

## More NP-Completeness

## Clique

We know:  $3SAT \leq_P CLIQUE$ ,  $CLIQUE \in NP$ , and  $3SAT$  is NP-complete. So,  $CLIQUE$  is NP-complete.

## Vertex Cover

A vertex cover of an undirected graph is a subset of nodes such that every edge touches a member of the subset.

$VERTEX-COVER = \{ \langle G, k \rangle \mid G \text{ has a vertex cover of size } k \}$ .

**Theorem.**  *$VERTEX-COVER$  is NP-complete.*

**Proof** Proving  $VERTEX-COVER \in NP$  is easy. Guess a bit for each node to decide whether or not select the node. Check whether precisely  $k$  nodes are selected, if so, check whether the set of  $k$  nodes is a vertex cover.

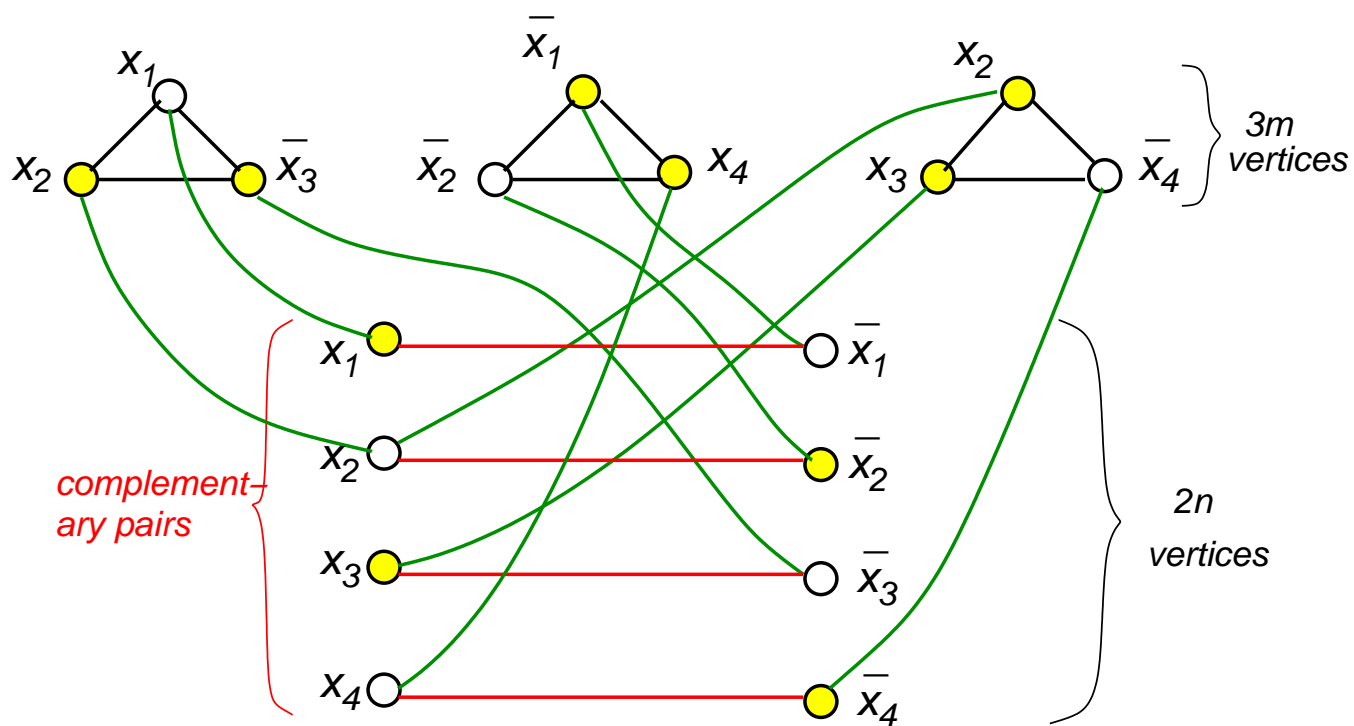
## Proof (cont'd)

Reduce *3SAT* to *VERTEX-COVER*.

Let  $\phi$  be an instance of *3SAT* with  $n$  variables and  $m$  clauses. Define the graph  $G$  as follows:

- **the nodes:** the literals  $v_i, \overline{v_i}, 1 \leq i \leq n$ , and  
their occurrences  $a_{i1}, a_{i2}, a_{i3} : 1 \leq i \leq m$ ,  
... a total of  $3m + 2n$  nodes
- **the edges:**  $(v_i, \overline{v_i}), 1 \leq i \leq n$ ;  
 $(a_{i1}, a_{i2}), (a_{i2}, a_{i3}), (a_{i3}, a_{i1}), 1 \leq i \leq m$ ;  
for each  $i, 1 \leq i \leq n$ , and  $j, 1 \leq j \leq 3$ , connect  $a_{ij}$  and its corresponding literal.

$$\phi = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_4) \wedge (x_2 \vee x_3 \vee \bar{x}_4)$$



Any vertex cover must include

- 2 nodes of each triangle (clause)
  - 1 member of each complementary pair of literals
- }  $2m+n$

### Proof (cont'd)

We claim that  $G$  has an  $(n + 2m)$  node vertex cover if and only if  $\phi$  is in  $3SAT$ .


There are precisely  $n$  edges of the type  $(v_i, \overline{v_i})$ ,  $1 \leq i \leq n$ . So an  $n$ -node vertex cover has to have at least one out of  $v_i$  and  $\overline{v_i}$  for every  $i$ .

There are precisely  $m$  triangles for the clauses. At least two nodes have to be selected from each triangle to cover the triangle edges. So  $2m$  nodes are needed.

## Proof (cont'd)

A selection of  $n + 2m$  nodes becomes a cover if and only if:

- (\*) **for each clause, the literal side of one “literal-occurrence” edge is selected, say  $l_k$  of  $(a_{ij}, l_k)$ , where  $l_k$  is a literal, and two nodes  $a_{ij'}$  and  $a_{ij''}$  are selected, where  $\{j, j', j''\} = \{1, 2, 3\}$ , and**
- (\*\*) **for each variable precisely one of  $v_i$  and  $\overline{v_i}$  is selected.**

So an  $n + 2m$  node vertex cover corresponds to an assignment that satisfies each clause. 

## Subset-Sum is NP-complete

*SUBSET-SUM* is the problem of, given a multiset of numbers  $z_1, \dots, z_m$  and a number  $S$ , whether there is subset  $y_1, \dots, y_t$  of  $z_i$ 's such that  $y_1 + \dots + y_t = S$ .

**Theorem.** *SUBSET-SUM is NP-complete.*

**Proof** Reduce *3SAT* to *SUBSET-SUM*. The construction is reminiscent of the reduction from *3SAT* to *VERTEX-COVER*, where the reduction generates a graph whose  $n + 2m$  node cover has a property that at least one “literal-occurrence” edge of each triangle is touched and the rest of the nodes in each triangle is touched.

## Proof (cont'd)

Let  $\phi$  be a formula of  $n$  variables and  $m$  clauses. Introduce decimal numbers  $y_1, \dots, y_n, z_1, \dots, z_n, c_1, \dots, c_m, d_1, \dots, d_m$ , each of at most  $n + m$  digits.

$y_i$ :  $y_i$  has a 1 at the  $(m+i)$ th digit and has a 1 at position  $j$  if  $x_i$  appears in the  $j$ th clause; all the other positions have a 0

$z_i$ :  $z_i$  has a 1 at the  $(m+i)$ th digit and has a 1 at position  $j$  if  $\overline{x_i}$  appears in the  $j$ th clause; all the other positions have a 0

$c_i, d_i$ :  $c_i$  has a 1 only at the  $i$ th position,  $d_i$  has a 1 only at the  $i$ th position,

$S$ :  $S$  is the number that has a 3 at every position between 1 and  $m$  and has a 1 at every position between  $m + 1$  and  $m + n$





### Proof (cont'd)

In order to generate  $S$ , exactly one of  $y_i$  and  $z_i$  has to be selected for every  $i$  so that the selection as a whole touches each bit position between 1 and  $m$  at least once (and at most three times). Such a selection is a satisfying assignment of  $\phi$ . 