Simplex

CSC 282
let $v$ be any vertex of the feasible region while there is a neighbor $v'$ of $v$ with better objective value: set $v = v'$

Each vertex is specified by a set of $n$ inequalities.

Two vertices are *neighbors* if they have 
$n - 1$ defining inequalities in common.

**Defining inequalities for A and C?**
Case 1: Vertex is Origin

- Origin is optimal iff all $c_i \leq 0$

- Otherwise:
  - Release some tight constraint $x_i$
  - Increase $x_i$ until some other inequality becomes tight

Increasing $x_2$ until it “runs into” constraint 3 stopping at $x_2 = 3$

\begin{align*}
\text{max } 2x_1 + 5x_2 \\
2x_1 - x_2 &\leq 4 \quad (1) \\
x_1 + 2x_2 &\leq 9 \quad (2) \\
-x_1 + x_2 &\leq 3 \quad (3) \\
x_1 &\geq 0 \quad (4) \\
x_2 &\geq 0 \quad (5)
\end{align*}
Case 2: Vertex is not the origin

- If not at the origin: transform coordinates so that the vertex is the origin

- New coordinate system $\mathbf{y}$ is a linear transformation of $\mathbf{x}$

- New objective function becomes $\max c_u + k^T \mathbf{y}$
  - $c_u$ is the value of the objective function at original vertex $u$
  - $k$ is the transformed cost vector
Initial LP:

\[
\begin{align*}
\text{max} \quad & 2x_1 + 5x_2 \\
2x_1 - x_2 & \leq 4 \quad (1) \\
x_1 + 2x_2 & \leq 9 \quad (2) \\
-x_1 + x_2 & \leq 3 \quad (3) \\
x_1 & \geq 0 \quad (4) \\
x_2 & \geq 0 \quad (5)
\end{align*}
\]

Current vertex: \{4, 5\} (origin).
Objective value: 0.

Move: increase \(x_2\).
\(5\) is released, \(3\) becomes tight. Stop at \(x_2 = 3\).

New vertex \{4, 3\} has local coordinates \((y_1, y_2)\):
\[
y_1 = x_1, \quad y_2 = 3 + x_1 - x_2
\]
Rewritten LP:

\[
\begin{align*}
\text{max} & \quad 15 + 7y_1 - 5y_2 \\
& y_1 + y_2 \leq 7 \quad (1) \\
& 3y_1 - 2y_2 \leq 3 \quad (2) \\
& y_2 \geq 0 \quad (3) \\
& y_1 \geq 0 \quad (4) \\
& -y_1 + y_2 \leq 3 \quad (5)
\end{align*}
\]

Current vertex: \{④, ③\}.
Objective value: 15.

Move: increase \(y_1\).
④ is released, ② becomes tight. Stop at \(y_1 = 1\).

New vertex \{②, ③\} has local coordinates \((z_1, z_2)\):

\[
\begin{align*}
  z_1 &= 3 - 3y_1 + 2y_2, \\
  z_2 &= y_2
\end{align*}
\]
Rewritten LP:
\[
\begin{align*}
\text{max } & 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2 \\
-\frac{1}{3}z_1 + \frac{5}{3}z_2 & \leq 6 \quad \text{(1)} \\
\quad z_1 & \geq 0 \quad \text{(2)} \\
\quad z_2 & \geq 0 \quad \text{(3)} \\
\frac{1}{3}z_1 - \frac{2}{3}z_2 & \leq 1 \quad \text{(4)} \\
\frac{1}{3}z_1 + \frac{1}{3}z_2 & \leq 4 \quad \text{(5)}
\end{align*}
\]

Current vertex: \( \{2, 3\} \).
Objective value: 22.

Optimal: all \( c_i < 0 \).

Solve \( 2, 3 \) (in original LP) to get optimal solution \( (x_1, x_2) = (1, 4) \).
Running Time of Simplex

• n variables, m constraints

• Each iteration is O(mn)
  • Calculating objective value: O(n)
  • Checking if a neighbor is feasible:
    • Naive approach O(mn^4)
    • Incremental algorithm amortized cost O(mn)
  • Moving to a neighbor: O(1)

• Worst case number of iterations $\binom{m+n}{n}$ exponential
Circuit Evaluation

- Given Boolean circuit and its inputs, compute the output
- Can be encoded as an LP
- Shows that LP is "P-complete" - as hard as any program in P
Circuit Satisfiability

- Given Boolean circuit, is there some set of inputs that makes the output 1?
- Cannot be encoded as an LP
- Can be encoded as an integer program
- Shows that integer programming is "NP-complete" - as hard as any program in NP