Question 1. [30 points]

Show $A \leq_m B$, where the alphabet $\Sigma = \{0, 1\}$, and

$A = \{x \in \{0, 1\}^* \mid |x| \text{ is even}\}$

$B = \{x \in \{0, 1\}^* \mid |x| \text{ is odd}\}$.

Answer:

Let $\sigma(x) = 0x$, or in other words, its input with a zero concatenated onto the front. $\sigma$ is clearly a recursive function.

If $x \in A$, $|x|$ is even. Thus $|0x|$ is odd, and so $0x \in B$. Thus $\sigma(x) \in B$.

If $x \notin A$, $|x|$ is odd. Thus $|0x|$ is even, and so $0x \notin B$. Thus $\sigma(x) \notin B$.

We have that $x \in A \iff \sigma(x) \in B$, and so $A \leq_m B$. 

Question 2. [70 points] Show $\mathbb{HP} \leq_m B$, where

$$B = \{ i \mid \overline{L(M_i)} \text{ is infinite} \}.$$

Answer:

$\sigma(i) = j$ where $j$ is the index of a Turing machine $M_j$ that does the following on input $y$: Simulate $M_i$ on $i$. $\sigma$ is clearly a recursive function.

If $i \in \mathbb{HP}$, $M_i$ rejects $i$. Thus $M_j$ accepts no strings, and so $L(M_j) = \emptyset$ and $\overline{L(M_j)} = \Sigma^*$, which is infinite. Thus $j \in B$.

If $i \notin \mathbb{HP}$, $M_i$ accepts $i$. Thus $M_j$ accepts all strings, and so $L(M_j) = \Sigma^*$ and $\overline{L(M_j)} = \emptyset$, which is finite. Thus $j \notin B$.

We have that $i \in \mathbb{HP} \iff \sigma(i) \in B$, so $\mathbb{HP} \leq_m B$. 