Instructions: There are six problems here. Do four. If you choose to answer more than four problems (you may if you want to), we will grade all that you answer, and will count the highest (that is not a typo; it really says and means “highest”) four scores.

On this exam, $HP$ denotes $\{i | M_i(i) \text{ accepts}\}$. Note carefully that the word there is not “halts” but is “accepts.”

Don’t just jot down random stuff (doing so may LOSE points)—think before answering (we’ll hand out scrap paper so you have scrap paper on which to doodle/plan/try-things).

Note that not all questions are necessarily identical in difficulty, so don’t, for example, make the mistake of spending all your time on one hard problem and leaving the others blank.

Don’t confuse “from” and “to” on reductions (if when trying to prove $A \leq_m B$ you accidentally just prove $B \leq_m A$, you’ll probably get no points). Note, for example, that there is a many-one reduction from $\emptyset$ to $HP$, but there exists no many-one reduction from $HP$ to $\emptyset$. (On this exam, “many-one” of course refers to recursive many-one reductions.)

Your handwriting must be clear and readable. We will not guess that some unclear character is what is needed to make your answer correct; your written characters must be clearly readable on their own.

Rules: Closed book, closed notes, no computers or calculators, use pen not pencil. An exception to the “closed notes” rule is that each person can have one sheet of self-prepared, hand-written notes.

Make sure to put all your answers that you want graded onto (not the scrap paper but rather) the stapled test sheets.

Scoring: To repeat: There are six problems here. Do four. If you choose to answer more than four problems (you may if you want to), we will grade all that you answer, and will count the highest four scores.

Since your grade will be based on four 70-point problems, there are 280 points available here (actually, 28 10-point chunks). This exam is all one superchunk. That is, each of the 28 10-point chunks will have the same value, namely the sum of your 4 highest problems divided by 28.
Question 1  [70 points] Prove by a clear Tarski-Kuratowski quantifier analysis that 
\{ \langle i, j \rangle \mid L(M_i) = L(M_j) \} \text{ is in } \Pi_2.
Question 2 [70 points] Let $A = \{ i \mid L(M_i) = \emptyset \}$. Let $B = \{ \langle j, k \rangle \mid L(M_j) = L(M_k) \}$. Give an explicit, direct, many-one reduction from $A$ to $B$ (i.e., $A \leq_m B$), and prove (in our 3-part way) that it is indeed is such a reduction.
Question 3 [70 points] Let $R = \{i \mid L(M_i) \neq \emptyset\}$. First, state Rice’s Theorem (the easy 1-part theorem, not the 3-part theorem). Second, either use that Rice’s Theorem to prove that $R$ is not recursive, or explain clearly why that Rice’s Theorem is not appropriate for this problem.
**Question 4** [70 points]

(a) [35 points] Prove that $\Sigma_1 \cap \Pi_1 = \Sigma_0$ (i.e., that if $A$ is both r.e. and co-r.e., then $A$ is recursive).
(b) [35 points] Give an explicit, direct, many-one reduction (note: on this particular problem, you do not have to include a correctness proof; just give a correct reduction) from \( \mathit{HP} \) to \( D = \{ i \mid L(M_i) \text{ is infinite} \} \).
Question 5 [70 points] Let $A$ and $B$ be as in problem 2. Namely, $A = \{i \mid L(M_i) = \emptyset\}$ and $B = \{(j,k) \mid L(M_j) = L(M_k)\}$. Give an explicit, direct, many-one reduction from $A$ to $B$ (i.e., $A \leq_m B$), and prove (in our 3-part way) that it is indeed is such a reduction.
**Question 6** [70 points] Prove that there is a set $A$, over the alphabet $\{0, 1\}$, such that (a) $A$ is an infinite set, (b) $A$ has no infinite recursively enumerable subset, and (c) for each $n \geq 486$ it holds that of the $2^n$ distinct strings of length $n$ at least 2010 of them are elements of $A$. (Note: If in your proof you prove that $A$ will contains at least 2010 distinct strings of length 2, there probably is a problem in your proof.)