

# Removing Independently Even Crossings

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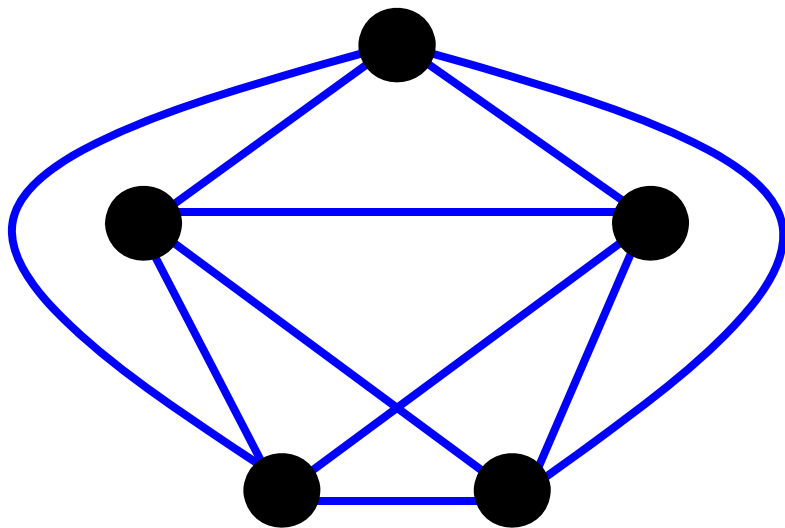
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# Crossing number

$\text{cr}(G)$  = minimum number of crossings  
in a drawing\* of  $G$



$$\text{cr}(K_5) = 1$$

\*(general position drawings, i.e., no intersections with 3 edges, edges don't cross vertices, edges do not touch)

# Crossing number

poorly understood, for example:

- don't know  $cr(K_n)$ ,  $cr(K_{m,n})$

Guy's conjecture:

$$cr(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

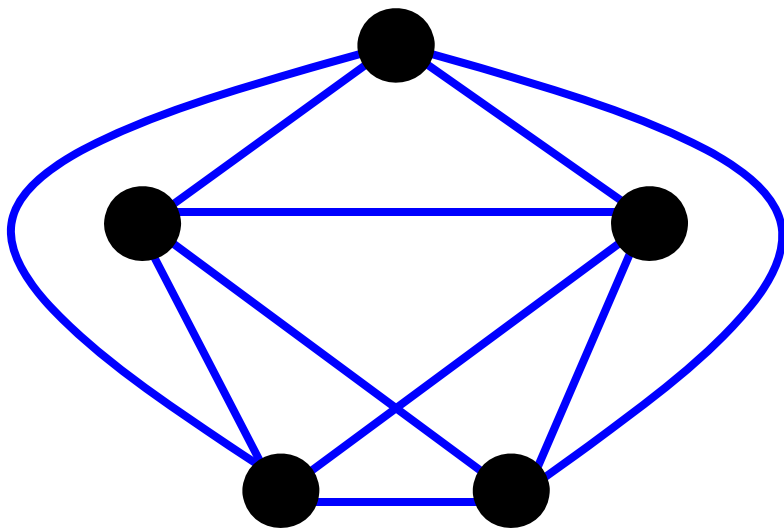
Zarankiewicz's conjecture:

$$cr(K_{m,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$$

- no approximation algorithm

# Pair crossing number

$\text{pcr}(G)$  = minimum number of pairs of edges that cross in a drawing\* of  $G$

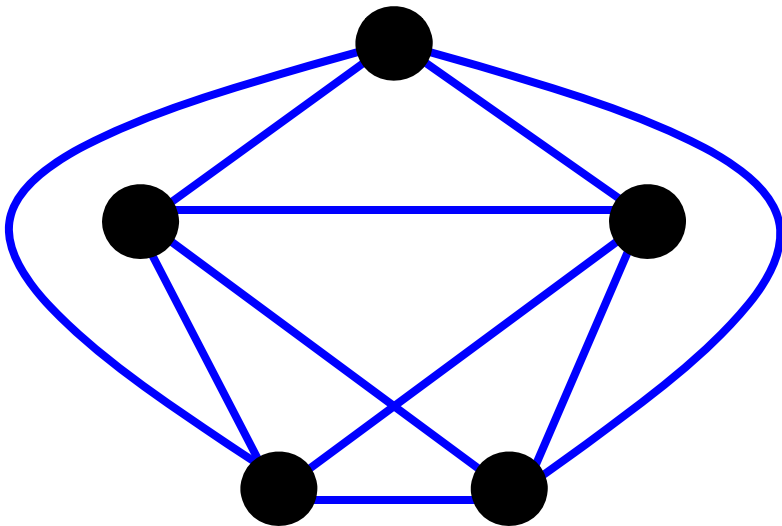


$$\text{pcr}(K_5) = 1$$

\*(general position drawings, i.e., no intersections with 3 edges, edges don't cross vertices, edges do not touch)

# Odd crossing number

$\text{ocr}(G)$  = minimum number of pairs of edges that cross **oddly** in a drawing\* of  $G$



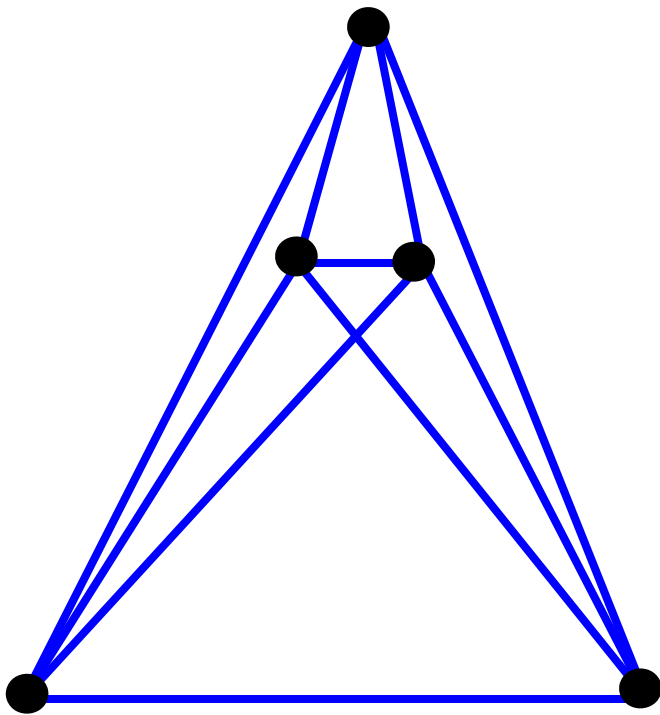
$$\text{ocr}(K_5) = 1$$

**oddly** = odd number of times

\*(general position drawings, i.e., no intersections with 3 edges, edges don't cross vertices, edges do not touch)

# Rectilinear crossing number

$\text{rcr}(G)$  = minimum number of crossings  
in a **planar straight-line**  
drawing of  $G$



$$\text{rcr}(K_5) = 1$$

# “Independent” crossing numbers

only **non-adjacent** edges contribute

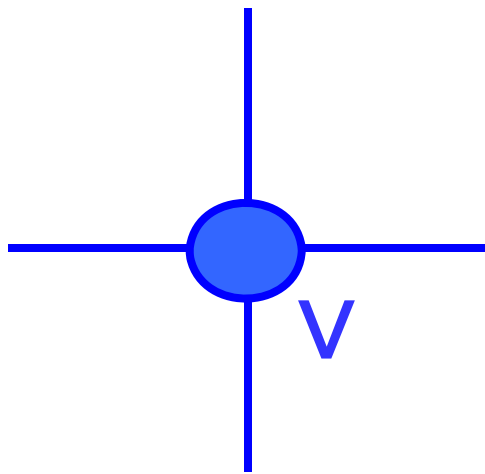
$iocr(G)$  = minimum number of pairs of **non-adjacent** edges that cross oddly in a drawing of  $G$

$ocr(G)$  = minimum number of pairs of edges that cross oddly in a drawing of  $G$

# “Independent” crossing numbers

only **non-adjacent** edges contribute

$iocr(G)$  = minimum number of pairs of **non-adjacent** edges that cross oddly in a drawing of  $G$



What should be the ordering of edges around  $v$ ?

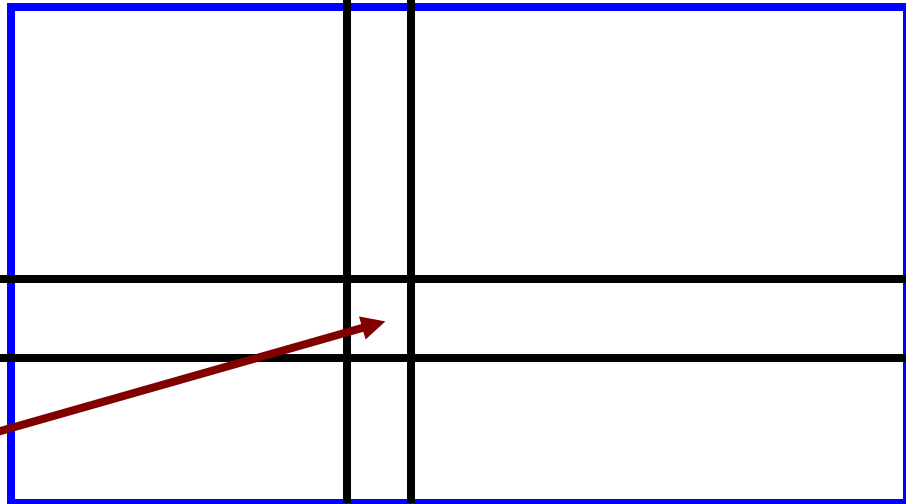
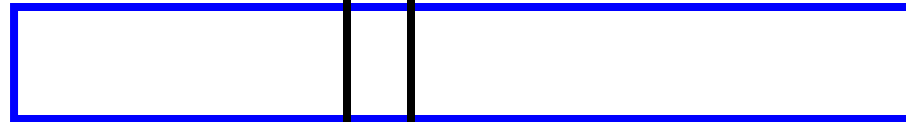
“independent”  $\Rightarrow$  does not matter!



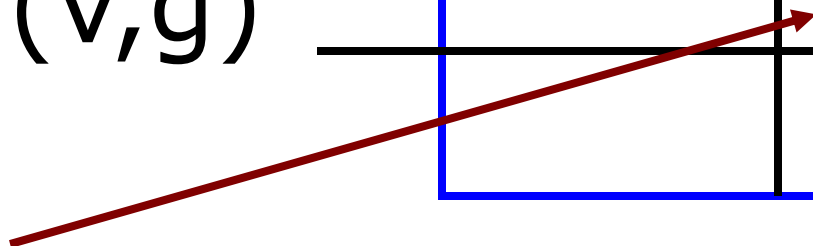
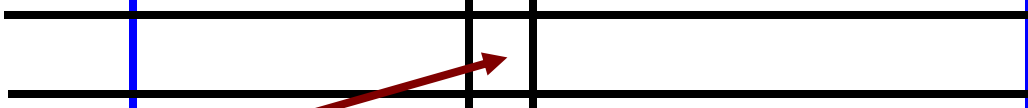
**iocr(G) = CVP**

$\{e_0, e_1\}$

any initial  
drawing



$(v, g)$



**1** if  $g=e_i$  and  $v$  is an endpoint of  $e_{1-i}$   
**0** otherwise

columns = pair of non-adjacent edges, e.g., for  $K_5$ , 15 columns  
rows = non-adjacent (vertex, edge), e.g., for  $K_5$ , 30 rows



# Crossing numbers

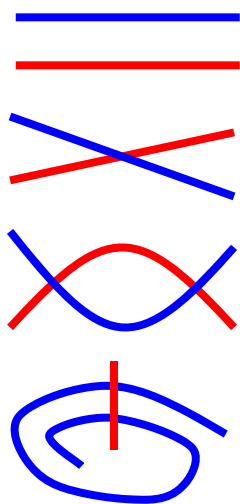
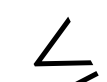
$iocr(G)$

$acr(G)$

$ocr(G)$

$pcr(G)$

$cr(G) \leq rcr(G)$



ocr

acr

pcr

cr

0

0

0

0

1

1

1

1

0

0

1

2

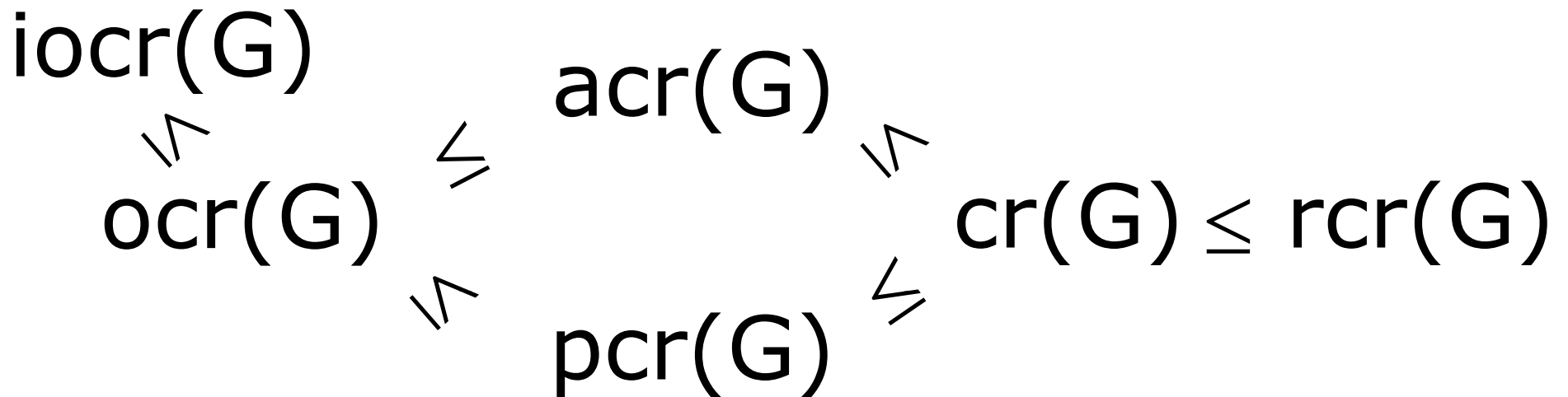
0

2

1

2

# Crossing numbers – amazing fact



$$iocr(G)=0 \Rightarrow rcr(G)=0$$

$$iocr(G)=0 \Rightarrow cr(G)=0 \quad (\text{Hanani}'34, \text{Tutte}'70)$$

$$cr(G)=0 \Rightarrow rcr(G)=0 \quad (\text{Steinitz, Rademacher}'34; \text{Wagner}'36; \text{Fary}'48; \text{Stein}'51)$$

# Crossing numbers – amazing fact

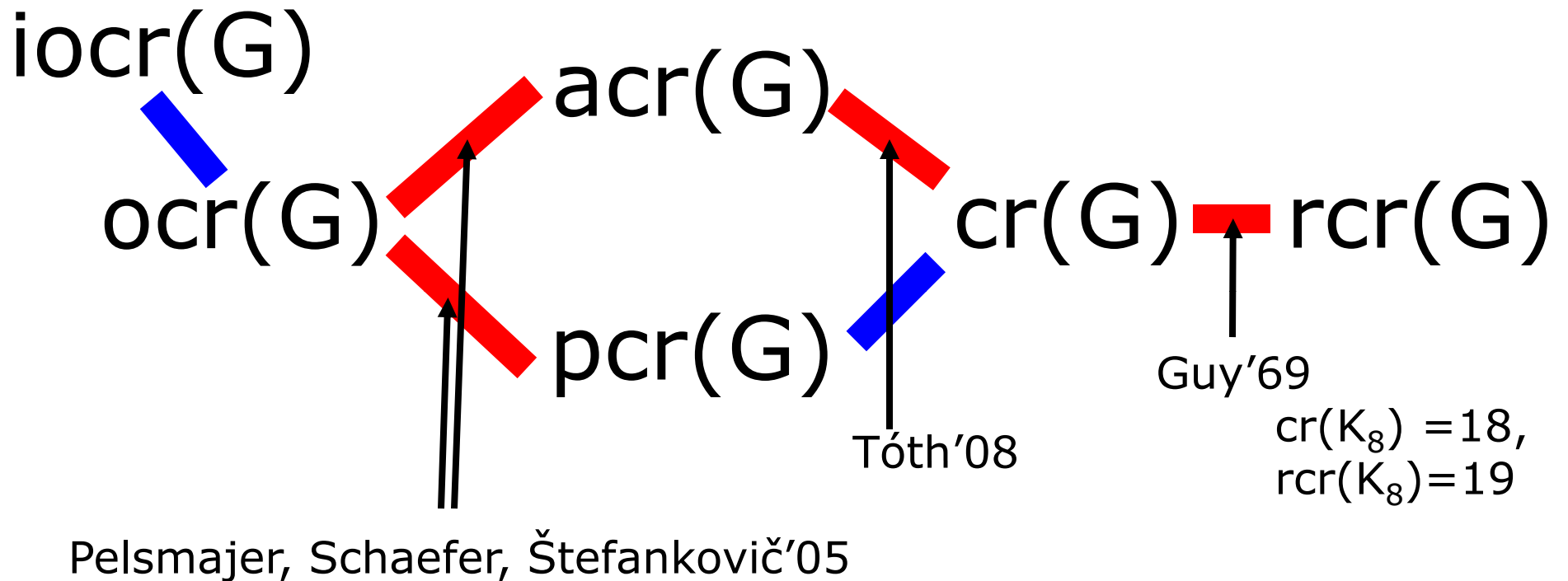
$$\begin{array}{ccccc} \text{iocr}(G) & & \text{acr}(G) & & \\ \swarrow & & & & \swarrow \\ \text{ocr}(G) & \leq & & & \text{cr}(G) \leq \text{rcr}(G) \\ \swarrow & & \text{pcr}(G) & & \swarrow \end{array}$$

$$\text{iocr}(G) \leq 2 \Rightarrow \text{rcr}(G) = \text{iocr}(G)$$

$$\text{iocr}(G) \leq 2 \Rightarrow \text{cr}(G) = \text{iocr}(G) \quad (\text{present paper})$$

$$\text{cr}(G) \leq 3 \Rightarrow \text{rcr}(G) = \text{cr}(G) \quad (\text{Bienstock, Dean'93})$$

# Crossing numbers - separation

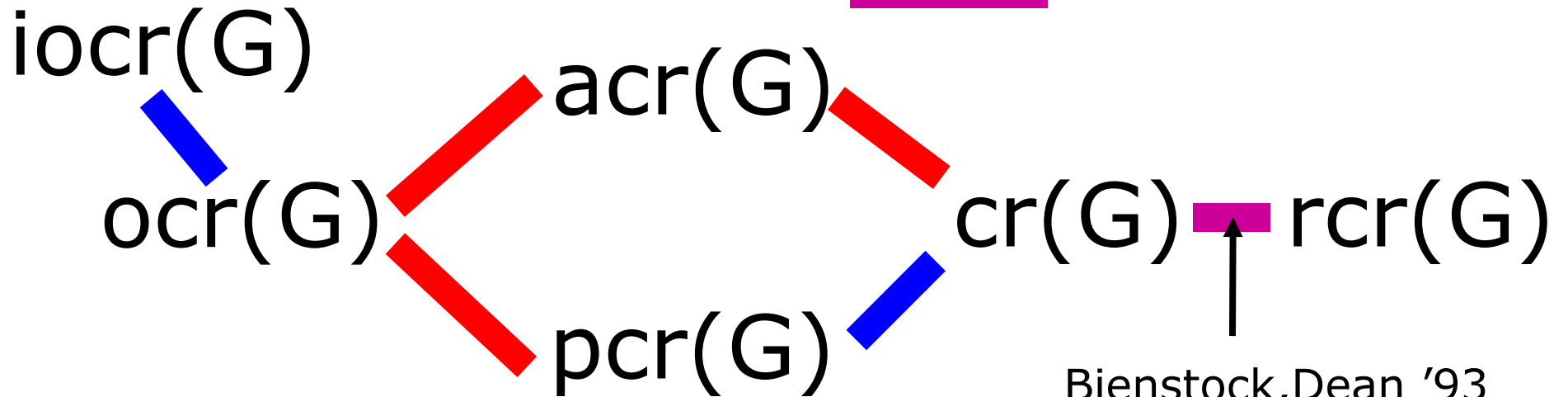


**red arrow** different

**blue arrow** maybe equal?

# Crossing numbers $\wedge$ separation

**BIG**



Bienstock, Dean '93  
 $(\forall k \geq 4)(\exists G)$   
 $cr(G)=4, rcr(G)=k$

 very different

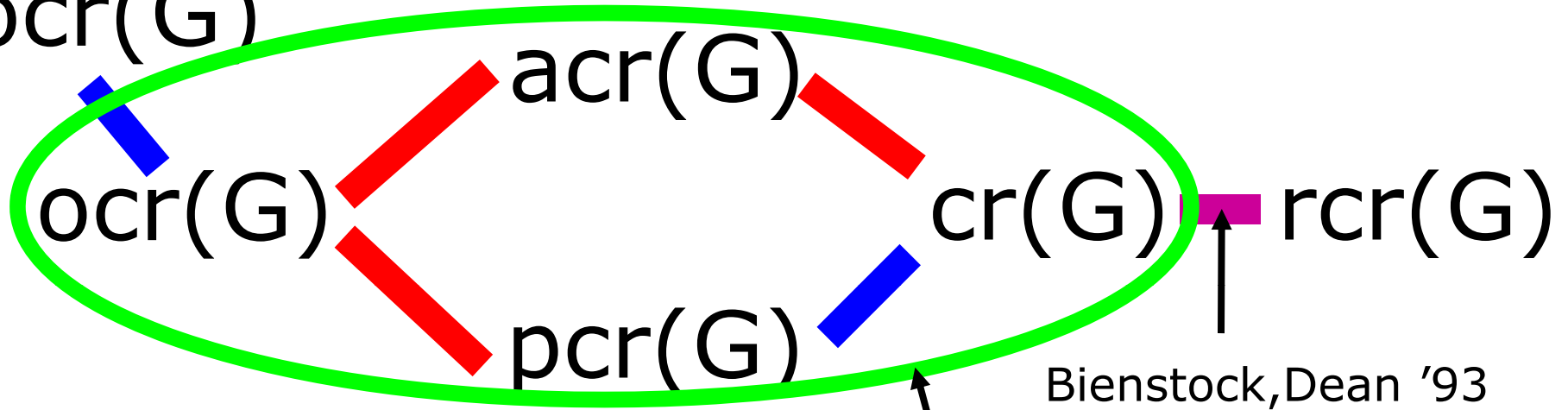
 different

 maybe equal?

# Crossing numbers $\wedge$ separation

**BIG**

$iocr(G)$



**polynomially related**

**very different**

**different**

**maybe equal?**

Bienstock, Dean '93  
 $(\forall k \geq 4)(\exists G)$   
 $cr(G)=4, rcr(G)=k$

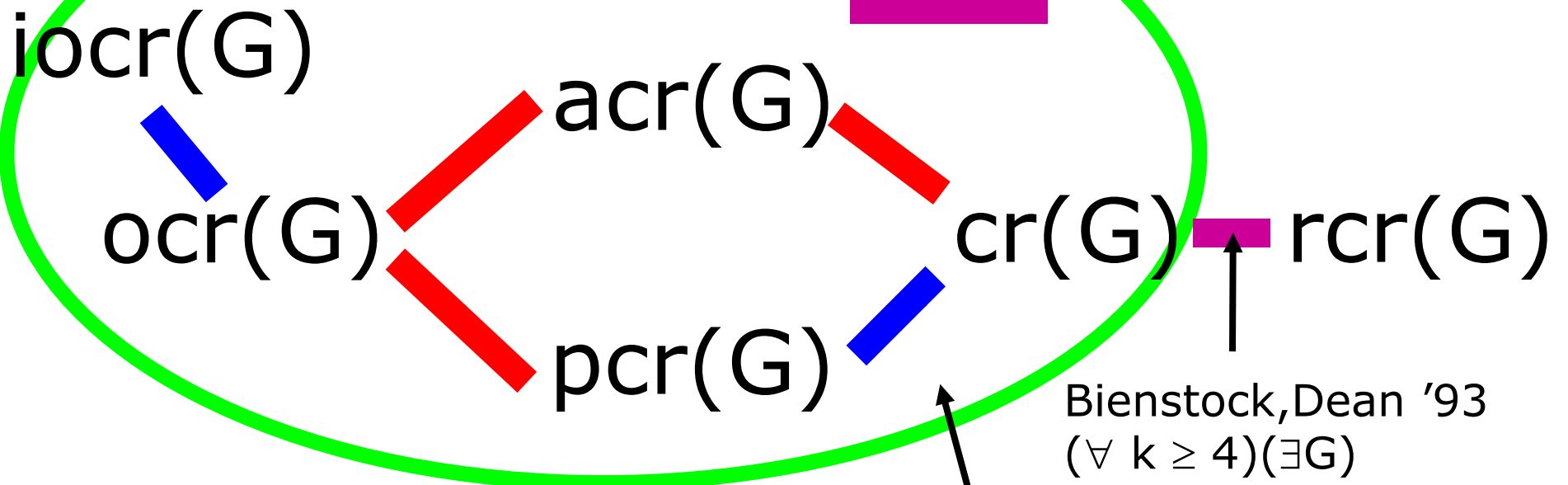
Pach, Tóth'00

$$cr(G) \leq \binom{2ocr(G)}{2}$$



# Crossing numbers $\Delta$ separation

**BIG**



Bienstock, Dean '93  
 $(\forall k \geq 4)(\exists G)$   
 $cr(G)=4, rcr(G)=k$

Pach, Tóth'00

$$cr(G) \leq \binom{2ocr(G)}{2}$$

our result

$$cr(G) \leq \binom{2iocr(G)}{2}$$

polynomially related

maybe equal?

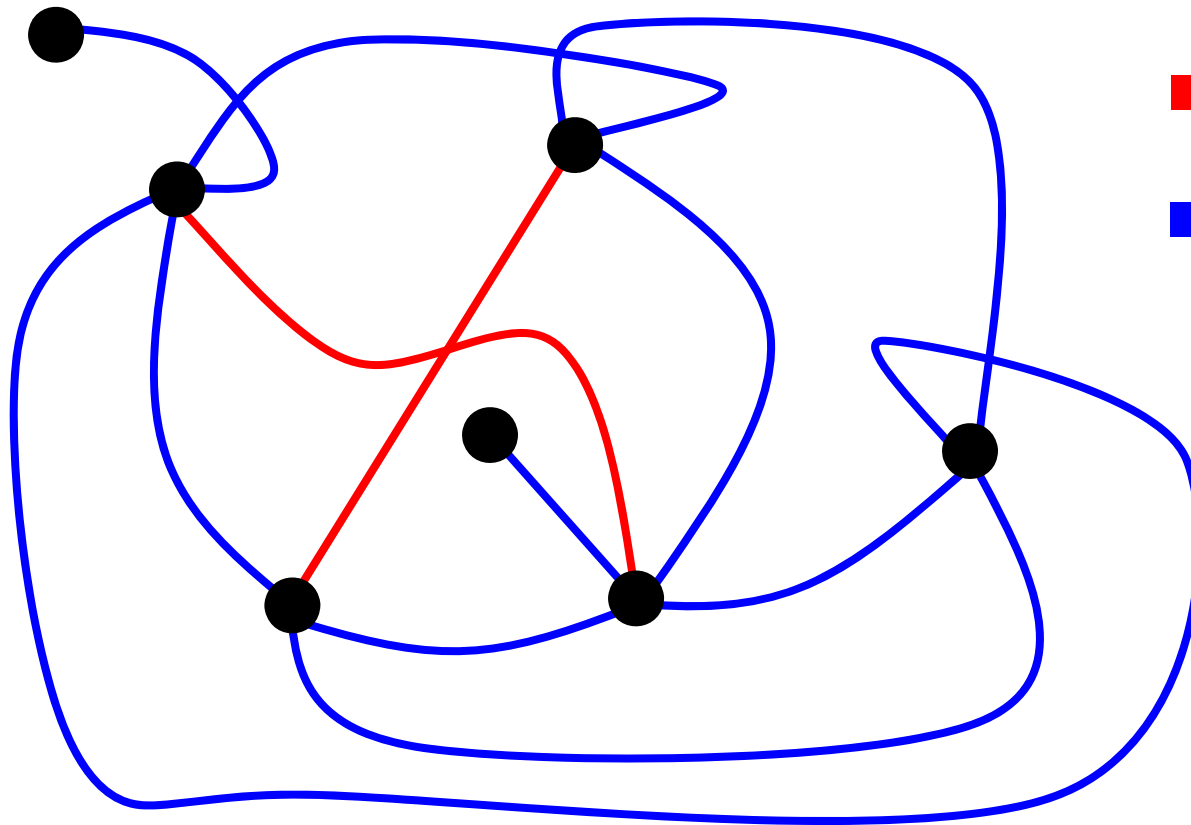
our result

$$\text{cr}(G) \leq \binom{2\text{iocr}(G)}{2}$$

$e$  is **bad** if  $\exists f$   
such that

- $e, f$  independent
- $e, f$  cross oddly

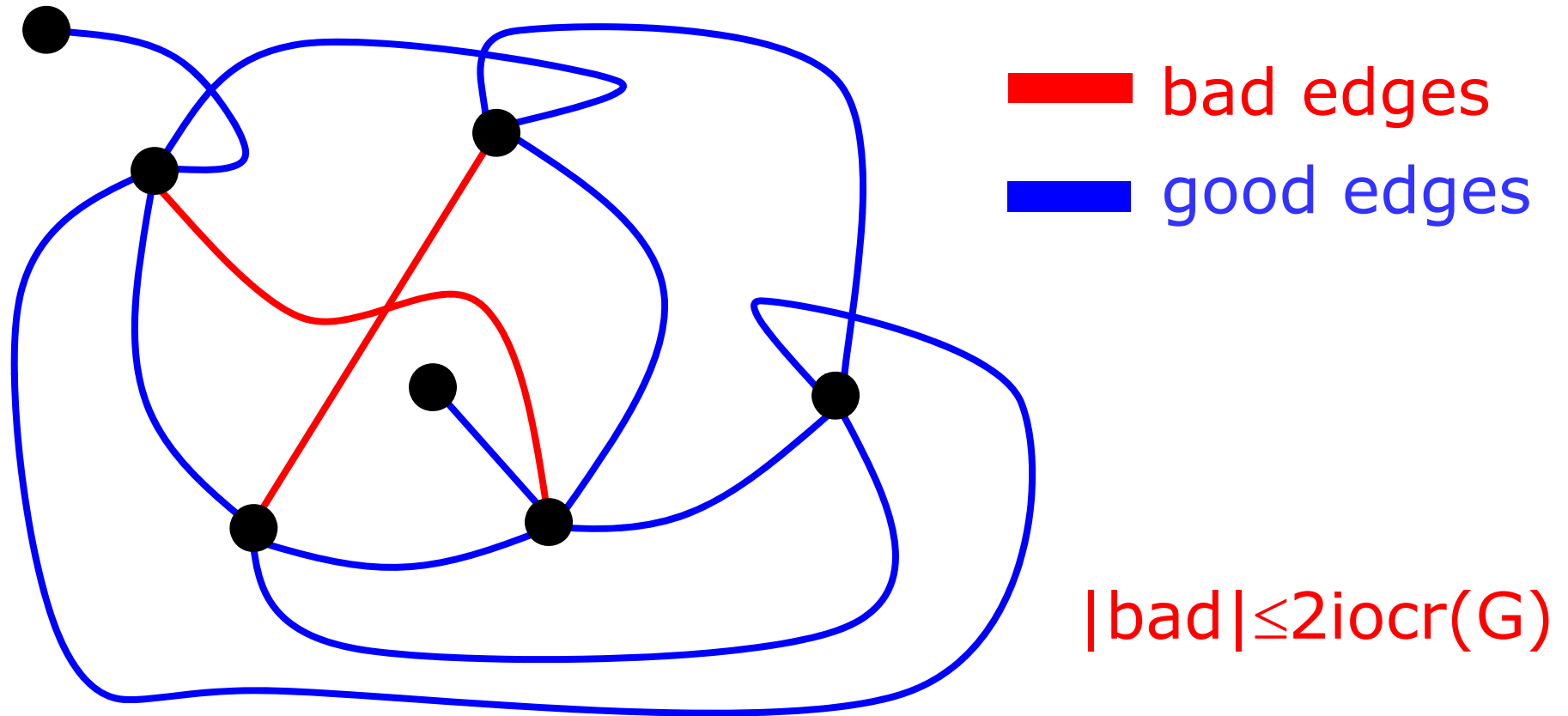
drawing  $D$  realizing  $\text{iocr}(G)$



**red** bad edges  
**blue** good edges

$$|\text{bad}| \leq 2\text{iocr}(G)$$

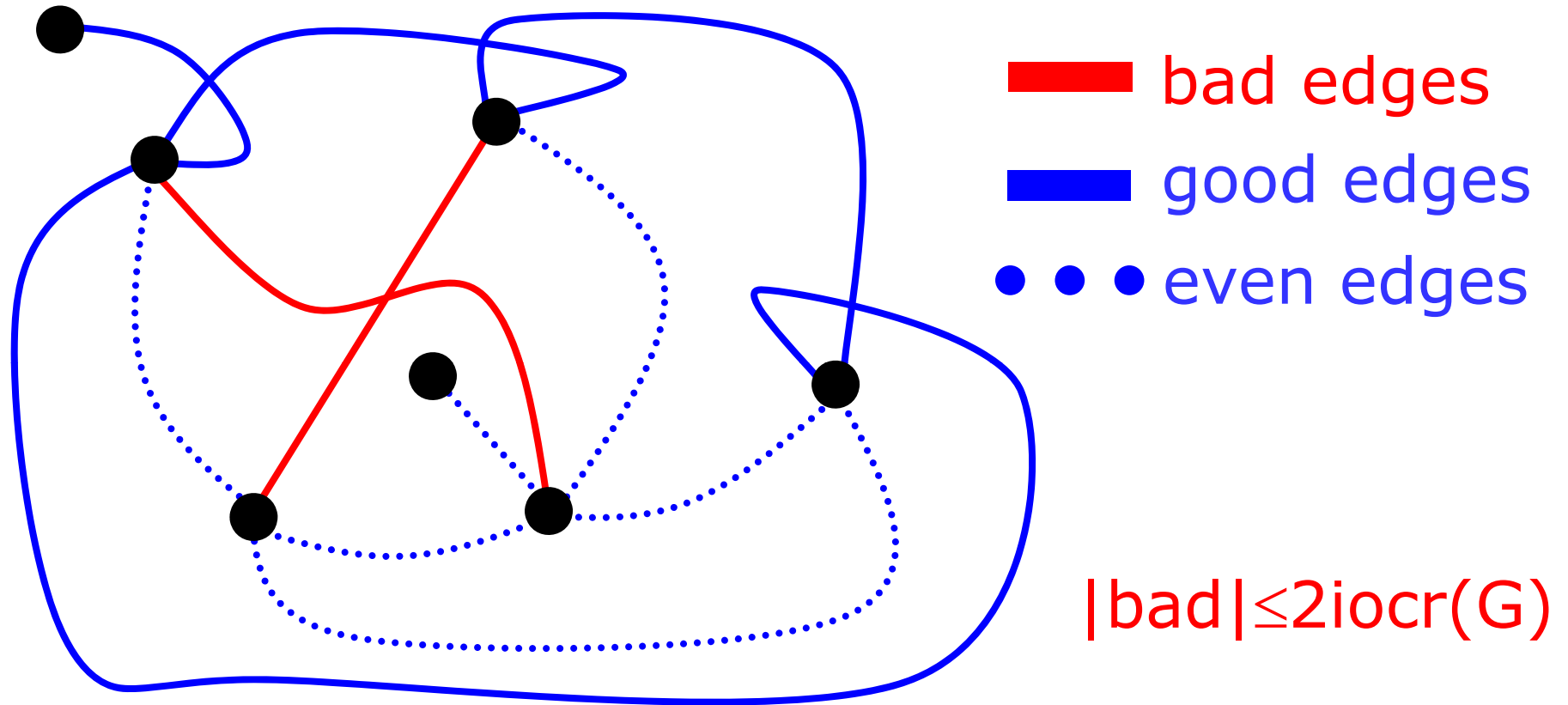
# drawing D realizing $iocr(G)$



GOAL: drawing  $D'$  such that

- good edges are intersection free
- pair of bad edges intersects  $\leq 1$  times

# drawing D realizing $iocr(G)$



GOAL: drawing  $D'$  such that

- good edges are intersection free
- pair of bad edges intersects  $\leq 1$  times

# drawing D realizing $iocr(G)$

— bad edges

Lemma (Pelsmajer, Schaefer, Stefankovic'07)

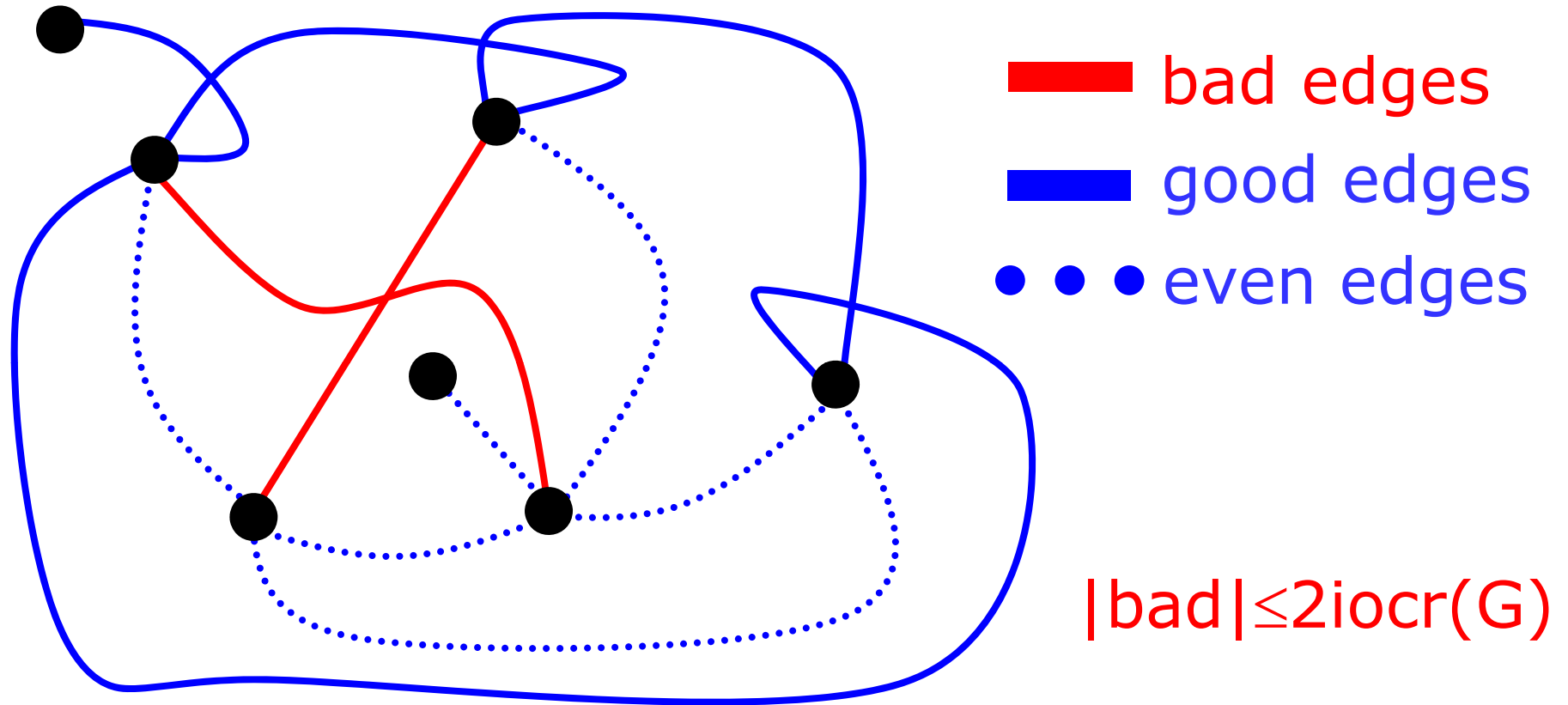
cycle  $C$  consisting of even edges



redrawing so that  $C$  is intersection free,  
no new odd pairs, same rotation system

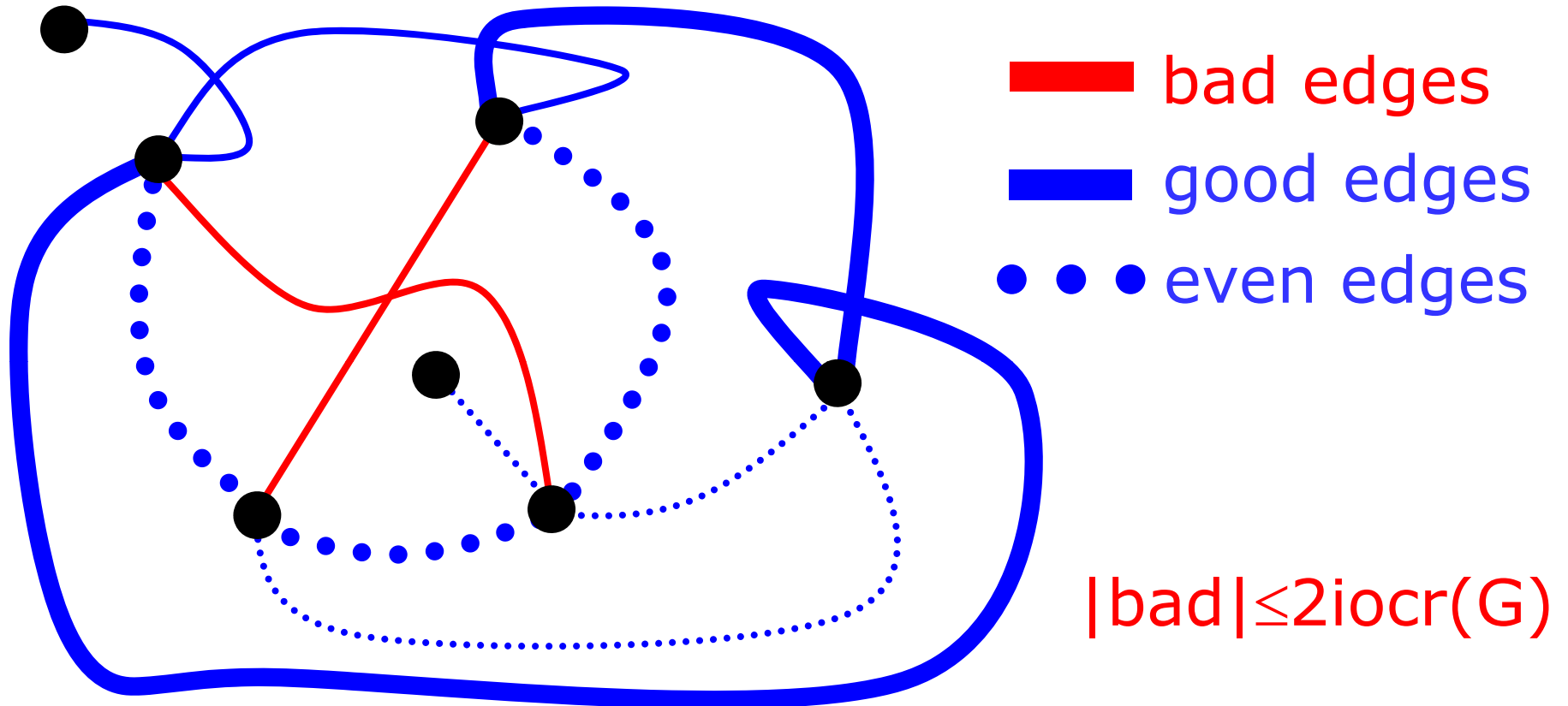
- pair of bad edges intersects  $\leq 1$  times

good  $\rightarrow$  even, locally



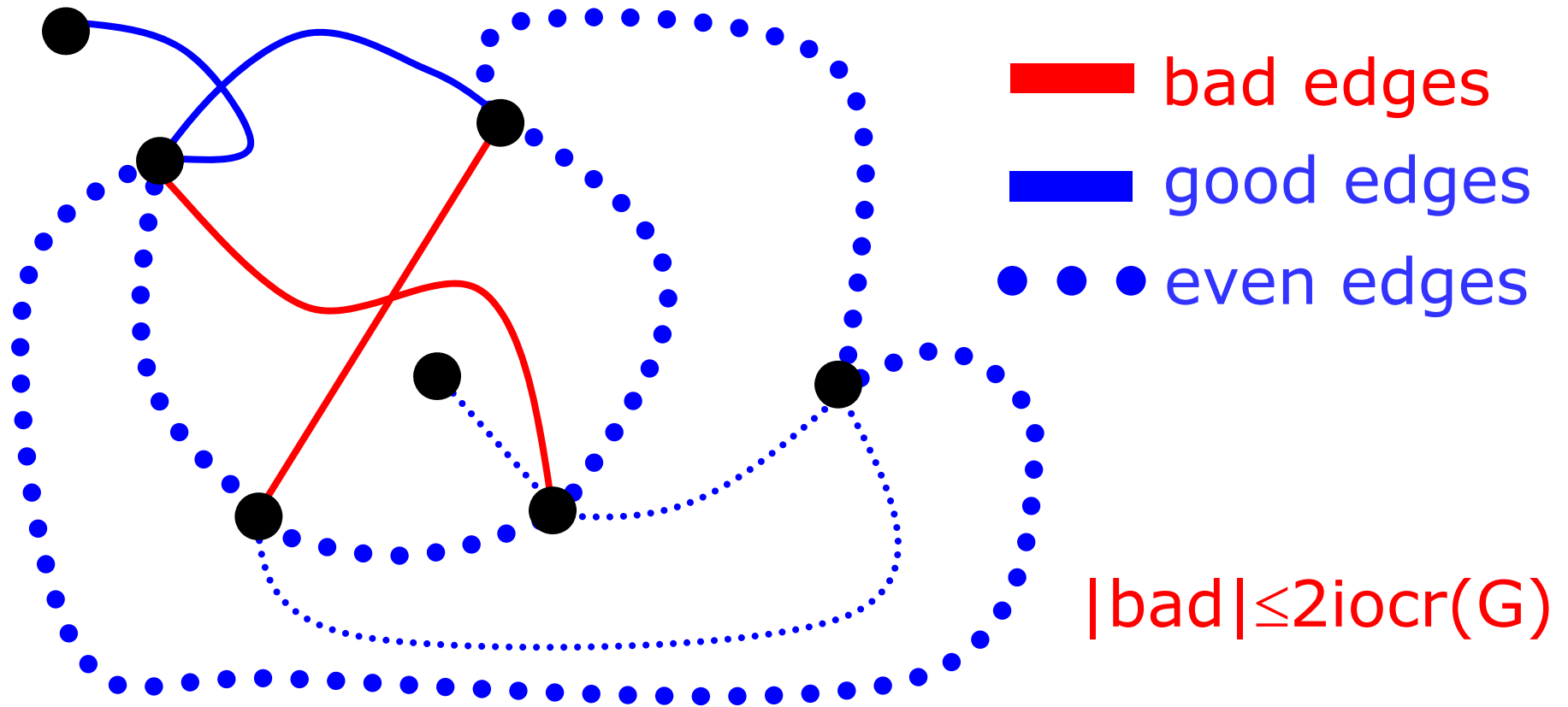
cycle of good edges  $\rightarrow$  cycle of even edges  
 $\rightarrow$  intersection free cycle

good  $\rightarrow$  even, locally



cycle of good edges  $\rightarrow$  cycle of even edges  
 $\rightarrow$  intersection free cycle

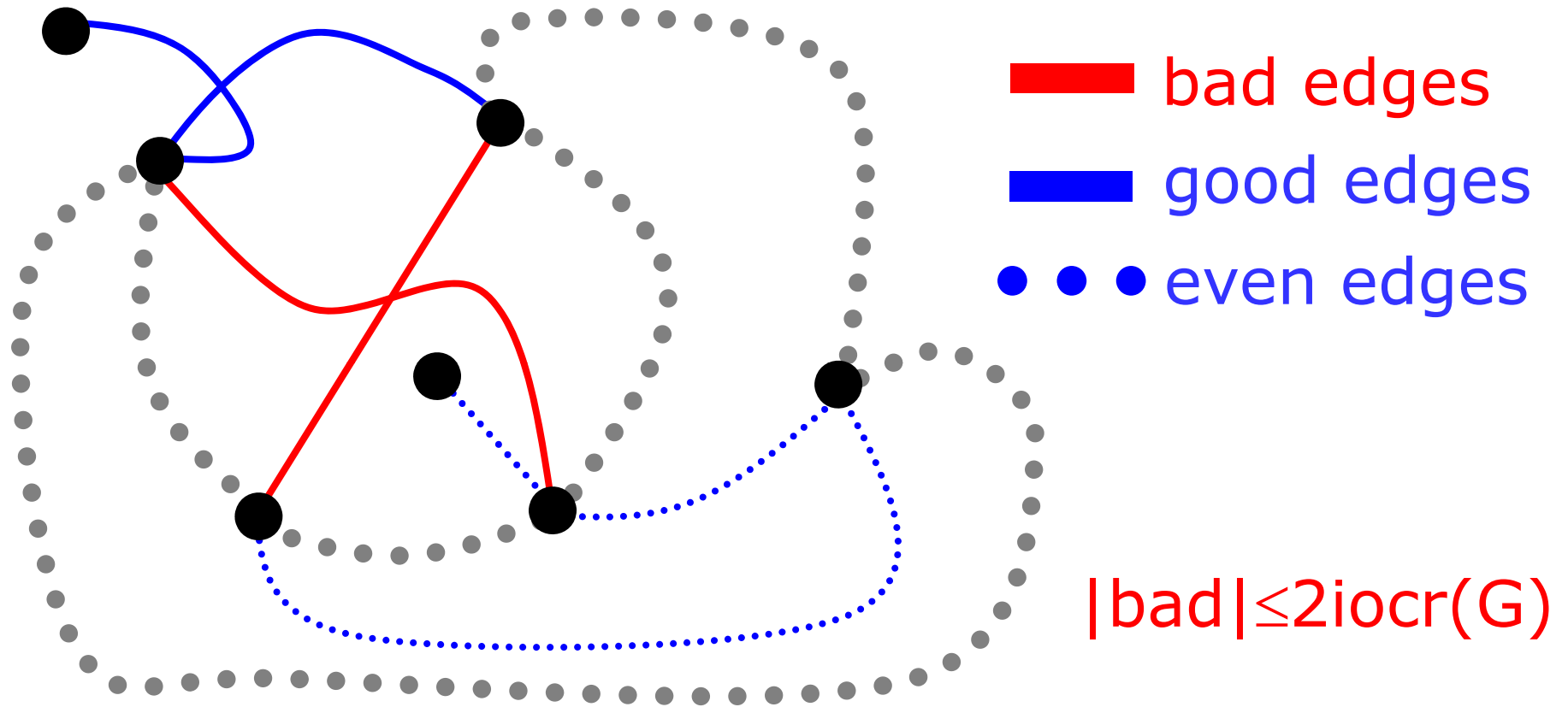
good  $\rightarrow$  even, locally



cycle of good edges  $\rightarrow$  cycle of even edges  
 $\rightarrow$  intersection free cycle

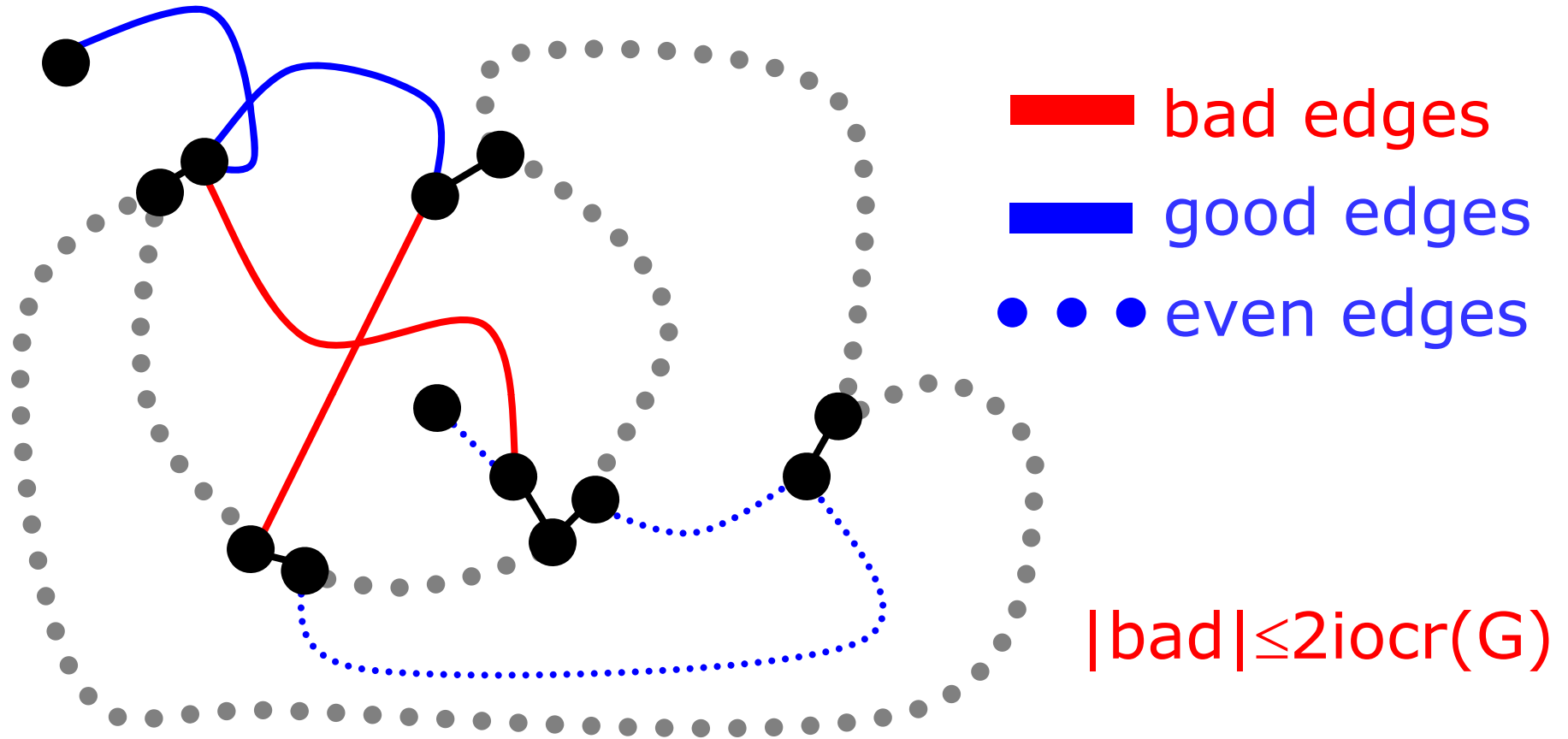


good  $\rightarrow$  even, locally



cycle of good edges  $\rightarrow$  cycle of even edges  
 $\rightarrow$  intersection free cycle  $\rightarrow$   
degree  $\leq 3$  vertices

good  $\rightarrow$  even, locally



cycle of good edges  $\rightarrow$  cycle of even edges  
 $\rightarrow$  intersection free cycle  $\rightarrow$   
degree  $\leq 3$  vertices

good  $\rightarrow$  even, locally

cycle of good edges  $\rightarrow$  cycle of even edges  
 $\rightarrow$  intersection free cycle  $\rightarrow$   
degree  $\leq 3$  vertices

repeat, repeat, repeat

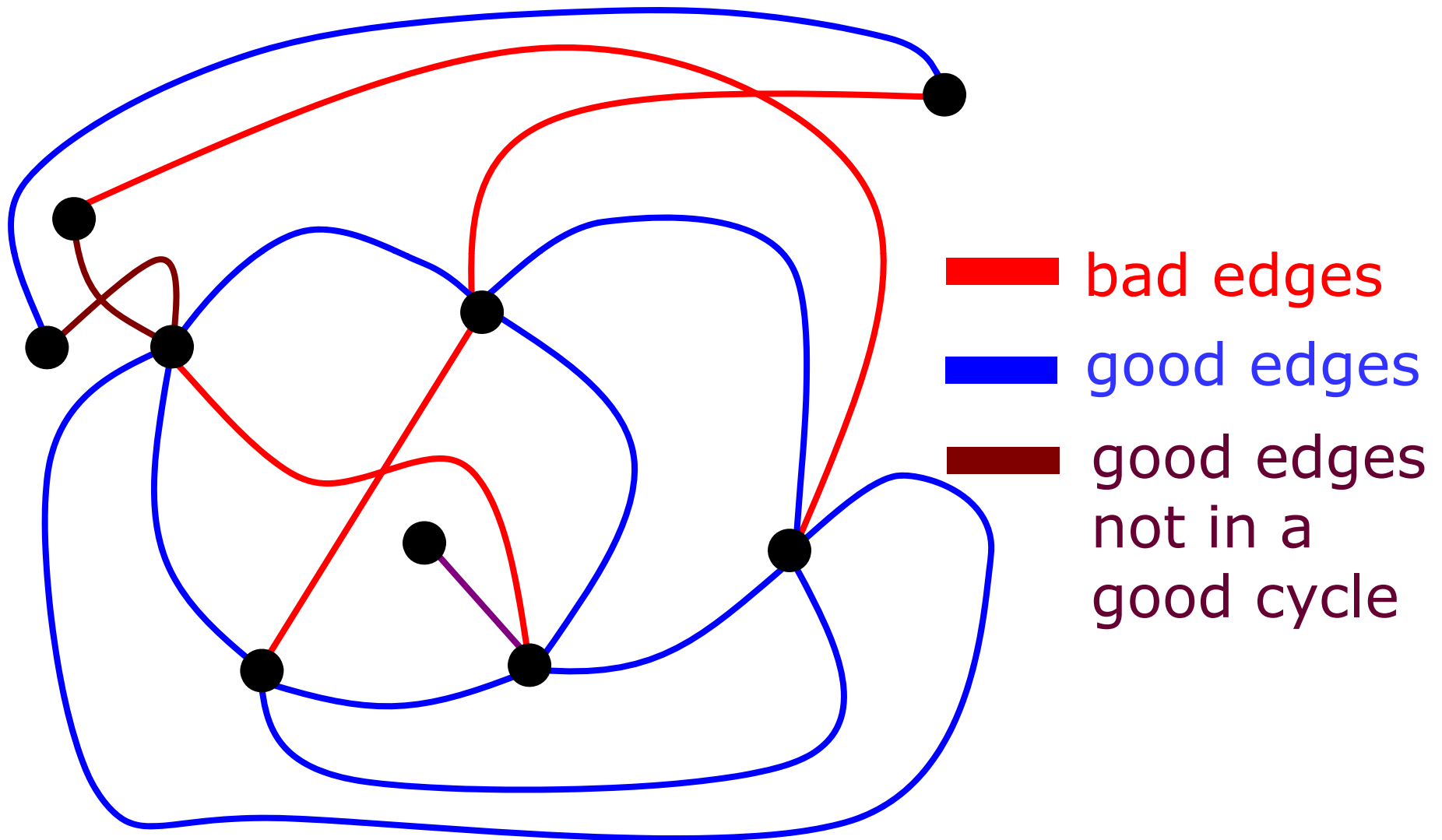
potentials decreasing:

$$\phi = \sum d_v^3$$

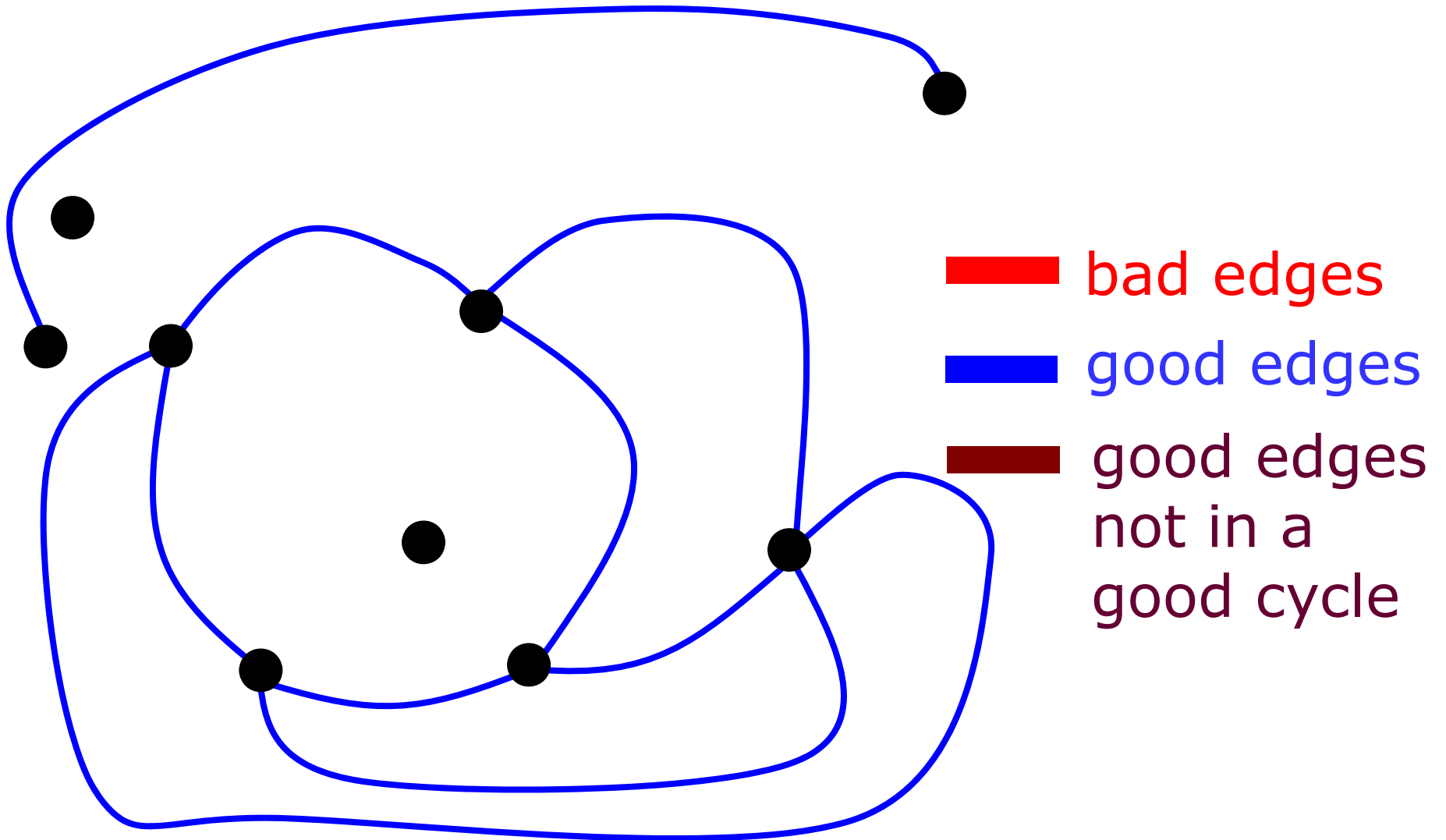
#good cycles with intersections

DONE  $\Rightarrow$  good edges in cycles  
are intersection free

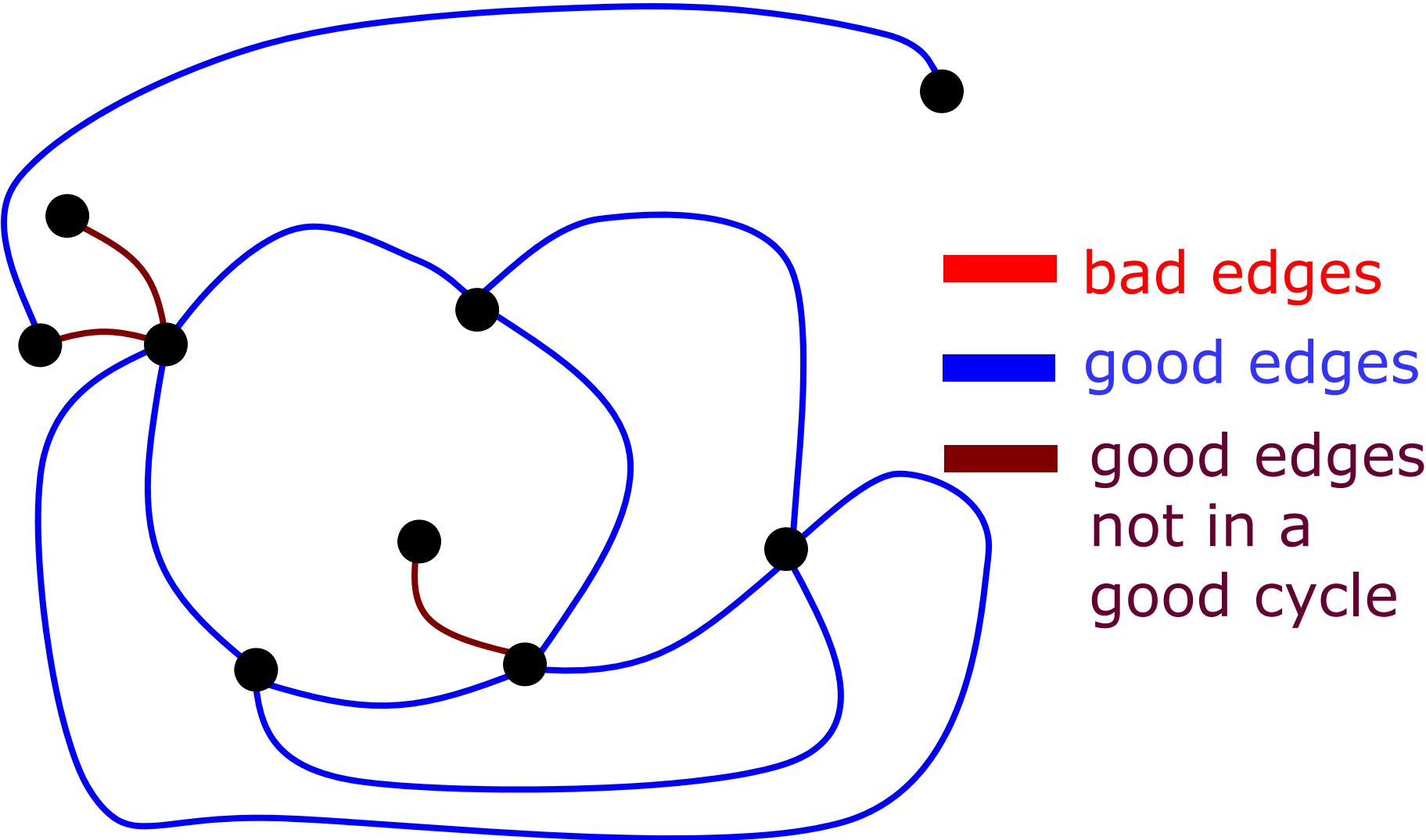
DONE  $\Rightarrow$  good edges in cycles  
are intersection free



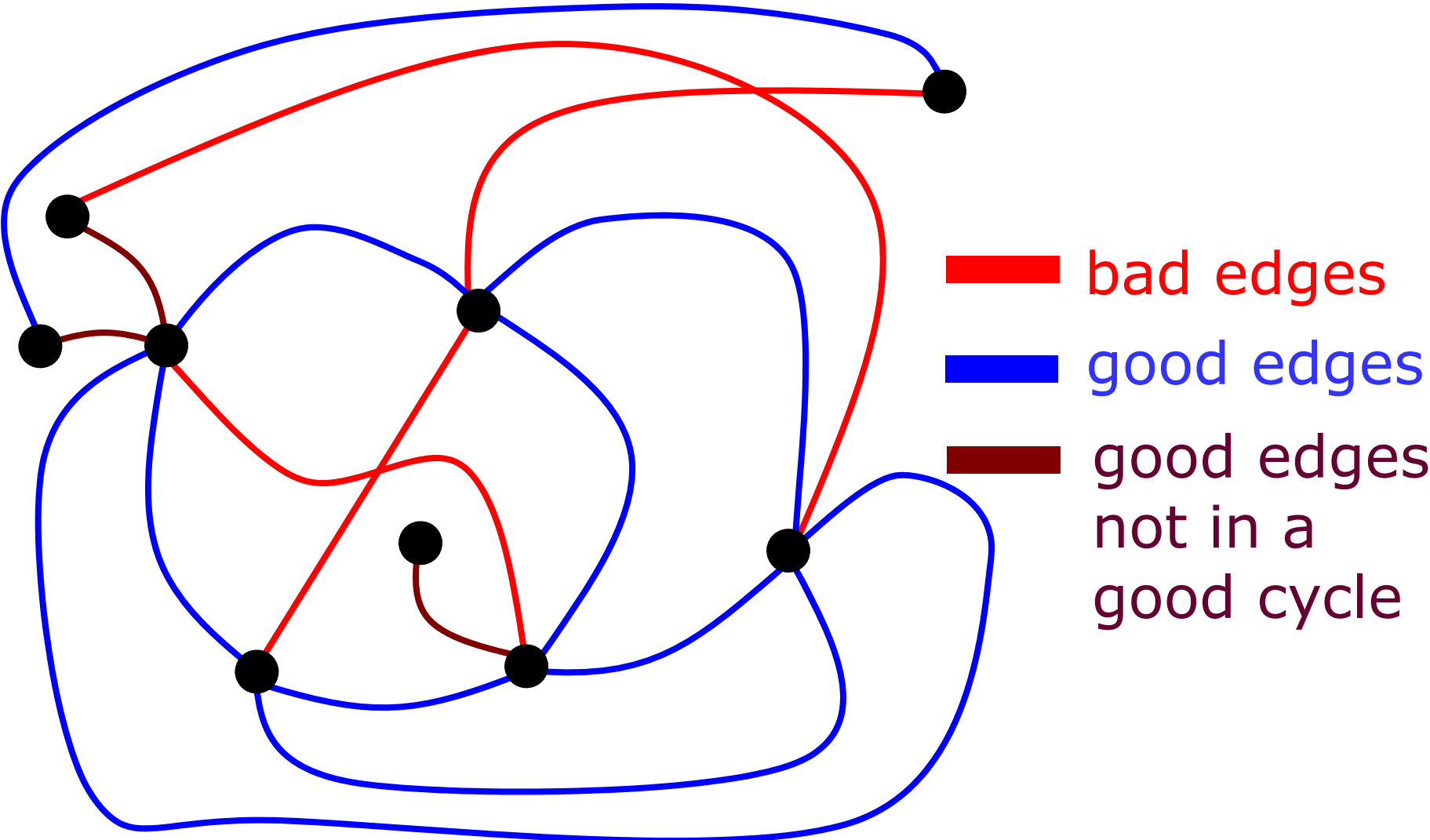
look at the blue faces



add violet good edges, no new faces

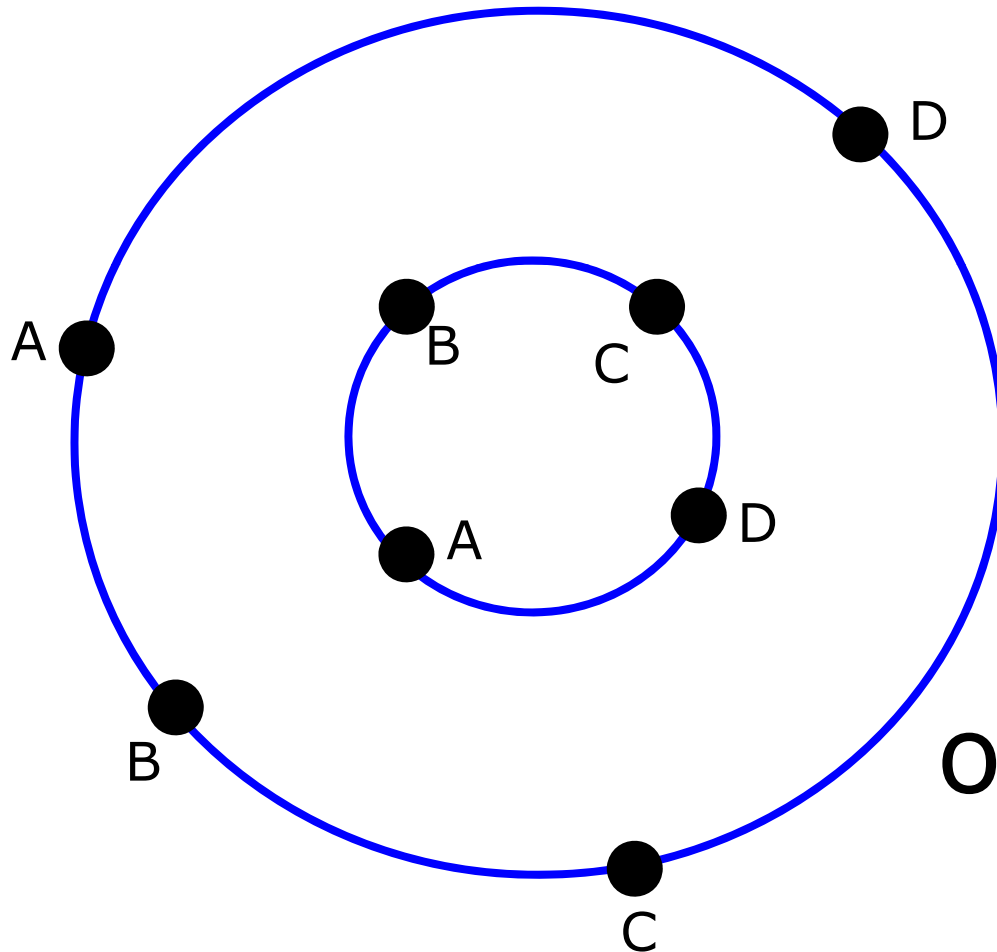


add bad edges in their faces ...



# Open problems

Is  $\text{pcr}(G) = \text{cr}(G)$  ?



on annulus?



# Open problems

Is  $\text{iocr}(G) = \text{ocr}(G)$  ?

(genus  $g$  strong Hannani-Tutte)

Does  $\text{iocr}_g(G) = 0 \Rightarrow \text{cr}_g(G) = 0$  ?

Is  $\text{cr}(G) = O(\text{iocr}(G))$  ?