Number Representation

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Number Representation

An n digit number can be represented in any base as

`MSD`	...	`LSD`
`n-1`	...	`0`

The value of the ith digit d is $d \times base^i$, where i starts at 0 and increases from right to left

Decimal (base 10) is the natural human representation, binary (base 2) is the natural computer representation

E.g. $1100_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 12_{10}$

Negative Number Representation

Sign-Magnitude - MSB represents sign (0 for +ve, 1 for -ve)

One's Complement ($1`s complement of x = 2^n - x - 1$)

Problem: Balanced representation, but two values for 0

Solution: Two's Complement ($2`s complement of x = 2^n - x$)

- single bit pattern for 0
- ensures that $x + (-x)$ is 0
- still keeps 1 in MSB for a -ve number (sign bit)
- 100... represents the most -ve number
- E.g. 4-bit 2's complement number
  $1100_2 = -1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = -4_{10}$

Exceptions

Overflow: number too large to be represented in n bits

Overflow condition for $O = A + B$:
$\overline{MSBA}.\overline{MSBB}.MSBO + MSBA.MSBB.\overline{MSBO}$

Detection of overflow ignored in C, required in Fortran

Memory addressing arithmetic on unsigned numbers as well

Therefore, MIPS provides both signed and unsigned arithmetic, with an exception/interrupt generated on overflow for the signed arithmetic

Shortcuts

2's complement = 1's complement + 1

2's complement representation of n digit number as $n + m$ digit number - sign extend

Recall Scientific Notation

$65.4 = 6.54 \times 10^1$ (normal form - 1 non-zero leading digit)
Mantissa = 6.54, Exponent = 1, Radix (base) = 10

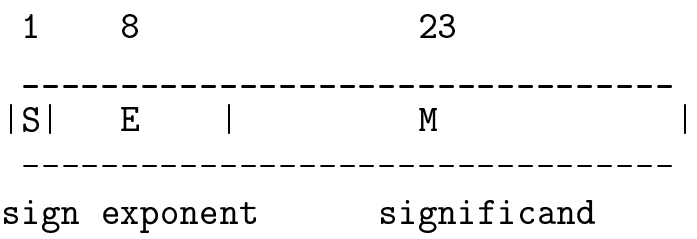
$-0.0654 = -6.54 \times 10^{-2}$

IEEE 754 Floating Point Representation

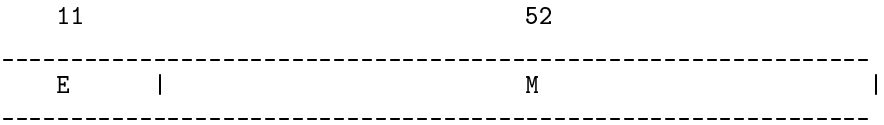
Sign and magnitude Mantissa representation

Biased notation for Exponent (0 represents most negative/smallest exponent)

Single precision - $\pm 1.M \times 2^{E-127}$



Double precision - $\pm 1.M \times 2^{E-1023}$



Trade-offs between range and accuracy

Floating Point Addition

Align decimal point by matching larger exponent

Add significands (possibly unnormalized)

Normalize - shift sum/adjust exponent

Round number to fit in significand field

Floating Point - Additional Bits for Accuracy

Borrow/carry bit - to take into account overflow (carry/borrow) in result

Guard bit - to take care of normalizing left shift

Round bit - for correct rounding

Sticky bit - fine-tune rounding - additional bit to the right of round that is set to 1 if any 1 bit falls off the end of the round digit

IEEE 754 Floating Point Rounding

Four modes:

1. Round to $+\infty$ - add 1 if round or sticky bit set and result ≥ 0
2. Round to $-\infty$ - add 1 if round or sticky bit set and result < 0
3. Round to 0 (truncate)
4. Round to nearest number (default) - round to even when exactly half-way
   - add 1 if round bit and least significant bit of result are set, or, if round bit and sticky bit are set
   - minimizes mean error introduced by rounding

Floating Point Multiplication

Exponent of product = sum of exponents - bias

Multiply significands - decimal point location = sum of digits to the right of decimals

Normalize - check for overflow/underflow

Round

Sign - +ve if both are the same, -ve otherwise

Zero, Infinity, NaN

Zero - 0|0...0|0...0

$+/-\infty$ - S|1...1|0...0

NaN - S|1...1|non-zero

Exceptions

IEEE standard specifies defaults and allows traps to permit user to handle exception

- Invalid operation: e.g., sqrt of -ve number, 0/0, ∞/∞ - result is NaN
- Overflow: result is $+/-\infty$ unless overflow exception is enabled
- Divide by 0: result is $+/-\infty$ if exception not enables
- Underflow: non-zero result underflows to 0
- Inexact: rounded result not the actual result

Converting 0.15_{10} to binary

$remainder$	$remainder \times 2$	leading digit
0.15	0.30	0
0.30	0.60	0
0.60	1.20	1
0.20	0.40	0
0.40	0.80	0
0.80	1.60	1
0.60	1.20	1
	Repeats...	

$0.15_{10} = 0.00100110011..._2$