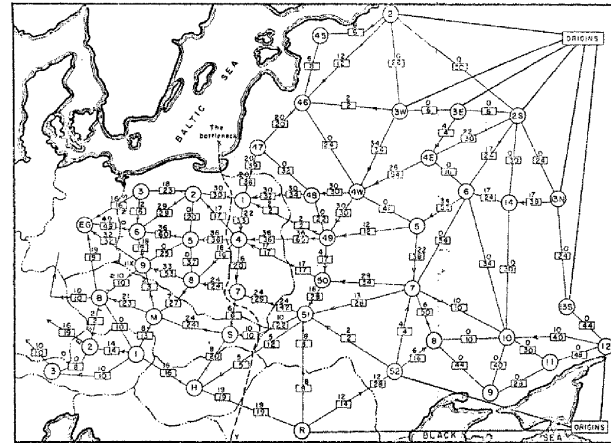


Chapter 7 Network Flow

PEARSON
Addison
Wesley
Slides by Kevin Wayne.
Copyright © 2005 Pearson-Addison Wesley.
All rights reserved.

Soviet Rail Network, 1955



Reference: *On the history of the transportation and maximum flow problems.*
Alexander Schrijver in Math Programming, 91: 3, 2002.

Maximum Flow and Minimum Cut

Max flow and min cut.

- Two very rich algorithmic problems.
- Cornerstone problems in combinatorial optimization.
- Beautiful mathematical duality.

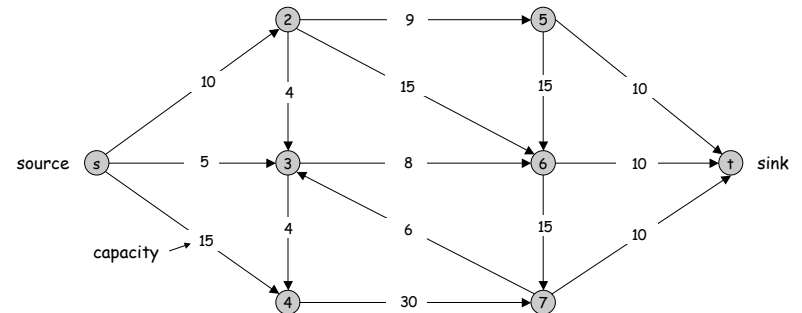
Nontrivial applications / reductions.

- Data mining.
- Open-pit mining.
- Project selection.
- Airline scheduling.
- Bipartite matching.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Network reliability.
- Distributed computing.
- Egalitarian stable matching.
- Security of statistical data.
- Network intrusion detection.
- Multi-camera scene reconstruction.
- Many many more ...

Minimum Cut Problem

Flow network.

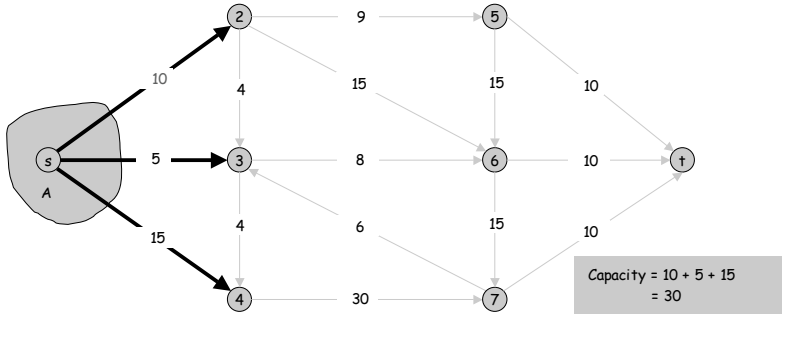
- Abstraction for material **flowing** through the edges.
- $G = (V, E)$ = directed graph, no parallel edges.
- Two distinguished nodes: s = source, t = sink.
- $c(e)$ = capacity of edge e .



Cuts

Def. An **s-t cut** is a partition (A, B) of V with $s \in A$ and $t \in B$.

Def. The **capacity** of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$

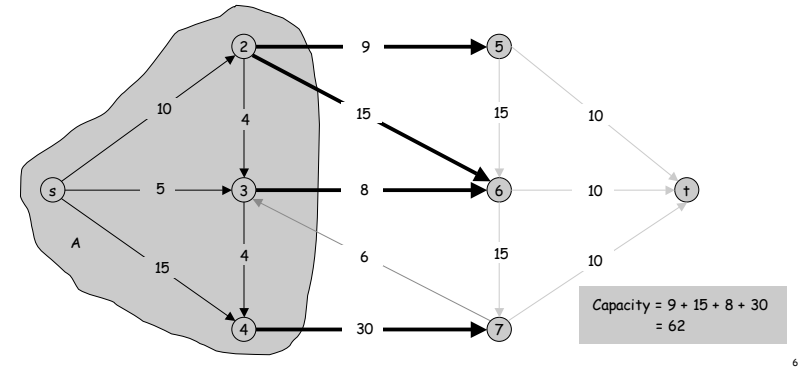


5

Cuts

Def. An **s-t cut** is a partition (A, B) of V with $s \in A$ and $t \in B$.

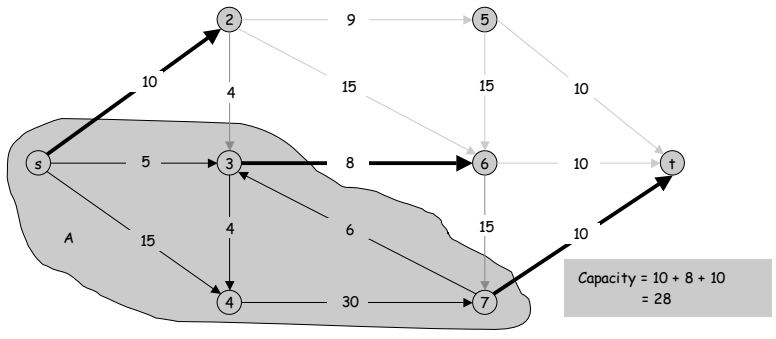
Def. The **capacity** of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



6

Minimum Cut Problem

Min s-t cut problem. Find an s-t cut of minimum capacity.



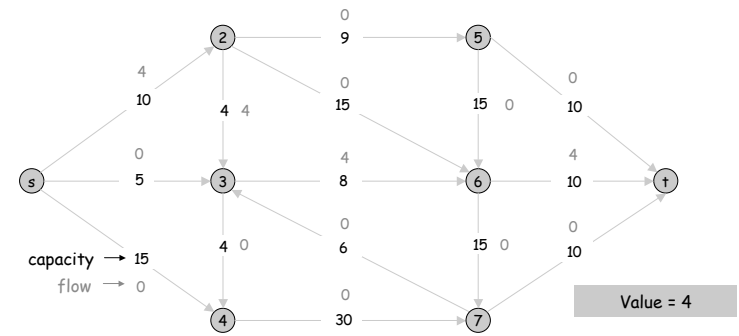
7

Flows

Def. An **s-t flow** is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

Def. The **value** of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.



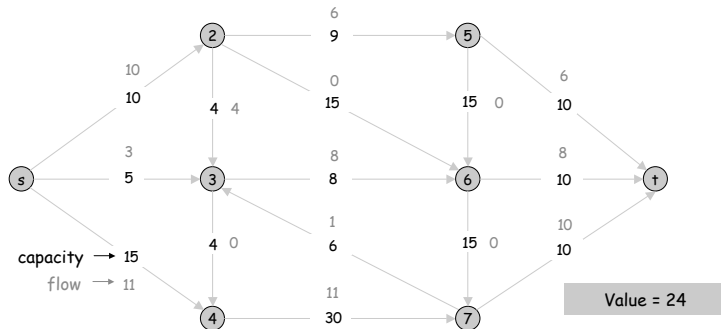
8

Flows

Def. An **s-t flow** is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

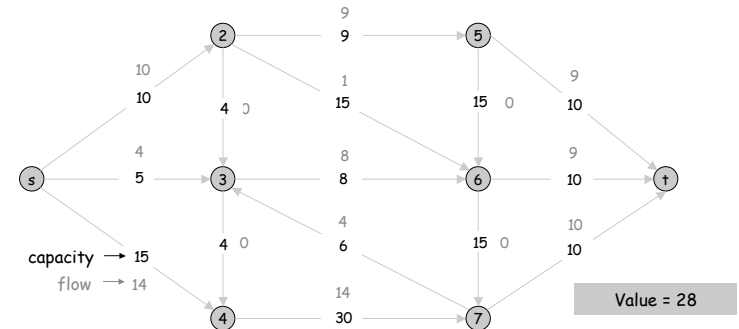
Def. The **value** of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.



9

Maximum Flow Problem

Max flow problem. Find s-t flow of maximum value.

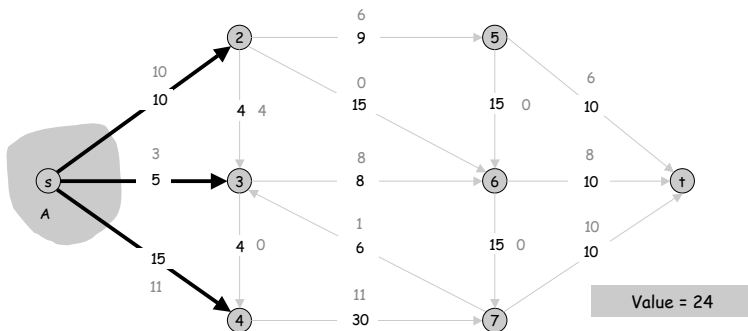


10

Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s .

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$

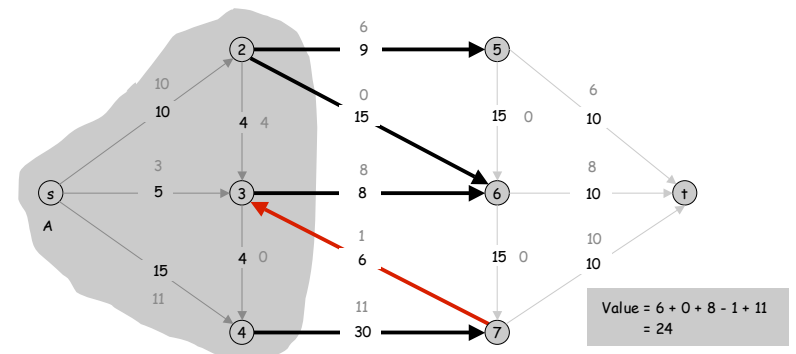


11

Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s .

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$

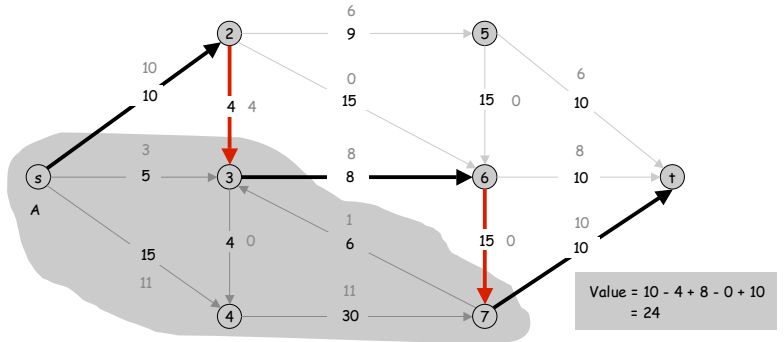


12

Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s - t cut. Then, the net flow sent across the cut is equal to the amount leaving s .

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



13

Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s - t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

$$\text{by flow conservation, all terms except } v = s \text{ are 0} \rightarrow = \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

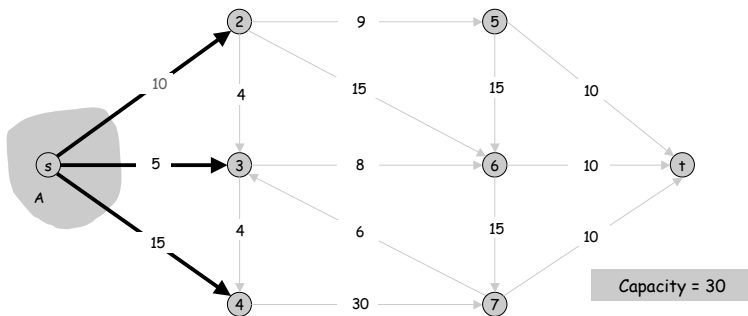
$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e).$$

14

Flows and Cuts

Weak duality. Let f be any flow, and let (A, B) be any s - t cut. Then the value of the flow is at most the capacity of the cut.

$$\text{Cut capacity} = 30 \Rightarrow \text{Flow value} \leq 30$$



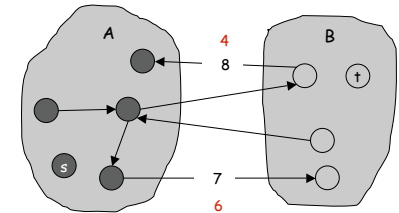
15

Flows and Cuts

Weak duality. Let f be any flow. Then, for any s - t cut (A, B) we have $v(f) \leq \text{cap}(A, B)$.

Pf.

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\leq \sum_{e \text{ out of } A} f(e) \\ &\leq \sum_{e \text{ out of } A} c(e) \\ &= \text{cap}(A, B) \quad \blacksquare \end{aligned}$$

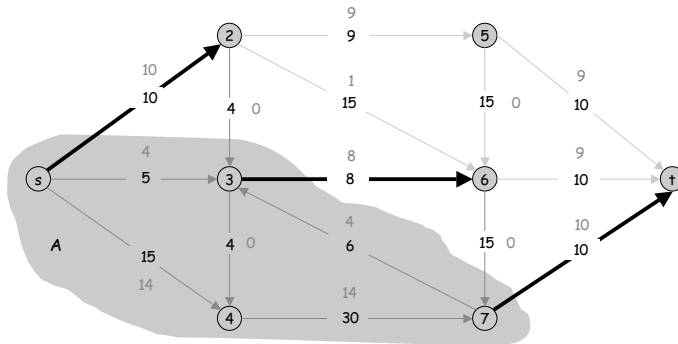


16

Certificate of Optimality

Corollary. Let f be any flow, and let (A, B) be any cut. If $v(f) = \text{cap}(A, B)$, then f is a max flow and (A, B) is a min cut.

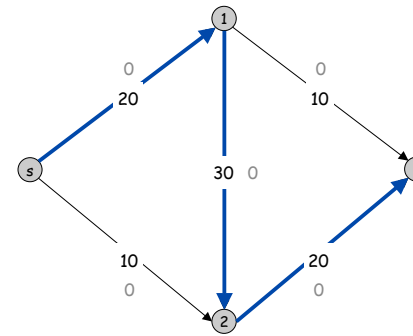
Value of flow = 28
Cut capacity = 28 \Rightarrow Flow value \leq 28



Towards a Max Flow Algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for all edge $e \in E$.
- Find an s - t path P where each edge has $f(e) < c(e)$.
- Augment flow along path P .
- Repeat until you get stuck.



Flow value = 0

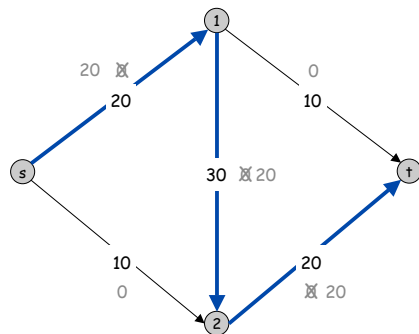
17

18

Towards a Max Flow Algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for all edge $e \in E$.
- Find an s - t path P where each edge has $f(e) < c(e)$.
- Augment flow along path P .
- Repeat until you get stuck.



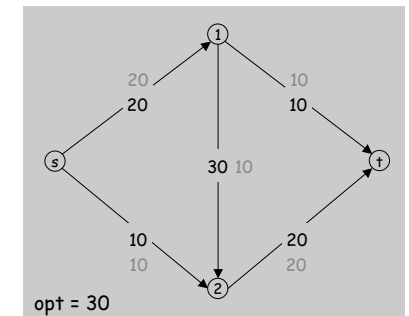
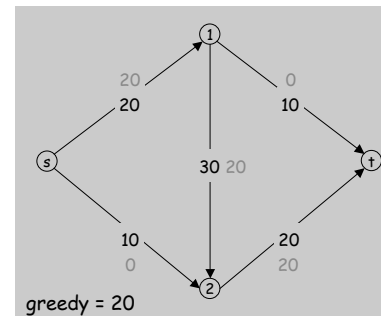
Flow value = 20

Towards a Max Flow Algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for all edge $e \in E$.
- Find an s - t path P where each edge has $f(e) < c(e)$.
- Augment flow along path P .
- Repeat until you get **stuck**.

← locally optimality \neq global optimality



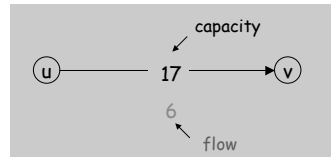
19

20

Residual Graph

Original edge: $e = (u, v) \in E$.

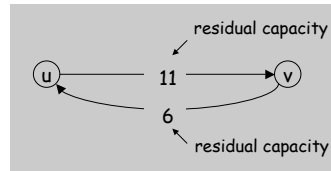
- Flow $f(e)$, capacity $c(e)$.



Residual edge.

- "Undo" flow sent.
- $e = (u, v)$ and $e^R = (v, u)$.
- Residual capacity:

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^R \in E \end{cases}$$

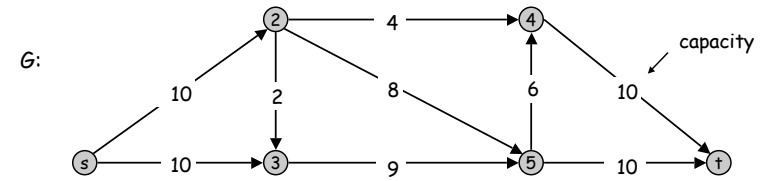


Residual graph: $G_f = (V, E_f)$.

- Residual edges with positive residual capacity.
- $E_f = \{e : f(e) < c(e)\} \cup \{e^R : c(e) > 0\}$.

21

Ford-Fulkerson Algorithm



22

Augmenting Path Algorithm

```
Augment(f, c, P) {
  b ← bottleneck(P)
  foreach e ∈ P {
    if (e ∈ E) f(e) ← f(e) + b
    else      f(eR) ← f(e) - b
  }
  return f
}
```

forward edge
reverse edge

```
Ford-Fulkerson(G, s, t, c) {
  foreach e ∈ E f(e) ← 0
  Gf ← residual graph

  while (there exists augmenting path P) {
    f ← Augment(f, c, P)
    update Gf
  }
  return f
}
```

23

Max-Flow Min-Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Ford-Fulkerson 1956] The value of the max flow is equal to the value of the min cut.

Proof strategy. We prove both simultaneously by showing the TFAE:

- There exists a cut (A, B) such that $v(f) = \text{cap}(A, B)$.
- Flow f is a max flow.
- There is no augmenting path relative to f .

(i) \Rightarrow (ii) This was the corollary to weak duality lemma.

(ii) \Rightarrow (iii) We show contrapositive.

- Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along path.

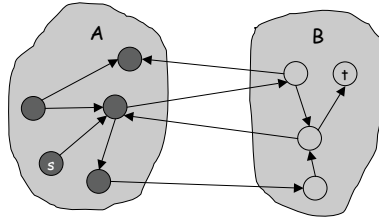
24

Proof of Max-Flow Min-Cut Theorem

(iii) \Rightarrow (i)

- Let f be a flow with no augmenting paths.
- Let A be set of vertices reachable from s in residual graph.
- By definition of A , $s \in A$.
- By definition of f , $t \notin A$.

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &= \sum_{e \text{ out of } A} c(e) \\ &= \text{cap}(A, B) \quad \blacksquare \end{aligned}$$



original network

25

Running Time

Assumption. All capacities are integers between 1 and C .

Invariant. Every flow value $f(e)$ and every residual capacities $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \leq nC$ iterations.

Pf. Each augmentation increase value by at least 1. \blacksquare

Corollary. If $C = 1$, Ford-Fulkerson runs in $O(mn)$ time.

Integrity theorem. If all capacities are integers, then there exists a max flow f for which every flow value $f(e)$ is an integer.

Pf. Since algorithm terminates, theorem follows from invariant. \blacksquare

26

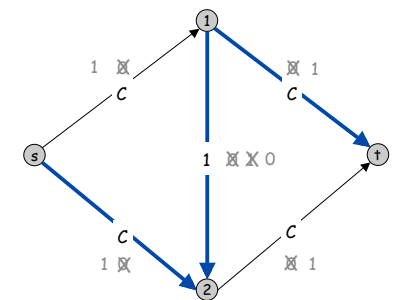
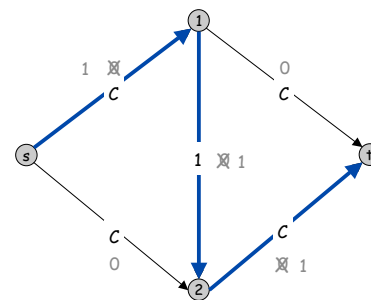
7.3 Choosing Good Augmenting Paths

Ford-Fulkerson: Exponential Number of Augmentations

Q. Is generic Ford-Fulkerson algorithm polynomial in input size?

$m, n,$ and $\log C$

A. No. If max capacity is C , then algorithm can take C iterations.



28

Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- If capacities are irrational, algorithm not guaranteed to terminate!

Goal: choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

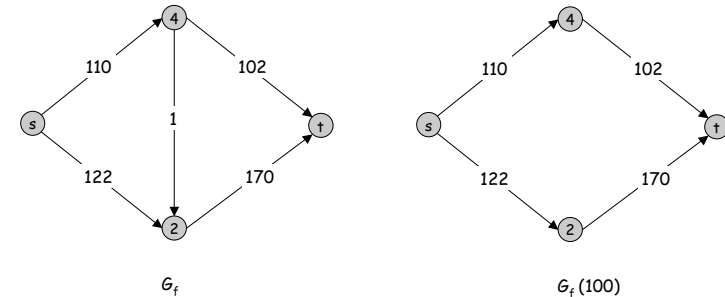
Choose augmenting paths with: [Edmonds-Karp 1972, Dinitz 1970]

- Max bottleneck capacity.
- Sufficiently large bottleneck capacity.
- Fewest number of edges.

Capacity Scaling

Intuition. Choosing path with highest bottleneck capacity increases flow by max possible amount.

- Don't worry about finding exact highest bottleneck path.
- Maintain scaling parameter Δ .
- Let $G_f(\Delta)$ be the subgraph of the residual graph consisting of only arcs with capacity at least Δ .



29

30

Capacity Scaling

```

Scaling-Max-Flow(G, s, t, c) {
  foreach e ∈ E f(e) ← 0
  Δ ← smallest power of 2 greater than or equal to C
  G_f ← residual graph

  while (Δ ≥ 1) {
    G_f(Δ) ← Δ-residual graph
    while (there exists augmenting path P in G_f(Δ)) {
      f ← augment(f, c, P)
      update G_f(Δ)
    }
    Δ ← Δ / 2
  }
  return f
}

```

31

Capacity Scaling: Correctness

Assumption. All edge capacities are integers between 1 and C .

Integrality invariant. All flow and residual capacity values are integral.

Correctness. If the algorithm terminates, then f is a max flow.

Pf.

- By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$.
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths. ■

32

Capacity Scaling: Running Time

Lemma 1. The outer while loop repeats $1 + \lceil \log_2 C \rceil$ times.

Pf. Initially $C \leq \Delta < 2C$. Δ decreases by a factor of 2 each iteration. ■

Lemma 2. Let f be the flow at the end of a Δ -scaling phase. Then the value of the maximum flow is at most $v(f) + m \Delta$. ← proof on next slide

Lemma 3. There are at most $2m$ augmentations per scaling phase.

- Let f be the flow at the end of the previous scaling phase.
- L2 $\Rightarrow v(f^*) \leq v(f) + m(2\Delta)$.
- Each augmentation in a Δ -phase increases $v(f)$ by at least Δ . ■

Theorem. The scaling max-flow algorithm finds a max flow in $O(m \log C)$ augmentations. It can be implemented to run in $O(m^2 \log C)$ time. ■

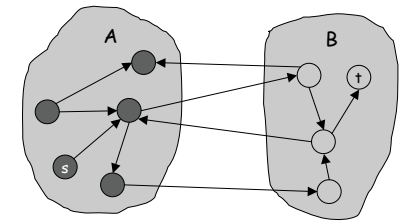
Capacity Scaling: Running Time

Lemma 2. Let f be the flow at the end of a Δ -scaling phase. Then value of the maximum flow is at most $v(f) + m \Delta$.

Pf. (almost identical to proof of max-flow min-cut theorem)

- We show that at the end of a Δ -phase, there exists a cut (A, B) such that $\text{cap}(A, B) \leq v(f) + m \Delta$.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of A , $s \in A$.
- By definition of f , $t \notin A$.

$$\begin{aligned}
 v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
 &\geq \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\
 &= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\
 &\geq \text{cap}(A, B) - m\Delta \quad \blacksquare
 \end{aligned}$$



original network