Non-context-Free Languages

The Pumping Lemma

Theorem. *(Pumping Lemma) Let L be context-free. There exists a positive integer p such that for every $w \in L$ of length at least p, w is divided into five pieces, $w = uvxyz$, such that*

- *for each $i \geq 0$, $uv^ixy^iz \in L$,*
- *$|vy| > 0$, and*
- *$|vxy| \leq p$.*

Proof Let $L = L(G)$ for some CNF grammar $G = (V, \Sigma, R, S)$. Let $m = \| V \|$ and $p = 2^m$. Let w, $|w| \geq p$, be in L and T be a derivation tree for w.

For any subtree R of T, its non-leaf nodes are all variables and its leaves are symbols with unique parents and form a substring of w.

Proof of Pumping Lemma (cont,d)

Claim. *In every subtree R of T with $\geq 2^{m-1}+1$ leaves there are two nodes α and β that are labeled by the same variable and are on the same downward path from the root to a leaf.*

Proof of Claim Let R be a subtree of T with $\geq 2^{m-1}+1$ leaves. Since the complete binary tree of depth $m-1$ has 2^{m-1} leaves, R has a **downward path of length** $\geq m$**.** The path has $\geq m+1$ **nodes.** Since there are **only** m **variables,** by the pigeonhole principle, the path has **two nodes with the same label.** ■Claim

Proof of Pumping Lemma (cont,d)

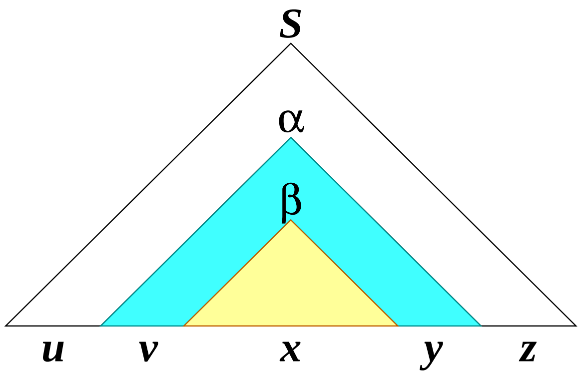
By the Claim there is a node in T whose label coincides with that of a descendant. Let α be one such node that is **the farthest from the root.**

Here **neither the left subtree nor the right subtree of α has more than 2^{m-1} leaves**; otherwise, by the Claim we would find, in one of the two subtrees, a pair of nodes on a downward path labeled by the same variable, which would contradict our assumption that α is the farthest.

Proof of Pumping Lemma (cont,d)

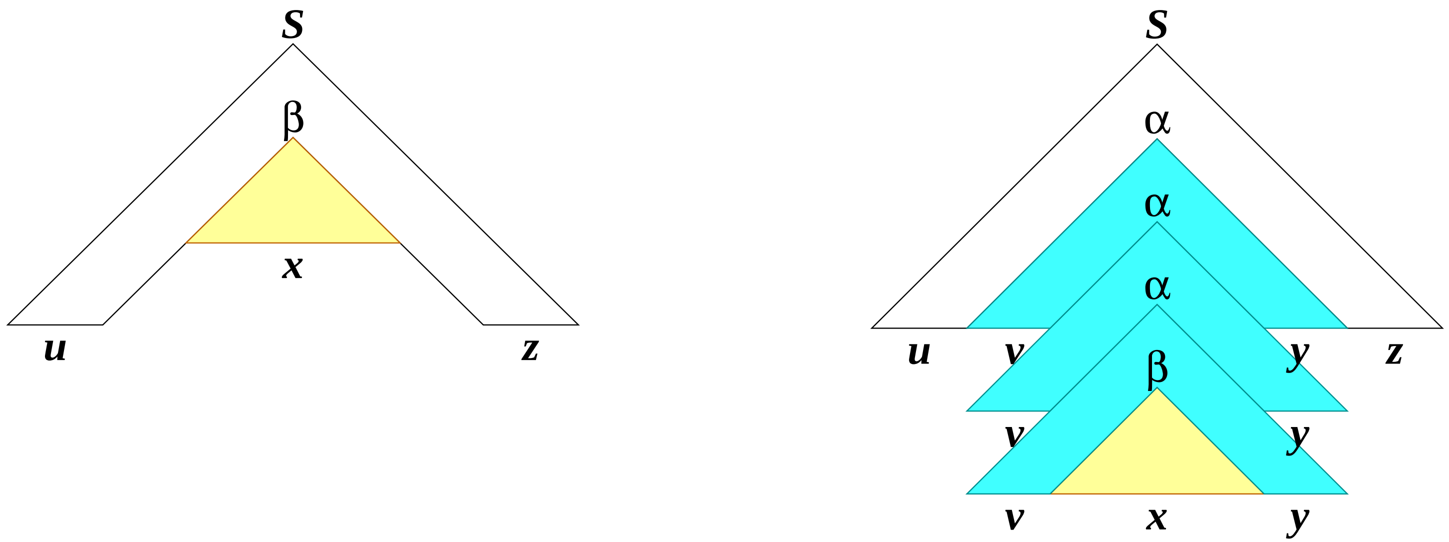
Let β be the descendant of α with the same label as α. **Replacing α by β**, or **repeatedly replacing β by α**, produces **a valid derivation tree.**

Let x be the substring under β, $r = vxy$ the substring under α, with v and y to the left and to the right of x, respectively, and $w = urz$ with u and z to the left and to the right of r, respectively.



Proof of Pumping Lemma (cont,d)

Then replacing α by β corresponds to eliminating v and y; and replacing β by α corresponds to inserting a v before x and a y after x. So, for every $i \geq 0$, $uv^ixy^iz \in L$.



Proof of Pumping Lemma (cont,d)

Since G does not have ϵ rules, either v or y is nonempty, so $|vy| > 0$. Since both left and right subtrees of α have at most 2^{m-1} leaves, α has at most 2^m leaves, and so $|vxy| \leq p$. This proves the lemma. $\blacksquare$

Example 1

$A = \{0^n 1^n 2^n \mid n \geq 0\}$ is not context free.

Proof Assume, on the contrary, that A is context free. By the Pumping Lemma there exists a constant p such that every $w \in A$ of length $\geq p$ is divided into $w = uvxyz$ such that $|vxy| \leq p$, $|vy| \geq 1$, and for every $i \geq 0$, $uv^ixy^iz \in A$.

Let $w = 0^p 1^p 2^p$. Since $|vxy| \leq p$, vxy is either in 0^*1^* or 1^*2^*. So it is not the case uv^2xy^2z has the same number of 0s, 1s, as 2s. ∎

Example 2

$B = \{a\#b\#c \mid a, b$ and c are binary numbers such that $a + b = c\}$ is not context free.

Proof Assume, on the contrary, that B is context free. Let p be the constant from the Pumping Lemma for B. Let $w = 10^p\#10^p\#10^{p+1}$, where $a = b = 2^p$ and $c = 2^{p+1}$. Let $uvxyz$ be the decomposition of w as in the lemma.

For "pumping" to be possible, v has to be a nonempty part of a or of b, and y a nonempty part of c. If v either is a part of a or contains the '1' of b, since $|vxy| \leq p$, y cannot contain a part of c. Thus, v is a part of b and $v \in 0^*$.

Proof Continued

If y contains the first symbol of c, then (pumping down) uxz is not in B because now c is 0 while $a = 2^p$.

If $y \in 0^*$, then (pumping up) $uv^2xy^2z \notin B$ because now the equation becomes $2^p + 2^q = 2^r$ for some $q > p$.

Thus, B is not context-free. ∎

Example 3

$C = \{ww \mid w \in \{0,1\}^*\}$ is not context free.

Proof Assume C is context free. Let p the constant from the pumping lemma for C.

Let $w = 0^p 1^p 0^p 1^p$, which is in C.

Let $w = uvxyz$ be the decomposition of w such that $|vy| > 0$, $|vxy| \leq p$, and for every $i \geq 0$, $uv^i xy^i z \in C$.

If v contains a symbol from the first 0^p then y cannot contain one from the second 0^p, so pumping (up or down) doesn't work. If v contains symbols from the first 1^p then y cannot contain one from the second 1^p, so pumping doesn't work. If v contains only symbols from the second $0^p 1^p$ then pumping does not work. ∎

Application

Corollary. *The class of context-free languages is not closed under intersection.*

Proof Let $L_1 = \{0^i 1^j 2^k \mid i = j\}$ and $L_2 = \{0^i 1^j 2^k \mid j = k\}$. Then L_1 and L_2 are both context-free. If the class were closed under intersection then $L_1 \cap L_2 = \{0^n 1^n 2^n \mid n \geq 0\}$ would be context-free. ∎

Corollary. *The class of context-free languages is not closed under complement.*