

Problem: Solve the recurrence $T(n) = T(n/3) + T(2n/3) + O(1)$.

We will show that the solution is $O(n)$. More precisely, we will show that the following precise mathematical statement is true.

Let $T : \mathbb{N} \mapsto \mathbb{R}^+$ be a function such that for every $n \in \mathbb{N}$

$$T(n) = T(\lceil n/3 \rceil) + T(\lceil 2n/3 \rceil) + O(1).$$

Then $T(n) = O(n)$.

Proof: Let C be the constant hidden in $O(1)$. Thus for every $n \in \mathbb{N}$ we have

$$T(n) \leq T(\lceil n/3 \rceil) + T(\lceil 2n/3 \rceil) + C. \tag{1}$$

Let

$$D = \max \left\{ C, \max_{k \in \{101, \dots, 300\}} \frac{T(k)}{k - 100} \right\} \tag{2}$$

We will show, by mathematical induction, that for all $n \in \mathbb{N}$ such that $n \geq 101$ we have

$$T(n) \leq D \cdot (n - 100) \tag{3}$$

base case:

For $n \in \{101, \dots, 300\}$ equation (3) is true by our choice of D (equation (2)).

inductive step:

Let $k \geq 301$. Assume that (3) is true for $n \in \{101, \dots, k - 1\}$. We have

$$101 \leq \lceil k/3 \rceil \leq 1 + k/3 \leq k - 1,$$

and

$$101 \leq \lceil 2k/3 \rceil \leq 1 + 2k/3 \leq k - 1.$$

Thus, by the induction hypothesis (that is, the assumption), we have that (3) is true for $n = \lceil k/3 \rceil$ and $n = \lceil 2k/3 \rceil$, that is,

$$T(\lceil k/3 \rceil) \leq D \cdot (\lceil k/3 \rceil - 100), \tag{4}$$

and

$$T(\lceil 2k/3 \rceil) \leq D \cdot (\lceil 2k/3 \rceil - 100). \tag{5}$$

Combining (1) with (4) and (5) we obtain

$$T(k) \leq D \cdot (\lceil k/3 \rceil - 100) + D \cdot (\lceil 2k/3 \rceil - 100) + C. \tag{6}$$

We estimate the right-hand side of (6) as follows

$$T(k) \leq D \cdot (k/3 - 99) + D \cdot (2k/3 - 99) + C = D \cdot (k - 197) + (C - D) \leq D \cdot (k - 100), \tag{7}$$

that is, (3) is true for $n = k$.

Hence, by induction (3) is true for all $n \geq 101$. □