NP-Completeness

The P=NP Problem

Is $\mathrm{P} = \mathrm{NP}$?

Identify the *most difficult* problems in $\mathrm{NP} \cdots \mathrm{NP}$-completeness

SAT

A **Boolean formula** is a formula of propositional logic, constructed from variables and Boolean operations ($\wedge$, $\vee$, $\neg$)

A Boolean formula is **satisfiable** if there exists some assignment to the variables that makes the formula evaluate to 1

Example: $\phi = (\overline{x} \vee y) \wedge (x \vee \overline{z})$ is satisfiable. A satisfying assignment is $x = 1, y = 1, z = 1$

$\phi = x \wedge \overline{x} \wedge y$ is not satisfiable.

The Main Theorem

The **satisfiability problem** is the problem of deciding whether a given input Boolean formula is satisfiable

$SAT = \{\phi \mid \phi$ is a satisfiable Boolean formula $\}$

Theorem. $SAT \in \mathrm{P}$ *if and only if* $\mathrm{P} = \mathrm{NP}$

Polynomial Time Reductions

Definition. *A function f is polynomial time computable if there exists a polynomial time TM that halts on each input x with only $f(x)$ on the tape.*

Definition. *A language A is polynomial time mapping reducible to a language B (write $A \leq_P B$) if A is mapping reducible to B via a polynomial time computable function.*

3SAT

A **literal** is a variable or its negation

A **clause** is the disjunction of some literals, e.g., $x_1 \vee \overline{x_3} \vee x_{17}$

A Boolean formula is in **conjunctive normal form** if it is the conjunction ($\wedge$) of some clauses

A **3CNF formula** is a formula in CNF-form in which each clause consists of three literals

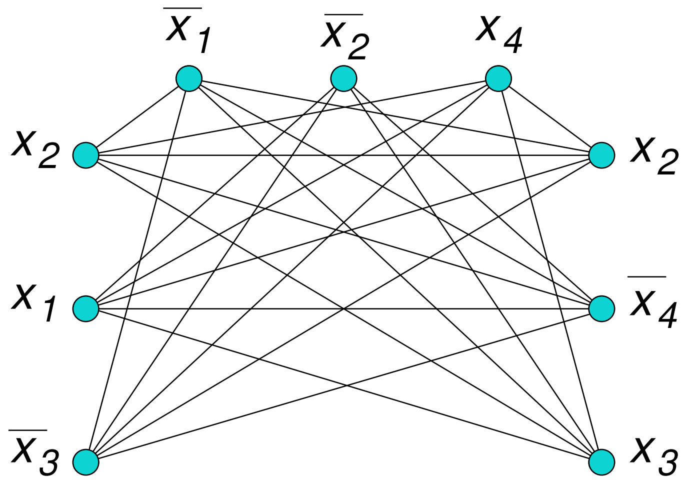
Theorem. *$3SAT$ is polynomial time reducible to $CLIQUE$.*

Proof of Theorem

Given a formula $\phi = \bigwedge_{i=1}^{k}(\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$ of k three-literal clauses construct a graph $G = (V, E)$, where $V = \{\langle i, j\rangle \mid 1 \leq i \leq k, 1 \leq j \leq 3\}$ and $E = \{(\langle i, j\rangle, \langle i', j'\rangle) \mid (i \neq i')$ and **the jth literal in the ith clause and the j'th literal in the i'th clause do not interfere with each other (i.e., $\ell_{i,j} \neq \overline{\ell}_{i',j'}$)**$\}$

The graph for the formula $\phi = (x_2 \vee x_1 \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_4} \vee x_3)$. (Here $k = 3$.)

Note that any distinct vertices forming part of a clique must be from distinct clauses.



Proof of Theorem (cont'd)

Claim. *G has a k-clique if and only if ϕ is satisfiable.*

$[\Rightarrow]$ Let S be a k-clique of G. Then S **has a node from each triple** so that **the selected nodes do not interfere with each other** as assignments.

$[\Leftarrow]$ Let A be a satisfying assignment. Select from each triple **a literal that is satisfied by A** to construct a set S. $\| S \| = k$ and it is a clique.

The mapping is **polynomial time computable.** ∎

NP-Completeness

Definition. *A language A is NP-complete if A is in NP and every language in NP is polynomial time reducible to A.*

Theorem. *If some language A is NP-complete and $A \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$.*

We may use the following to prove something is NP-complete.

Theorem. *If a language A is NP-complete, $B \in \mathrm{NP}$, and $A \leq_P B$, then B is NP-complete.*

Proof. *$\leq_P$ is transitive.*

SAT is NP-Complete

Theorem. *SAT is NP-complete. (S. Cook 1971, L. Levin 1973)*

Proof $SAT \in \mathrm{NP}$. Consider a two-tape NTM that, on an input ϕ of n variables, **guesses and writes an assignment A** on Tape 2 using nondeterministic moves, accepts if $\phi(A) = 1$, and rejects $\phi(A) = 0$. Such a machine decides SAT and can be polynomial time.

For the other part, suppose A is in NP. We show $A \leq_P SAT$.

Since $A \in \mathrm{NP}$, there is some $k > 0$ and some n^k-time one-tape NTM N such that $L(N) = A$. We will show that a deterministic TM can construct a formula within a polynomial time bound for any given input to N, describing accepting computations of N, where that formula is satisfiable iff there is such an accepting computation.

Tableau

Let w be an input of length n. Define a **tableau** for N on w to be an $n^k \times n^k$ table whose entries are from $\{\#\} \cup Q \cup \Gamma$ such that

1. **the columns 1 and n^k are all $\#$,**
2. **each row is a configuration of N,**
3. **the first row is the initial configuration of N on w,** and
4. **for each i, $2 \leq i \leq n^k$, the ith row results from the $(i-1)$st row applying one move of N**

A tableau is **accepting** if

5. **the last row is an accepting configuration**

Encoding of Tableau

Let $C = Q \cup \Gamma \cup \{\#\}$ and for each i, j, let $cell[i, j]$ denote the jth element in row i.

For each i, j, $1 \leq i, j \leq n^k$, and $s \in C$, let $x_{i,j,s}$ represent condition $cell[i, j] = s$.

Encode Conditions 1 through 5 into Boolean formulas ϕ_1 through ϕ_5 and let x reduce to the conjunction of the five formulas.

There is a one-to-one correspondence between the set of satisfying assignments and the set of all accepting tableaus

Condition 1 “#’s”

$$\bigwedge_{1\le i\le n^k} \left(x_{i,1,\#} \wedge x_{i,n^k,\#}\right).$$

Condition 5 “Accepting”

$$\bigvee_{1\le j\le n^k} x_{n^k,j,q_{\mathrm{accept}}}$$

Condition 3 “Initial”

$$x_{1,2,q_0} \wedge \left(\bigwedge_{1\le j\le n} x_{1,2+j,w_j}\right) \wedge \left(\bigwedge_{n+1\le j\le n^k-3} x_{1,2+j,\sqcup}\right)$$

Condition 2 "configuration"

The conjunction of the following two for all i, $1 \le i \le n^k$:

A_i**:** $cell[i, j]$ **has a unique value for all** j**:**

$$\bigwedge_{1 \le i,j \le n^k} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{s,t \in C, s \ne t} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}}) \right) \right]$$

B_i**:** **There is precisely one** j **such that** $cell[i, j] \in Q$

$$\bigwedge_{1 \le i \le n^k} \left[\left(\bigvee_{\substack{1 \le j \le n^k, \\ s \in Q}} x_{i,j,s} \right) \wedge \left(\bigwedge_{\substack{1 \le j,\ell \le n^k, \\ j \ne \ell}} \left(\left(\bigwedge_{s \in Q} \overline{x_{i,j,s}} \right) \vee \left(\bigwedge_{s \in Q} \overline{x_{i,\ell,s}} \right) \right) \right) \right]$$

Condition 4 "step"

The conjunction of the following for all i, $2 \leq i \leq n^k$, j, $2 \leq j \leq n^k - 1$:

In the 2 block consisting of $cell[i-1, j-1], cell[i-1, j], cell[i-1, j+1], cell[i, j-1], cell[i, j], cell[i, j+1]$, if

(α_{ij}) $cell[i-1, j]$ *is a state*, then

(β_{ij}) **the values of the 6 cells correspond to one legal move of N**;

and if

(α'_{ij}) $cell[i-1, j-1], cell[i-1, j], cell[i-1, j+1]$ *are not states*, then

(β'_{ij}) $cell[i, j] = cell[i-1, j]$

Code this as $\overline{\alpha_{ij}} \vee \beta_{ij}$ and $\overline{\alpha'_{ij}} \vee \beta'_{ij}$.

The entire formula is of length bounded by some polynomial in n. ∎

3SAT is NP-Complete

The formula in the tableau method can be **converted to an equivalent 3CNF formula**

Conversion rules:

1. $(x \wedge y) \wedge z$ is equivalent to $x \wedge y \wedge z$
2. $(x \vee y) \vee z$ is equivalent to $x \vee y \vee z$
3. $(x \vee y \vee z \vee u)$ is equivalent to $(x \vee y \vee w) \wedge (w \equiv (z \vee u)$. The second term is equivalent to $(\overline{w} \vee z \vee u) \wedge (\overline{z} \vee w) \wedge (\overline{u} \vee w)$

Repeat literals in clauses with < 3 literals to make the number of literals equal to three.