1 Easy Problems

1. [early Wed. Brandon] Draw a DFA accepting the following language, over the alphabet \( \{a, b\} \):
   \[ L = \{ w \mid \text{the number of symbols in } w \text{ is odd, i.e., the string } w \text{ is of odd length.} \} \]. Then specify the DFA formally, as a 5-tuple, giving each part of the 5-tuple.

2. [early Wed. Shir] Draw a DFA accepting the following language, over the alphabet \( \{a, b\} \):
   \[ L = \{ w \mid \text{the number } a\text{'s in } w \text{ is at most two and the number of } b\text{'s in } w \text{ is at least two} \} \]. Then specify the DFA formally, as a 5-tuple, giving each part of the 5-tuple.

3. [late Wed.] Draw an NFA accepting the language of all strings, over the alphabet \( \{a, b\} \), that end with \( aba \) or with \( bab \). Then specify the NFA formally, as a 5-tuple, giving each part of the 5-tuple.

4. [early Thurs.] Draw an NFA accepting the language of all strings, over the alphabet \( \{a, b\} \), that have abba as a substring.

5. [late Thurs. Brandon] Draw a 2-state DFA accepting the set \( \{a, b\}^* \), over the alphabet \( \{a, b, c\} \).

6. [late Thurs. Shir] Draw a DFA accepting all strings of length not equal to 7 over the alphabet \( \{a\} \).

2 Harder Problems

1. [early Wed. Brandon] Draw a DFA accepting the following language, over the alphabet \( \{0, 1\} \):
   All strings that when interpreted as binary integers are multiples of 7. (What is the integer related to a string for this problem? Ignore leading 0's. Treat the empty string as if it is interpreted as the value 0. So 00000 represents 0 and 0101 represents 5 and 101 represents 5.) (Hint: A very clear way to approach this is to be inspired by long division. The reason I say that that it is a very clear way is that the long-division inspired attack gives you a way to more generally solve division problems, across whatever bases and across whatever positive integer you want to divide by.) Now, after solving this, do you see how to change your answer to capture exactly the strings that interpreted as binary integers are 3 greater than a multiple of 7 (i.e., the integer is congruent, modulo 7, to 3)?

2. [early Wed. Shir] Draw an NFA accepting the following language, over the alphabet \( \{0, 1\} \):
   All strings \( w \) such that \( w \) contains 011 as a substring and also contains 100 as a substring.
   (Warning: we have not said which comes first, and, also, it is legal for them to overlap.)
3. [late Wed.] Let $\Sigma$ be the standard 26-letter alphabet $\{a, b, \ldots, z\}$. $L = \{a_1a_2a_1a_2 \mid a_1 \in \Sigma$ and $a_2 \in \Sigma\}$. Describe an DFA for $L$ (a picture-sketch is ok, as long as it is quite clear). Argue that each DFA accepting $L$ must contain at least $26^2$ states. (Do this by arguing that if any two of the $26^2$ two-symbol strings lead to the same state as each other, the DFA will make an error on some four-symbol string.) (Note that by generalizing this, we can easily show, for example, natural lower bounds of, for example, $26^{1776}$ states for an appropriate similar language.)

### 3 Very Challenging Problems

1. [early Thurs.] AND [late Thurs. Brandon] Let $\Sigma = \{0, 1\}$. Let $A$ be an arbitrary regular set over alphabet $\Sigma$. Consider the language of first halves of strings in $A$: $L = \{w \mid (\exists y)[|w| = |y| \land wy \in A]\}$. Prove that $L$ is regular. (Hint: Describe an NFA for $L$.) (Food for thought: What about first 1/3s? Middle thirds? Final thirds? Thirteenth seventeenths? Answer: They’re all regular.)

2. [late Thurs. Shir] Let $\Sigma = \{0, 1\}$. Let $A$ be an arbitrary regular set over alphabet $\Sigma$. Let $B$ be an arbitrary set over alphabet $\Sigma$ (it might not be regular!). Consider the language $L = \{w \mid (\exists y \in B)[wy \in A]\}$. Prove that $L$ is regular. (Hint: Describe an NFA for $L$.) (Food for thought: Is $L = \{w \mid (\exists y \in B)[ywy \in A]\}$ also regular? Yes, by extending the same basic insight.)

3. [Not assigned to any group and we’re not likely to go over it as it draws on facts that you may well not know and that are beyond what we will cover. But if you just for fun want sort of a “Part D,” here it is. And unless there is not time, I’ll in class probably verbally give you very briefly a sketch of the flavor of a solution.] Give an NFA (let us say in the Sipser model of NFAs) having at most 280 states (for extra fun, try to get it to be even smaller than that... one can in fact get it into the 60s) for the language $\{0^k \mid k \in \{0, 1, 2, \ldots, 999, 1001, 1002, 1003, \ldots\}\}$. 