

# CONTROL THEORY

- o Huge Field
- o Practical ... Abstract
- o Many idiosyncratic engineering analysis tricks?
- o Linear Systems - common approx.
- o Discrete  $\leftrightarrow$  Continuous  
(Digital)  $\leftrightarrow$  (Analog)
- o Common Applications
  - Setpoint control (e.g. thermostat)
  - Fly by wire
  - Auto pilot
  - Anti-shake, stabilization
  - ...

$n^{\text{th}}$  order linear differential eqn.

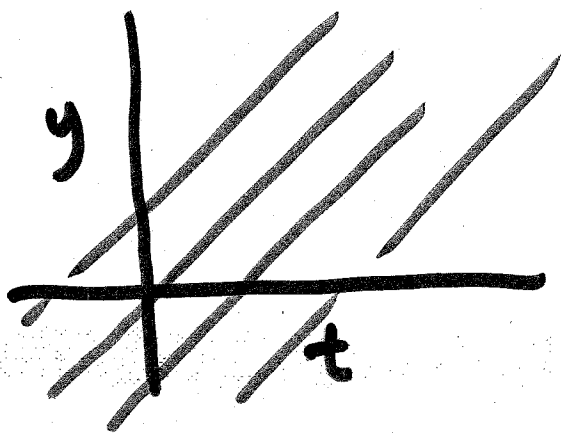
constant or time-varying coefficients

$$a_n(t) \frac{d^n y}{dt^n} + a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1(t) \frac{dy}{dt} + a_0(t) y = g(t).$$

For complete solution, need initial condition(s).

E.g.  $\frac{dy}{dt} = 1$

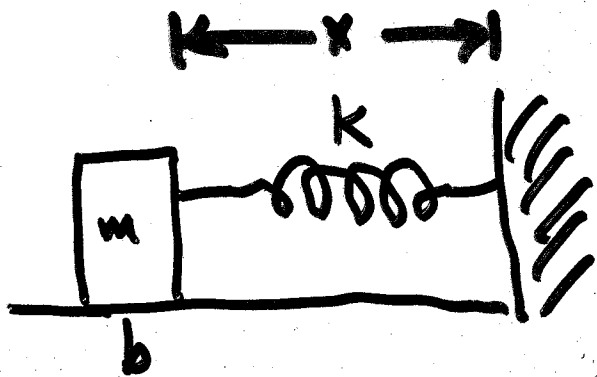
$$y(t) = t + b$$



I.C. e.g. what is  $y$  when  $t=0$ ?

Linear Systems (Dudek Ch 4.)

# 2<sup>nd</sup> - Order System (constant coeffs)



$m, k, b$  parameters  
(the coeffs) determine  
behavior.

With no outside applied forces:

$$m \ddot{x} + b \dot{x} + kx = 0 \quad \leftarrow \text{Homogeneous eqn.}$$

$F = ma$  friction spring

Solution depends on ICs. of position  
and velocity. There are 2 ICs:

2<sup>nd</sup> order system.

# Characteristic Equation Method

$$a_n \frac{d^n z}{dt^n} + a_{n-1} \frac{d^{n-1} z}{dt^{n-1}} + \dots + a_0 z = 0$$

Suppose  $z = e^{st}$  for some constant  $s$   
is a solution. Then...

$$a_n s^n e^{st} + a_{n-1} s^{n-1} e^{st} + \dots + a_0 e^{st} = 0$$

$e^{st} > 0$  so divide it out

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_0 = 0 \quad (*)$$

$e^{st}$  is solution  $\Rightarrow s$  satisfies  $(*)$

also  $(!)$

$s$  satisfies  $(*) \Rightarrow e^{st}$  is solution.

NOTE CAN ADD PLUS, MINUS BY CONSTANT, ADD CONSTANT.  $\sqrt{4}$

E.G.

$$\frac{dy}{dt} = ay$$

$$s - a = 0$$

$s = a$  is a solution:

$$y(t) = Ke^{at}$$

$K$  is a const. determined by I.C.

as in:

$$R \frac{dq}{dt} = -\frac{1}{C} q$$

$$\frac{dq}{dt} = -\frac{1}{RC} q$$

E.G.

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0$$

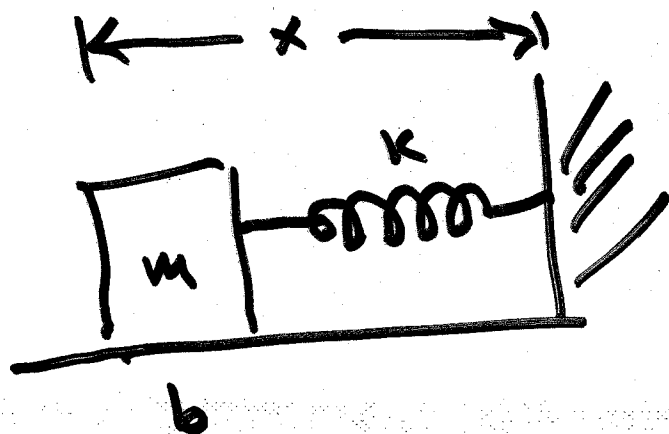
$$s^2 - 5s + 6 = 0 \quad (s-2)(s-3) = 0$$

so  $s = 2, s = 3$  are solns.

$$y(t) = c_1 e^{2t} + c_2 e^{3t}$$

5

# 2<sup>nd</sup> Order Systems Again



$m, b, k > 0$   
 $\Rightarrow$  physical  
 $\Rightarrow$  stable

$$m \ddot{x} + b \dot{x} + kx = 0$$

$$m s^2 + b s + k = 0$$

$$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4km}}{2m}$$

can be  
complex.

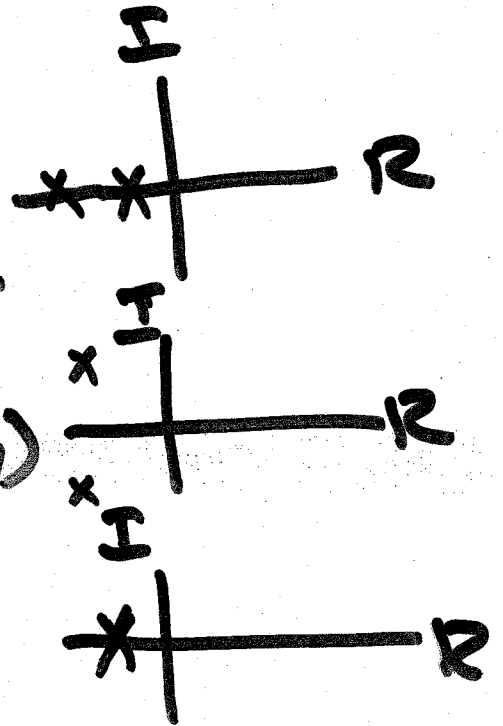
$$s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4km}}{2m}$$

$s_1, s_2$  are so-called poles of system.  
(control-theory speak).

Describe its behavior. Dominant  
poles  $\Rightarrow$  can approx with  $n^{\text{th}}$ -order syst.  
as 2<sup>nd</sup>-order. [6]

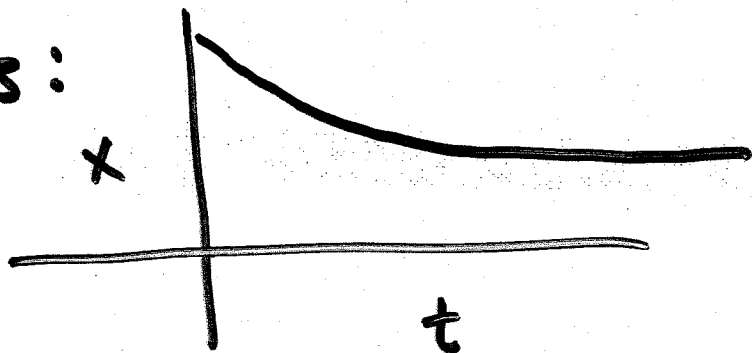
$s_1, s_2$  can be

- real and unequal  
 $b^2 > 4mk$ : friction dominates.  
 overdamped
- complex  
 $b^2 < 4mk$ : stiffness (spring) dominates = oscillatory.
- real and equal  
 $b^2 = 4mk$

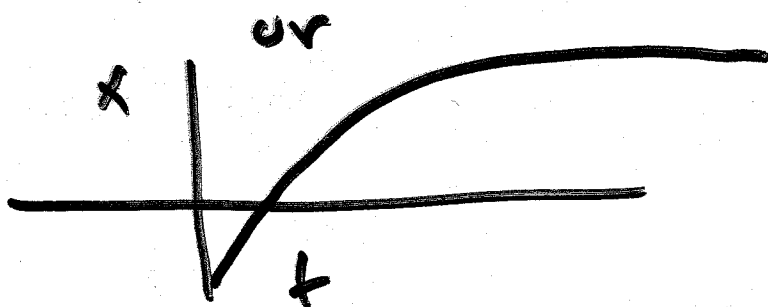


fastest non-oscillatory response  
critically damped.

Real Roots:



what do  
oscillations  
have to do  
with  
 $\Delta t$ ?



$$e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ &\quad + ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \dots \\ &= \cos x + i \sin x \end{aligned}$$

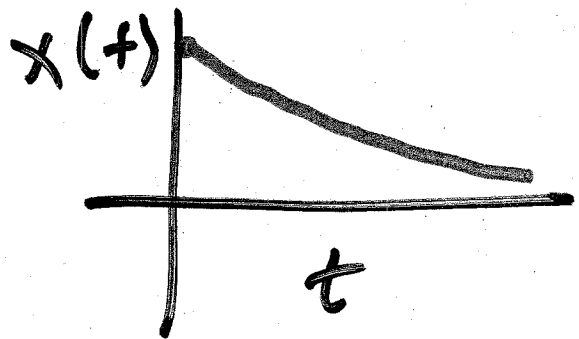
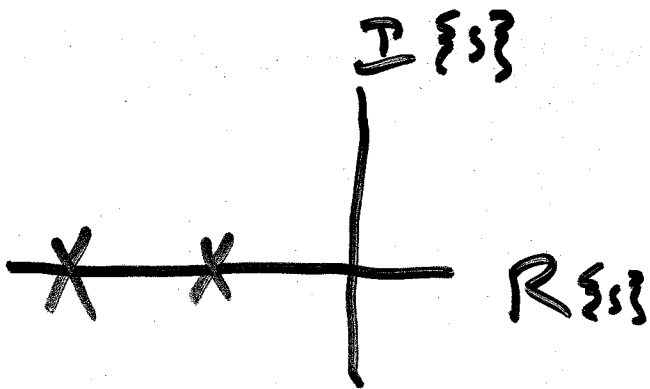
(Euler's Formula)



# Real & Unequal Roots

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

$s_1, s_2$  from the quadratic formula solution to char. eq., recall?

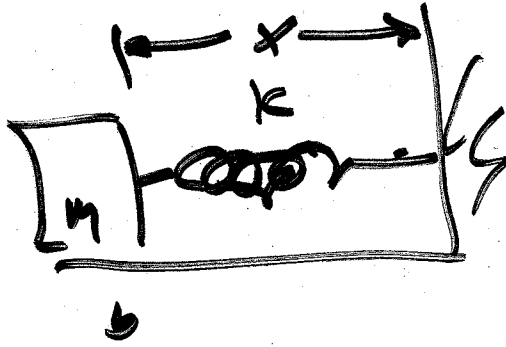


Sluggish motion - overdamped.

If one pole is much greater in magnitude than other, can ignore it.

"bigger"  $s \rightarrow e^{-st} \rightarrow$  zero faster.

E.G.



$$m = 1$$

$$b = 5$$

$$k = 6$$

release at rest block from  $x = -1$

$$s^2 + 5s + 6 = 0$$

$$s_1 = -2, s_2 = -3$$

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

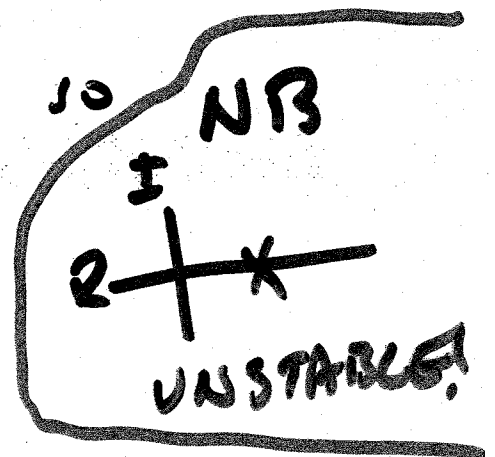
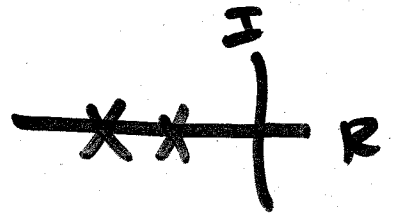
$$x(0) = -1, \dot{x}(0) = 0$$

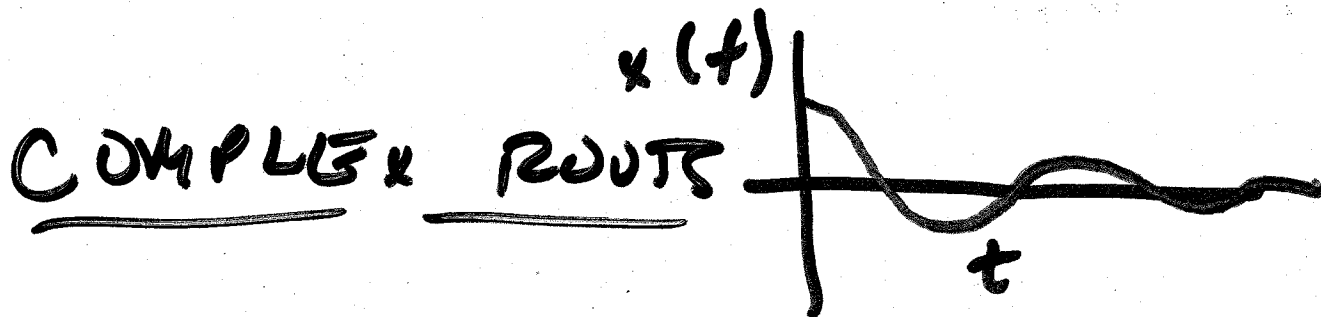
$$c_1 + c_2 = -1$$

$$-2c_1 - 3c_2 = 0$$

$$\text{so } c_1 = -3, c_2 = 2, \text{ so}$$

$$x(t) = -3e^{-2t} + 2e^{-3t}$$





$$s_1 = \lambda + i\mu$$

$$s_2 = \lambda - i\mu$$

complex  
conjugates

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

Euler's formula:

$$e^{ix} = \cos x + i \sin x$$

and

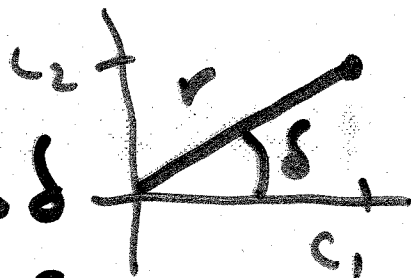
$$x(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$$

(good exercise)

$c_1, c_2$  depend on IC.

Further: say  $c_1 = r \cos \delta$

$$c_2 = r \sin \delta$$

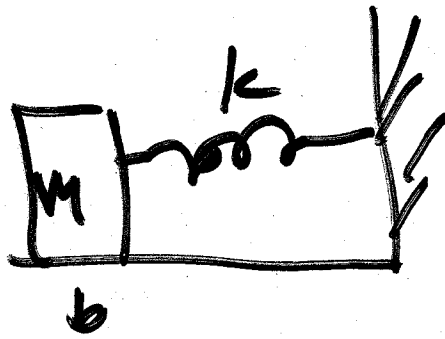


$$x(t) = r e^{\lambda t} \cos(\mu t - \delta)$$

$$r = \sqrt{c_1^2 + c_2^2} \quad \delta = \text{Atan2}(c_2, c_1)$$

□  
||

Ex.



$$m=1 \quad b=1 \quad k=1 \quad x=-1$$

$$s^2 + s + 1 = 0$$

$$s_j = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$x(t) = e^{-\frac{t}{2}} \left[ c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t \right]$$

$$x(0) = -1, \quad \dot{x}(0) = 0$$

$$\text{so } c_1 = -1, \quad -\frac{1}{2}c_1 - \frac{\sqrt{3}}{2}c_2 = 0$$

$$\text{or } c_1 = -1, \quad c_2 = \frac{\sqrt{3}}{3}$$

$$x(t) = e^{-\frac{t}{2}} \left[ -\cos \frac{\sqrt{3}}{2} t - \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}}{2} t \right]$$

$$x(t) = \frac{2\sqrt{3}}{3} e^{-\frac{t}{2}} \cos \left( \frac{\sqrt{3}}{2} t + 120^\circ \right)$$

12

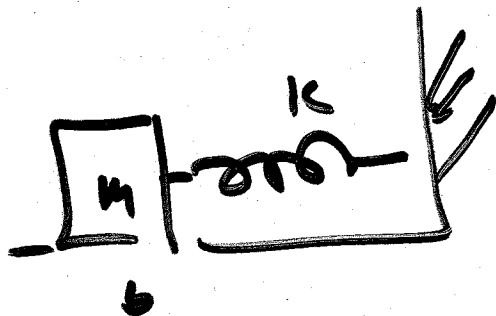
## REAL & EQUAL ROOTS

$$x(t) = c_1 e^{s_1 t} + c_2 t e^{s_2 t} \quad \left. \begin{array}{l} \text{ck. by.} \\ \text{subst.} \end{array} \right\}$$

$$s_1 = s_2 = -\frac{b}{2m}, \text{ so}$$

$$x(t) = (c_1 + c_2 t) e^{-\frac{b}{2m} t}$$

E.g.



$$m=1, \quad b=4, \quad k=4 \quad \begin{array}{l} \star = -1 \\ \text{initially} \end{array}$$

$$s^2 + 4s + 4$$

$$s_1 = s_2 = -2$$

$$x(t) = (c_1 + c_2 t) e^{-2t}$$

$$c_1 = -1$$

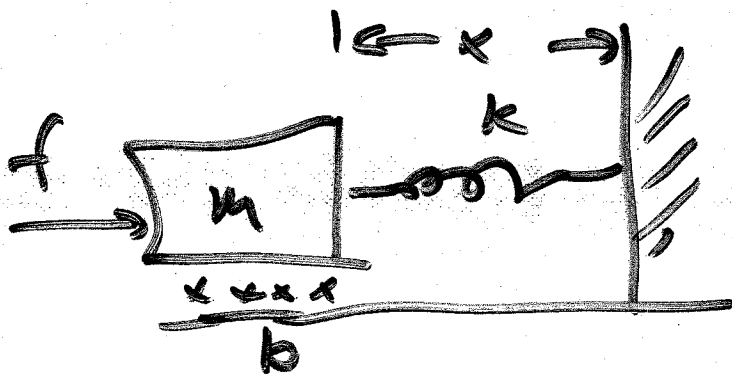
$$-2c_1 + c_2 = 0; \quad c_2 = -1, \quad c_2 = -2$$

$$\rightarrow x(t) = (-1 - 2t) e^{-2t}$$

# CONTROL OF 2<sup>ND</sup>-ORDER SYSTEMS

Suppose our 2<sup>nd</sup> order system isn't what we want. It's too lively or too sluggish... Maybe there's no spring.

- Add sensors, actuator for applying force  $f$  on block.



Assume  
position  $x$   
velocity  $\dot{x}$   
sensors.

Propose this: A CONTROL LAW:

$$f = -k_p x - k_v \dot{x}$$

## POSITION RESOLUTION

$$f = -k_p x - k_v \dot{x}$$

Clearly that's what this is doing!

(there are clearly other things controllers can do, like following a trajectory).

So...

$$m \ddot{x} + b \dot{x} + kx = -k_p x - k_v \dot{x}$$

$$m \ddot{x} + (b + k_v) \dot{x} + (k + k_p) x = 0$$

$$m \ddot{x} + b' \dot{x} + k' x = 0$$

So by picking CONTROL GAINS  $k_v, k_p$ ,  
can make any system we like!

E.g.  
e.g.  $b' = 2\sqrt{mk'}$  get critical  
damping

if  $m=b=k=1$ , how make the  
system critically damped with  
closed-loop stiffness (equivalent spring)  
16?

if  $k' = 16.0$ ,  $b' = 2\sqrt{mk'} = 8$

$k=1$ ,  $b=1$ ,

$k_p = 15$

$k_v = 7$



MODEL-BASED CONTROL  
( "computed torque", e.g. )  
(State of the art.)

A.K.A. control-law partitioning

- o model-based ~~partition~~ partition
- o servo partition
- o put system parameters in the model partition, so servo is independent.
- o generalizes to Non-linear systems.

$$m\ddot{x} + b\dot{x} + kx = f$$

If we know  $m, b, k$ ; why not use the model-based controller to turn system into a unit, frictionless, springless mass? THAT would be easy to control.

model-based control law:

$$f = \alpha f' + \beta$$

$\alpha, \beta$  are functions or constants, chosen so that if  $f'$  is the input, system behaves like unit mass.

$$m\ddot{x} + b\dot{x} + kx = \alpha f' + \beta$$

if  $f'$  should make system act like unit mass,  $\ddot{x} = f'$ .

So ...  $\alpha = m$

$$\beta = b\dot{x} + kx.$$

Now we carry on as before treating this as open-loop dynamics of a system:

$$f' = -k_v \dot{x} - k_p x \quad : \text{control law}$$

$$\ddot{x} + k_v \dot{x} + k_p x = 0$$

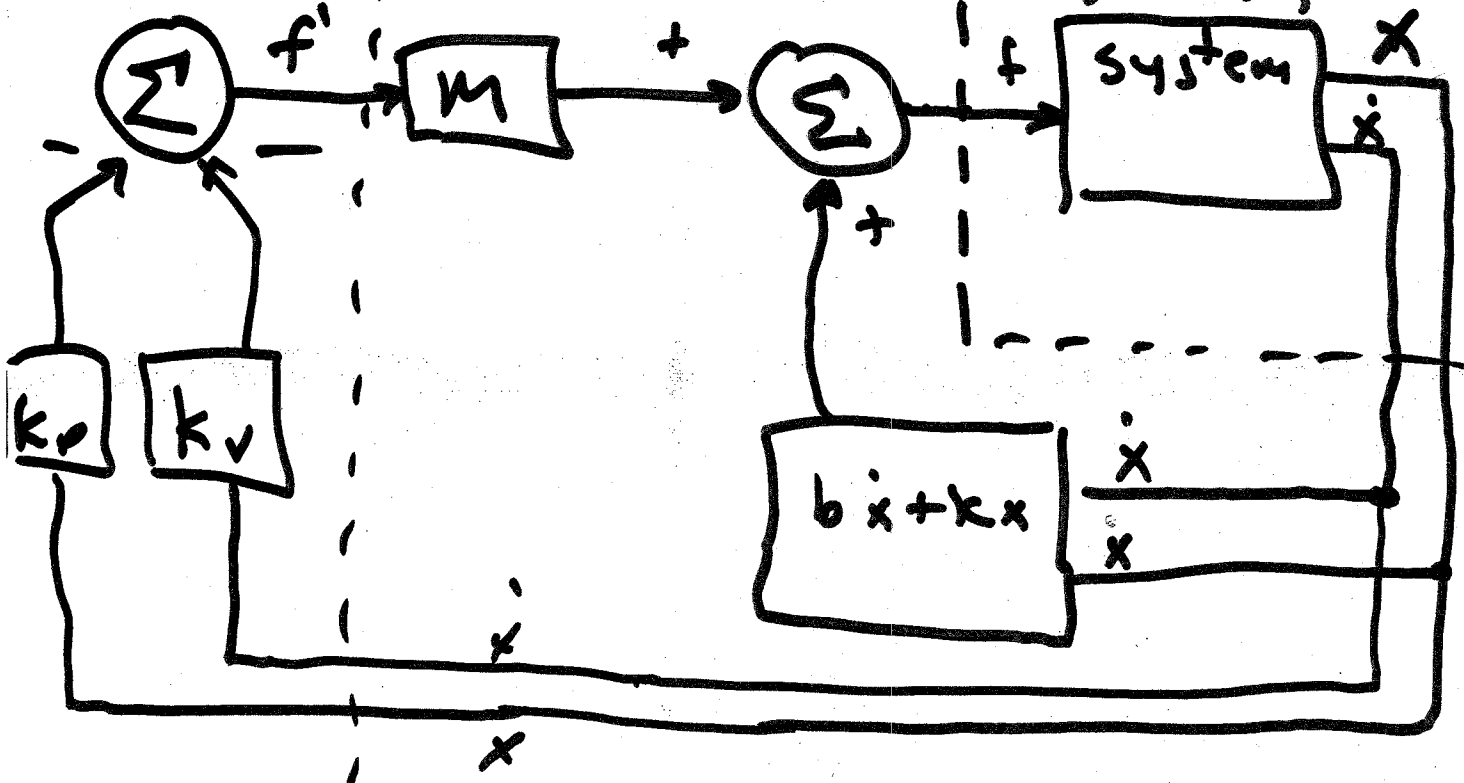
$$k_v = 2\sqrt{k_p}$$

for critical damping  
... no  $m, k, b!$

SERVO

MODEL-BASED

SYSTEM + SENSORS



CLOSED-LOOP CONTROL

w) PARTIALLY CONTROL