

1 Schedule

Part 1 of the homework is **due Sep 24**.
Part 2 of the homework is **due Sep 29**.
The **QUIZ** will be on **Tuesday, Sep. 29**.

2 List of algorithms covered in the class

(B-basic, I-intermediate, A-advanced):

- I: Karatsuba-Offman (p.47, DSV).
- B: Mergesort (p.50, DSV).
- I: Linear-time deterministic selection (p.189, CLRS).
- A: Expected-linear-time selection (p.53, DSV).
- A: FFT (p.57, DSV).

3 Basic material

Important concepts, problems, theorems, and algorithms:

- Merge-sort, linear-time select algorithm (median).
- Evaluation, interpolation, convolution.

Testing method:

- Find the unique polynomial of degree d which has given values on a set of $d + 1$ points (problems with $d \leq 3$).
- Apply master's theorem to recurrence $T(n) = aT(n/b) + O(n^d)$.
- Trace the execution of the expected linear-time select algorithm (p.54 of DPV).
- Trace the first step of the deterministic linear-time select algorithm.
- Compute the convolution of two sequences (multiply two polynomials).
- Evaluate a polynomial on the n -th roots of unity (fourier transform), for $n = 2, 4$.

Example problems:

2.1 (due Sep 24) Find the unique polynomial of degree 2 such that $p(-1) = 2$, $p(0) = 3$, and $p(1) = 6$.

2.2 (due Sep 24) Solve the following recurrences: $T(n) = 3T(n/2) + O(n)$, $T(n) = 3T(n/2) + O(n^2)$.

2.3 (due Sep 24) Let $A[1..10] = [7, 2, 4, 9, 5, 1, 3, 10, 6, 8]$. Assume that we run the randomized algorithm with the following parameters: $\text{SELECT}(A, 3)$. Suppose that the following sequence of v 's picked by the algorithm is 7,2,4. Trace the execution of the algorithm.

2.4 (due Sep 24) Let

$$A[1..25] = [7, 2, 23, 4, 9, 5, 1, 24, 3, 10, 6, 8, 17, 12, 14, 19, 25, 22, 15, 11, 13, 20, 16, 21, 18].$$

Suppose that we use the deterministic linear-time **select** algorithm on A . Compute the pivot used by the algorithm to split A .

2.5 (due Sep 24) Compute the convolution of $(1, 0, 0, 1, 0, 1)$ with $(0, 1, 0, 0, 1)$. (DEFINITION: convolution of two sequences a_0, \dots, a_n and b_0, \dots, b_m is a sequence c_0, \dots, c_{m+n} , where

$$c_k = \sum_{i=\max(0, k-m)}^{\min(k, n)} a_i b_{k-i},$$

i. e., c_0, \dots, c_{m+n} is the coefficient sequence of the polynomial $(a_0 + a_1x + \dots + a_nx^n)(b_0 + b_1x + \dots + b_mx^m)$).

2.6 (due Sep 24) The convolution of sequence $(1, 2, 3, 4)$ with an unknown sequence S is sequence $(5, 11, 17, 23, 4)$. Compute the sequence S .

2.7 (due Sep 24) Evaluate the polynomial $x^3 + 3x^2 - x + 1$ on the 4-th roots of unity. (DEFINITION: n -th roots of unity are $\exp(2\pi ik/n)$, where $k = 0, \dots, n - 1$. For example, 4-th roots of unity are: $i, -1, -i, 1$.)

4 Additional homework

2.8 (due Sep 24) We are given an array A with n elements and a number C . Assume that the sum of the elements in A is larger than C . We would like to compute the size of the smallest subset of A whose elements sum to at least C . (For example, if $A = [8, 3, 9, 2, 7, 1, 5]$ and $C = 18$ then the answer is 3; the set is $\{7, 8, 9\}$.) Give a linear-time algorithm for this problem. (HINT: use the linear-time SELECT algorithm)

2.9 (due Sep 24) We are given k sorted lists A_1, \dots, A_k . The total number of elements in the lists is n . We would like to merge the lists into one sorted list B (the number of elements in B will be n). Give an algorithm which solves this problem in time $O(n \log k)$.

2.10 (due Sep 24) We are given two sorted arrays A and B , each containing n elements. Assume that the arrays do not contain duplicates, and the elements in A are different from elements in B . We would like to compute the median of $A \cup B$. For example if $A = [1, 2, 3, 4, 5]$ and $B = [6, 7, 8, 9, 10]$ then the median is 5; if $A = [1, 3, 5, 7, 9]$ and $B = [2, 4, 6, 8, 10]$ then the median is again 5. Give an $O(\log n)$ algorithm for this problem.

2.11 (due Sep 29) We are given an array of integers $A[1..n]$. We would like to determine whether there exists an integer x which occurs in A more than $n/2$ times (i. e., whether A has a majority element). Give an algorithm which runs in time $O(n)$. (HINT: use the linear-time SELECT algorithm.)

Example: For $A = [3, 1, 2]$ the answer is NO. For $A = [3, 1, 3]$ the answer is YES.

2.12 (due Sep 29) We are given an array of integers $A[1..n]$. We would like to determine whether there exists an integer x which occurs in A more than $n/3$ times. Give an algorithm which runs in time $O(n)$. (HINT: use the linear-time SELECT algorithm)

2.13 (due Sep 29) We are given an array of integers $A[1..n]$ which is almost sorted in the following sense: for all $i \in \{1, \dots, n - k\}$ we have $A[i] \leq A[i + k]$. Give an algorithm which sorts the array A . Your algorithm should run in time $O(n \log k)$.

2.14 (due Sep 29) We are given two arrays of integers $A[1..n]$ and $B[1..n]$, and a number X . Design an algorithm which decides whether there exist $i, j \in \{1, \dots, n\}$ such that $A[i] + B[j] = X$. Your algorithm should run in time $O(n \log n)$.

2.15 (due Sep 29) Let A and B be two sets. Their sum $A + B$ is defined to be

$$A + B := \{a + b \mid a \in A, b \in B\},$$

for example if $A = \{1, 4, 6\}$ and $B = \{2, 5\}$ then $A + B = \{3, 6, 8, 9, 11\}$. Give $O(n \log n)$ algorithm whose input is two sets $A, B \subseteq \{1, \dots, n\}$ and the output is set $A + B$.

5 Additional problems from the book (do not turn in)

Try to solve the following problems. A few of them will be on the quiz. We will go over the ones that you choose in the problem sessions.

- 2.4, 2.5, 2.8, 2.10, 2.12, 2.16, 2.17, 2.18, 2.21, 2.26, 2.27, 2.28, 2.29, 2.30, 2.31, 2.34.