Hamiltonian Cycle
Hamiltonian Cycle Problem

Hamiltonian Cycle

Given a directed graph $G$, is there a cycle that visits every vertex exactly once?

Such a cycle is called a Hamiltonian cycle.
Hamiltonian Cycle is NP-complete

**Theorem**

*Hamiltonian Cycle is NP-complete.*

**Proof.** First, HamCycle ∈ **NP**. Why?

Second, we show 3-SAT \(\leq_P\) Hamiltonian Cycle.

Suppose we have a black box to solve Hamiltonian Cycle, how do we solve 3-SAT?

In other words: how do we encode an instance \(I\) of 3-SAT as a graph \(G\) such that \(I\) is satisfiable exactly when \(G\) has a Hamiltonian cycle.

Consider an instance \(I\) of 3-SAT, with variables \(x_1, \ldots, x_n\) and clauses \(C_1, \ldots, C_k\).
Reduction Idea (very high level):

- Create some graph structure (a “gadget”) that represents the variables
- And some graph structure that represents the clauses
- Hook them up in some way that encodes the formula
- Show that this graph has a Ham. cycle iff the formula is satisfiable.
Direction we travel along this chain represents whether to set the variable to \textbf{true} or \textbf{false}. \\

\begin{itemize}
\item \textbf{true}
\item \textbf{false}
\end{itemize}
Add a new node for each clause:

$C_k$
Connect it this way if $x_i$ in $C_k$

$C_j$
Connect it this way if $x_i$ in $C_k$

Direction we travel along this chain represents whether to set the variable to **true** or **false**.
Connecting up the paths

\[ s \xrightarrow{\lambda} x_1 \xrightarrow{\lambda} \ldots \xrightarrow{\lambda} x_n \xrightarrow{\lambda} t \]

Connections between \( x_1, x_2, \ldots, x_n \) and \( s \) and \( t \) are indicated by arrows.
Connecting up the paths

\[ x_1 \rightarrow C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_k \]

\[ s \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \cdots \rightarrow x_n \rightarrow t \]
A Hamiltonian path encodes a truth assignment for the variables (depending on which direction each chain is traversed).

For there to be a Hamiltonian cycle, we have to visit every clause node.

We can only visit a clause if we satisfy it (by setting one of its terms to true).

Hence, if there is a Hamiltonian cycle, there is a satisfying assignment.
Hamiltonian Path

**Hamiltonian Path:** Does $G$ contain a path that visits every node exactly once?

How could you prove this problem is NP-complete?
Hamiltonian Path: Does $G$ contain a path that visits every node exactly once?

How could you prove this problem is NP-complete?

Reduce Hamiltonian Cycle to Hamiltonian Path.

Given instance of Hamiltonian Cycle $G$, choose an arbitrary node $v$ and split it into two nodes to get graph $G'$:

Now any Hamiltonian Path must start at $v'$ and end at $v''$. 
Hamiltonian Path

$G''$ has a Hamiltonian Path $\iff$ $G$ has a Hamiltonian Cycle.

$\implies$ If $G''$ has a Hamiltonian Path, then the same ordering of nodes (after we glue $v'$ and $v''$ back together) is a Hamiltonian cycle in $G$.

$\impliedby$ If $G$ has a Hamiltonian Cycle, then the same ordering of nodes is a Hamiltonian path of $G'$ if we split up $v$ into $v'$ and $v''$. □

Hence, Hamiltonian Path is NP-complete.
Traveling Salesman Problem

Given \( n \) cities, and distances \( d(i,j) \) between each pair of cities, does there exist a path of length \( \leq k \) that visits each city?

Notes:

- We have a distance between every pair of cities.
- In this version, \( d(i,j) \) doesn’t have to equal \( d(j,i) \).
- And the distances don’t have to obey the triangle inequality \( d(i,j) \leq d(i,k) + d(k,j) \) for all \( i,j,k \).
TSP large instance

- TSP visiting 24,978 (all) cities in Sweden.
- Solved by David Applegate, Robert Bixby, Vašek Chvátal, William Cook, and Keld Helsgaun
- Lots more cool TSP at http://www.tsp.gatech.edu/
**Thm.** Traveling Salesman is NP-complete.

TSP seems a lot like Hamiltonian Cycle. We will show that

\[ \text{Hamiltonian Cycle} \leq_p \text{TSP} \]

To do that:

**Given:** a graph \( G = (V, E) \) that we want to test for a Hamiltonian cycle,

**Create:** an instance of TSP.
Creating a TSP instance

A TSP instance $D$ consists of $n$ cities, and $n(n - 1)$ distances.

**Cities** We have a city $c_i$ for every node $v_i$.

**Distances** Let $d(c_i, c_j) = \begin{cases} 1 & \text{if edge } (v_i, v_j) \in E \\ 2 & \text{otherwise} \end{cases}$
TSP Reduction

Theorem

\[ G \text{ has a Hamiltonian cycle } \iff D \text{ has a tour of length } \leq n. \]

Proof. If \( G \) has a Ham. Cycle, then this ordering of cities gives a tour of length \( \leq n \) in \( D \) (only distances of length 1 are used).

Suppose \( D \) has a tour of length \( \leq n \). The tour length is the sum of \( n \) terms, meaning each term must equal 1, and hence cities that are visited consecutively must be connected by an edge in \( G \). \( \square \)

Also, TSP \( \in \mathbf{NP} \): a certificate is simply an ordering of the \( n \) cities.
Hence, TSP is NP-complete.

Even TSP restricted to the case when the $d(i,j)$ values come from actual distances on a map is NP-complete.