1, 2, 6, 7, 13, 14, 16, 25, 37, 38, 40.

For part “A” your goal should be to write a completely formal proof, delving into the definitions. The following solved exercises may give you an idea on how much detail is expected. Do NOT use limits (the goal here is to practice “bare-hand” technique).

The following definition of  $O$  notation should be used (note that this one is from the DPV book and coincides with the Discrete Mathematics book definition for functions whose values are positive).

**Definition 2.1** Let  $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$  (i.e.,  $f$  maps natural numbers  $\mathbb{N} = \{1, 2, \dots\}$  to positive reals). We say  $f(x) = O(g(x))$  if there exists  $C \in \mathbb{R}^+$  such that

$$(\forall x \in \mathbb{N}) f(x) \leq Cg(x). \quad (1)$$

The following definition of  $\Theta$  notation is used.

**Definition 2.2** Let  $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$ . We say  $f(x) = \Theta(g(x))$  if there exist  $C_1, C_2 \in \mathbb{R}^+$  such that

$$(\forall x \in \mathbb{N}) C_1g(x) \leq f(x) \leq C_2g(x). \quad (2)$$

**1a)** The function  $f(x) = 10$  satisfies  $f(x) = O(x)$ .

We will show that for  $C = 12$ ,  $f(x) = 10$ , and  $g(x) = x$  the condition (1) is satisfied. Indeed, for all  $x \in \mathbb{N}$  we have  $10 < 12x$ .  $\square$

**2e)** The function  $f(x) = 2^x$  does NOT satisfy  $f(x) = O(x^2)$ .

We will use the following fact

$$(\forall x \geq 0) 2^x \geq x. \quad (3)$$

Suppose that  $f(x) = O(x^2)$ , i.e., there exists  $C$  such that the condition (1) is satisfied (with  $f(x) = 2^x$  and  $g(x) = x^2$ ). Thus we have

$$(\forall x \in \mathbb{N}) 2^x \leq Cx^2. \quad (4)$$

Using (3) we obtain for all  $x \geq 0$

$$2^x = 2^{x/3} 2^{x/3} 2^{x/3} \geq \frac{x^2}{9} 2^{x/3}. \quad (5)$$

Combining (5) and (4) we obtain

$$(\forall x \in \mathbb{N}) 2^{x/3} \leq 9C. \quad (6)$$

Clearly (6) is false for  $x > 3 \log_2(9C)$ , a contradiction. Thus our assumption that  $f(x) = O(x^2)$  is incorrect. Hence  $f(x) \neq O(x^2)$ .  $\square$

**5)** Show  $(x^2 + 1)/(x + 1) = O(x)$ .

For  $x \geq 1$  we have

$$x^2 + 1 \leq x^2 + x. \quad (7)$$

Dividing both sides of (7) by  $x + 1$  (which is positive for  $x \geq 1$ ) we obtain

$$\frac{x^2 + 1}{x + 1} \leq x,$$

for  $x \geq 1$ . Hence (1) is satisfied with  $n_0 = 1$  and  $C = 1$ . □

**36)** Does  $f(x) = O(g(x))$  imply  $2^{f(x)} = O(2^{g(x)})$ ?

NO. For example take  $f(x) = 2x$  and  $g(x) = x$ . Then  $f(x) = O(g(x))$  (with  $C = 2$ ), but there exists no  $C'$  such that for all  $x \in \mathbb{N}$

$$2^{2x} \leq C' \cdot 2^x,$$

since this would imply that for all  $x \in \mathbb{N}$  we have  $2^x \leq C'$ . □

### 3 Part I (induction)

Solve the following problems from part I: 6, 13, 18, 21, 22, 34, 39.

For part “I” the your goal should be follow the idea of inductive proof. The following solved exercises may give you an idea on you should structure your proofs.

**7)** Using mathematical induction show

$$(\forall n \in \mathbb{N}) 1^2 + 2^2 + \dots + n^2 = n(n+1)(2n+1)/6. \tag{8}$$

BASE CASE: For  $n = 1$  the left-hand side is 1 and the right-hand side is 1.

INDUCTION STEP: Suppose we showed the claim for  $n = k$ , i.e., we showed

$$1^2 + 2^2 + \dots + k^2 = k(k+1)(2k+1)/6. \tag{9}$$

Now we need to show the claim for  $n = k + 1$ , i.e.,

$$1^2 + 2^2 + \dots + k^2 + (k+1)^2 = (k+1)(k+2)(2k+3)/6. \tag{10}$$

Using (9) we obtain

$$\begin{aligned} 1^2 + 2^2 + \dots + k^2 + (k+1)^2 &= k(k+1)(2k+1)/6 + (k+1)^2 = \\ &= (2k^3 + 9k^2 + 13k + 6)/6 = (k+1)(k+2)(2k+3)/6. \end{aligned} \tag{11}$$

We showed that for every  $k$  (9) implies (10). Thus (8) follows by mathematical induction. □

**20)** Using mathematical induction show:

$$(\forall n \in \mathbb{N}) 3 \mid (n^3 + 2n). \tag{12}$$

BASE CASE: For  $n = 1$  the right-hand side is 3, which is divisible by 3.

INDUCTION STEP: Suppose we showed the claim for  $n = k$ , i.e., we showed

$$3 \mid (k^3 + 2k). \tag{13}$$

This means that there exists an integer  $C$  such that

$$k^3 + 2k = 3C. \tag{14}$$

We have

$$((k+1)^3 + 2(k+1)) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3k^2 + 3k + 3 = 3(C + k^2 + k + 3). \quad (15)$$

Note that  $C' = C + k^2 + k + 3$  is an integer and  $((k+1)^3 + 2(k+1)) = 3C'$ . Thus 3 divides  $((k+1)^3 + 2(k+1))$  and the claim is true for  $n = k + 1$ , i.e.,

$$3 \mid ((k+1)^3 + 2(k+1)). \quad (16)$$

We showed that for every  $k$  (13) implies (16). Thus (12) follows by mathematical induction.  $\square$