Phase Transition for Glauber Dynamics for Independent Sets on Regular Trees

Ricardo Restrepo*† Daniel Stefankovic‡ Juan C. Vera§ Eric Vigoda¶ Linji Yang¶

Abstract

We study the effect of boundary conditions on the relaxation time of the Glauber dynamics for the hard-core lattice gas model on the n-vertex regular b-ary tree of height h. The hard-core model is defined on independent sets weighted by an activity (or fugacity) λ on trees. Reconstruction studies the effect of a 'typical' boundary condition, i.e., fixed assignment to the leaves, on the root. The threshold for when reconstruction occurs (and a typical boundary influences the root in the limit $h \to \infty$) has been of considerable recent interest since it appears to be connected to the efficiency of certain local algorithms on locally tree-like graphs. The reconstruction threshold occurs at $\omega \approx \ln b/b$ where $\lambda = \omega(1+\omega)^b$ is a convenient re-parameterization of the model.

We prove that for all boundary conditions, the relaxation time τ in the non-reconstruction region is fast, namely $\tau = O\left(n^{1+o_b(1)}\right)$ for any $\omega \leq \ln b/b$. In the reconstruction region, for all boundary conditions, we prove $\tau = O\left(n^{1+\delta+o_b(1)}\right)$ for $\omega = (1+\delta)\ln b/b$, for every $\delta > 0$. In contrast, we construct a boundary condition, for which the Glauber dynamics slows down in the reconstruction region, namely $\tau = \Omega\left(n^{1+\delta/2-o_b(1)}\right)$ for $\omega = (1+\delta)\ln b/b$, for every $\delta > 0$. The interesting part of our proof is this lower bound result, which uses a general technique that transforms an algorithm to prove reconstruction into a set in the state space of the Glauber dynamics with poor conductance.

*School of Mathematics, Georgia Institute of Technology, Atlanta GA 30332. Email: restrepo@math.gatech.edu. Research supported in part by NSF grant CCF-0910584.

†Universidad de Antioquia, Departamento de Matematicas, Medellin, Colombia. Scholarship '200 years'.

‡Department of Computer Science, University of Rochester, Rochester, NY 14627. Email: stefanko@cs.rochester.edu. Research supported in part by NSF grant CCF-0910415.

§Department of Econometrics and Operations Research, Tilburg University, Tilburg, The Netherlands. Email: j.c.veralizcano@uvt.nl.

¶School of Computer Science, Georgia Institute of Technology, Atlanta GA 30332. Email: {vigoda,ljyang}@gatech.edu. Research supported in part by NSF grant CCF-0830298 and CCF-0910584.

1 Introduction

There has been much recent interest in possible connections between equilibrium properties of statistical physics models and efficiency of local Markov chains for studying these models (see, e.g., [3, 10, 21, 22, 23, 31]). In this paper we study the hard-core model and establish new connections between the so-called reconstruction threshold in statistical physics with the convergence time of the single-site Markov chain known as the Glauber dynamics.

The hard-core model is studied in statistical physics as model of a lattice gas (see, e.g., Sokal [30]), and in operations research as a model of communication network (see Kelly [16]). It is a natural combinatorial problem, corresponding to counting and randomly sampling weighted independent sets of an input graph $G = (V, E)$. Let $\Omega = \Omega(G)$ denote the set of independent sets of G. Each set is weighted by an activity (or fugacity) $\lambda > 0$. For $\sigma \in \Omega$, its weight is $Z(\sigma) = \lambda^{|\sigma|}$ where $|\sigma|$ is the number of vertices in the set σ. The Gibbs measure is defined over Ω as $\mu(\sigma) = Z(\sigma)/Z$ where $Z = \sum_{\sigma\in\Omega} Z(\sigma)$ is the partition function.

This paper studies the hard-core model on trees, in some cases with a boundary condition. Let T_h denote the complete tree of height h with branching factor b. For concreteness we are assuming the root has b children, but our results, of course, easily extend to allow $b+1$ children for the root, the so-called Bethe lattice. Let n denote the number of vertices in T_h, and let L denote the leaves of the tree. A boundary condition is an assignment Γ to the leaves, where in the case of the hard-core model, Γ specifies a subset of the leaves L that are in the independent set. Then, let $\Omega_\Gamma = \{\sigma \in \Omega : \sigma(L) = \Gamma\}$ be the set of independent sets of T_h that are consistent with Γ, and the Gibbs measure $\mu_{h,\Gamma}$ is defined with respect to Ω_Γ, i.e., it is the projection of μ onto Ω_Γ.

The (heat bath) Glauber dynamics is a discrete time Markov chain (X_t) for sampling from the Gibbs distribution μ for a given graph $G = (V, E)$ and activity λ. We view $\Omega \subset \{0,1\}^V$ where for $X_t \in \Omega$, $X_t(v) = 1$ iff v is in the independent set. The transitions $X_t \to X_{t+1}$ of the Glauber dynamics are defined as:

- Choose a vertex v uniformly at random;
- For all $w \neq v$ set $X_{t+1}(w) = X_t(w)$;
- If all of the neighbors of v are unoccupied, set $X_{t+1}(v) = 1$ with probability $\lambda/(1+\lambda)$, otherwise set $X_{t+1}(v) = 0$.

When a boundary condition Γ is specified, the state space is restricted to Ω_Γ. For the case of the complete tree T_h (possibly with a boundary condition Γ) it is straightforward to verify that the Glauber dynamics is ergodic with unique stationary distribution μ_h (or $\mu_{h,\Gamma}$ when a boundary condition is specified). Thus, the Glauber dynamics is a natural algorithmic process for sampling from the Gibbs distribution. We study the relaxation time of the dynamics, which is defined as the inverse of the spectral gap of the transition matrix. See Section 2 for a more detailed definition of the relaxation time.

The Gibbs distribution describes the equilibrium state of the system, and the Glauber dynamics is a model of how the physical system reaches equilibrium [11, 21]. Thus, it is interesting to understand connections between properties of the equilibrium state (i.e., the Gibbs distribution) and properties of how the system reaches equilibrium (i.e., the Glauber dynamics). Models from statistical physics are designed to study phase transitions in the equilibrium state. A phase transition is said to occur when a small change in the microscopic parameters of the system (in the case of the hard-core model that corresponds to λ) causes a dramatic change in the macroscopic properties of the system.

A well-studied phase transition is uniqueness/non-uniqueness of infinite volume Gibbs distributions. This phase transition corresponds to whether there exists a sequence of boundary conditions for which, roughly speaking, the root is "influenced" by the leaves in the limit $h \to \infty$. For the hard-core model on the complete tree, Kelly [16] showed that the uniqueness threshold is at $\lambda_u = b^b/(b-1)^{b+1}$ (namely, uniqueness holds iff $\lambda < \lambda_u$).

There are interesting connections between the uniqueness threshold λ_u and the efficiency of algorithms on general graphs. In particular, Weitz [32] showed a deterministic fully-polynomial approximation scheme to estimate the partition function for any graph with constant maximum degree b for activities $\lambda < \lambda_u$. Recently, Sly [29] showed that it is NP-hard (unless $NP = RP$) to approximate the partition function for activities λ satisfying $\lambda_u < \lambda < \lambda_u + \epsilon_b$ for some small constant ϵ_b.

We are interested in the phase transition for reconstruction/non-reconstruction. This corresponds to whether a "typical" boundary influences the root in the limit $h \to \infty$, whereas uniqueness/non-uniqueness considered the worst boundary condition. To construct a typical boundary, we consider the independent set on the leaves of T_h generated by the following broadcast process. This process constructs an independent set σ on the infinite tree in a top-down manner. Let ω be the real positive solution of $\lambda = \omega(1+\omega)^b$. Consider the infinite complete tree with branching factor b, and construct σ as follows. We first include the root r in σ with probability $\omega/(1+\omega)$ and leave it out with probability $1/(1+\omega)$. Then for each vertex v, once the state of its parent $p(v)$ is determined, if $p(v) \notin \sigma$ then we add v into σ with probability $\omega/(1+\omega)$ and leave it out with probability $1/(1+\omega)$; if $p(v) \in \sigma$ then we leave v out of σ. Let σ_h denote the configuration of σ on level h, and let ν_h denote the broadcast measure on T_h.

Reconstruction addresses whether σ_h influences the configuration at the root r. In words, we first generate σ using the broadcasting measure, then we fix σ_h and resample a configuration τ on T_h from the Gibbs distribution $\mu_{h,\Gamma}$ with boundary condition $\Gamma = \sigma_h$. Of course, for finite h, the configuration at the root r in τ has a bias to the initial configuration $\sigma(r)$. Non-reconstruction is said to hold if the root is unbiased in expectation in the limit $h \to \infty$. More precisely, reconstruction holds if and only if:

$$\lim_{h\to\infty} \mathrm{E}_{\sigma\sim\nu_h}\left[\left|\mu_{h,\sigma_h}(r \in \tau) - \frac{\omega}{1+\omega}\right|\right] > 0. \tag{1.1}$$

There are many other equivalent conditions to the above definition of reconstruction, see Mossel [25] for a more extensive survey.

We refer to the reconstruction threshold as the critical ω_r such that for all $\omega < \omega_r$ non-reconstruction holds and for all $\omega > \omega_r$ reconstruction holds. The existence of the reconstruction threshold follows from Mossel [26, Proposition 20], and, by recent work of Bhatnagar et al [4] and Brightwell and Winkler [6], it is known that $\omega_r = (\ln b + (1+o(1)) \ln\ln b)/b$.

Our interest in the reconstruction threshold is its apparent connections to the threshold for the efficiency of certain local algorithms on locally-tree like graphs, such as sparse random graphs $G(n, c/n)$ for constant $c > 1$, planar graphs, and trees. For colorings and independent sets, the reconstruction threshold on the tree is believed to be intimately connected to the threshold for the efficiency of local algorithms. The evidence in support of that belief, is that the the geometry of the space of solutions on sparse random graphs appears to change dramatically near (and possibly at) the reconstruction threshold, see [1, 12, 17, 24]. The results of [13] for the Glauber dynamics of colorings on planar graphs, sug-

gests that the reconstruction threshold may have connections to the mixing time of the Glauber dynamics on planar graphs. In addition, reconstruction for the Ising and Potts models has applications in phylogenetics [8].

Our interest in this paper is on establishing more detailed connections between the reconstruction threshold and the relaxation time of the Glauber dynamics for trees. Berger et al [3] proved that for the tree T_h with boundary condition Γ such that $\mu_{h,\Gamma} = \nu_h$, $O(n)$ relaxation time for all h implies non-reconstruction. For the Ising model and colorings the boundary condition is empty, i.e., ν_h corresponds to the free boundary condition. Hence, for these models, the result of [3] says that reconstruction implies relaxation time $\omega(n)$. For the hard-core model it is not clear if there is a boundary condition Γ for the finite tree which has the same measure as the broadcasting process, i.e., $\mu_{h,\Gamma} = \nu_h$. This is discussed further in Section 3.

It was recently established for the Ising model [3, 22, 7] and for k-colorings [31] that on the tree T_h with free boundary condition, the relaxation is $O(n)$ in the non-reconstruction region and there is a slow down in the reconstruction region. Our starting point was addressing whether a similar phenomenon occurs in the hard-core model. Martinelli et al [23] showed that for the hard-core model on T_h with free boundary condition the relaxation time is $O(n)$ for all λ (and the mixing time is $O(n \log n)$). Hence, for the hard-core model, unlike in the Ising and colorings models, the Glauber dynamics on the tree with free boundary condition does not have connections to the reconstruction threshold. Our interest was whether there is a boundary condition for which there is such a connection.

We prove there is a connection by constructing a boundary condition for which the relaxation time slows down at the reconstruction threshold. Here is the formal statement of our results.

THEOREM 1.1. *For the Glauber dynamics on the hard-core model with activity $\lambda = \omega(1+\omega)^b$ on the complete tree T_h with n vertices, height h and branching factor b, the following hold:*

1. **For all $\omega \leq \ln b/b$:**
   For every boundary condition,
   $$\Omega(n) \leq T_{\text{relax}} \leq O(n^{1+o_b(1)}).$$
2. **For all $\delta > 0$ and $\omega = (1+\delta)\ln b/b$:**
   (a) *For every boundary condition,*
   $$T_{\text{relax}} \leq O(n^{1+\delta+o_b(1)}).$$
   (b) *There exists a sequence of boundary conditions for all $h \to \infty$ such that,*
   $$T_{\text{relax}} = \Omega(n^{1+\delta/2-o_b(1)}).$$

REMARK 1.1. *More precisely, we show that there is a function $g(b) = O(\ln\ln b/\ln b) = o(1)$ such that for every b, the lower bound in Part 2b is $\Omega(n^{1+\delta/2-g(b)})$, and there is a function $f(b) = O((\ln\ln b)^2/\ln b) = o(1)$ such that for every b, the upper bound in Part 1 is $O(n^{1+f(b)})$ and in Part 2a is $O(n^{1+\delta+f(b)})$.*

The upper bound improves upon Martinelli et al [23] who showed $O(n)$ relaxation time (and $O(n \log n)$ mixing time) for $\lambda < 1/(\sqrt{b}-1)$ for all boundary conditions. Note, $\lambda = 1/\sqrt{b}$ is roughly equivalent to $\omega \approx \frac{1}{2}\ln b/b$ which is below the reconstruction threshold. Our main result extends the fast mixing up to the reconstruction threshold, and shows the slow-down beyond the reconstruction threshold. Our lower bound in the reconstruction region uses a general approach that transforms an algorithm showing reconstruction into a set with poor conductance, which implies the lower bound on the relaxation time. This framework captures the proof approach used in [31].

There are two major difficulties we were facing: one is to figure out a proper subset of the state space which identifies a poor conductance bound of the Glauber dynamics. The conductance of such a subset should be sensitive to the boundary conditions, as we already know that the Glauber dynamics is rapid mixing under properly chosen boundary conditions (see, e.g., [23]). And such a lower bound on the relaxation time should closely match its upper bound. The other difficulty, once we realized that the relaxation time of the Glauber dynamics can be nontrivially lower bounded under a nonuniform hard-core model (see Section 3 for details), is to prove that, when reconstruction happens, such a nonuniform model can be approximated (in the measure sense), by an appropriate sequence of boundary conditions. As a result, then we are able to show Theorem 2b via a conductance argument.

In Section 2 we formally define various terms and present the basic tools used in our proofs. The lower bound (Part 2b of Theorem 1.1) is presented in Sections 3, 4 and 5. Section 3 outlines the approach. We then prove an analogue of Theorem 1.1 in Section 4 for the broadcasting model and use it in Section 5 to prove Part 2b of Theorem 1.1. The argument for the upper bounds stated in Theorem 1.1 is presented in Section 6.

2 Background

Let $P(\cdot,\cdot)$ denote the transition matrix of the Glauber dynamics. Let $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{|\Omega|}$ be the eigenval-

ues of the transition matrix P. The spectral gap c_{gap} is defined as $1-\gamma$ where $\gamma = \max\{\gamma_2, |\gamma_{|\Omega|}|\}$ denotes the second largest eigenvalue in absolute value. The relaxation time T_{relax} of the Markov chain is then defined as c_{gap}^{-1}, the inverse of the spectral gap. Relaxation time is an important measure of the convergence rate of a Markov chain (see, e.g., Chapter 12 in [19]).

To lower bound the relaxation time we analyze conductance. The conductance of a Markov chain with state space Ω and transition matrix P is given by $\Phi = \min_{S\subseteq\Omega}\{\Phi_S\}$, where Φ_S is the conductance of a specific set $S \subseteq \Omega$ defined as

$$\Phi_S = \frac{\sum_{\sigma\in S}\sum_{\eta\in\bar{S}}\pi(\sigma)P(\sigma,\eta)}{\pi(S)\pi(\bar{S})}.$$

Thus, a general way to find a good upper bound on the conductance is to find a set S such that the probability of "escaping" from S is relatively small. The well-known relationship between the relaxation time and the conductance was established in [18] and [27], and we will use the form $T_{\text{relax}} = \Omega(1/\Phi)$ for proving the lower bounds.

3 Lower Bound Approach

First note that the lower bound stated in Part 1 of Theorem 1.1, namely, $T_{\text{relax}} = \Omega(n)$, is trivial for all ω. For example, by considering the set $S = \{\sigma \in \Omega : r \notin \sigma\}$ of independent sets which do not contain the root, $\Phi(S) = \Omega(1/n)$ since we need to update r to leave S.

We begin by explaining the high level idea of the non-trivial lower bound in Part 2b of Theorem 1.1. To that end, we first analyze a variant of the hard-core model in which there are two different activities, the internal vertices have activity λ and the leaves have activity ω. The resulting Gibbs distribution is identical to the measure ν_h defined in Section 1 for the broadcasting process. Thus we refer to the following model as the broadcasting model.

For the tree $T_h = (V, E)$, we look at the following equivalent definition of the distribution ν_h over the set Ω of independent sets of T_h. For $\sigma \in \Omega$, let

$$Z'(\sigma) = \lambda^{|\sigma\cap V\setminus L|}\omega^{|\sigma\cap L|},$$

where L are the leaves of T_h and ω is, as before, the positive solution to $\omega(1+\omega)^b = \lambda$. Let $\nu_h(\sigma) = Z'(\sigma)/Z'$ where $Z' = \sum_{\sigma\in\Omega} Z'(\sigma)$ is the partition function. By simple calculations, the following proposition holds.

PROPOSITION 3.1. *The measure ν_h defined by the hard-core model with activity λ for internal vertices and ω for leaves is identical to the measure defined by the broadcasting process.*

Proof. In fact, we just need to verify that in the hard-core model with activity λ for internal vertices and ω for leaves, the probability p_v of a vertex v being occupied conditioning on its parent is unoccupied is $\omega/(1+\omega)$. This can be proved by induction. The base case is v being a leaf, which is obviously true by the Markovian property of the Gibbs measure. If v is not a leaf, by induction, the probability p_v has to satisfy the following equation

$$p_v = (1-p_v)\frac{\lambda}{(1+\omega)^b},$$

which solves to $p_v = \omega/(1+\omega)$.

The result of Berger et al [3] mentioned in Section 1 implies that the relaxation time of the Glauber dynamics on the broadcasting model is $\omega(n)$. We will prove a stronger result, analogous to the desired lower bound for Part 2b of Theorem 1.1.

THEOREM 3.1. *For all $\delta > 0$, the Glauber dynamics for the broadcasting model on the complete tree T_h with n vertices, branching factor b and $w = (1+\delta)\ln b/b$ satisfies the following:*

$$T_{\text{relax}} = \Omega(n^{1+\delta/2-o_b(1)}),$$

where the $o_b(1)$ function is $O(\ln\ln b/\ln b)$.

REMARK 3.1. *We can show a similar upper bound on the relaxation time for the Glauber dynamics in this setting as in Theorem 1.1. Moreover, we can show the same upper bound for the mixing time by establishing a tight bound between the inverse log-Sobolev constant and the relaxation time as was done for colorings in Tetali et al [31].*

We will prove Theorem 3.1 via a general method that relates any reconstruction algorithm (or function) with the conductance of the Glauber dynamics. A *reconstruction algorithm* is a function $A : \Omega(L) \to \{0,1\}$ (ideally efficiently computable) such that $A(\sigma_h)$ and $\sigma(r)$ are positively correlated. Basically, the algorithm A takes the configurations at the leaves L as the input and tries to compute the configuration at the root. When the context is clear, we write $A(\sigma)$ instead of $A(\sigma_h)$. Under the Gibbs measure ν_h, the *effectiveness* of A is the following measure of the covariance between the algorithm A's output and the marginal at the root of the actual measure:

$$r_{h,A} = \min_{x\in\{0,1\}}[\nu_h(A(\sigma)=\sigma(r)=x) - \nu_h(A(\sigma)=x)\nu_h(\sigma(r)=x)].$$

If it is the case that $\liminf_{h\to\infty} r_{h,A} = c_0 > 0$ for some positive constant c_0 depending only on ω and b, then

we say that it is an *effective reconstruction algorithm*. In words, an effective algorithm, is able to recover the spin at the root, from the information at the leaves, with a nontrivial success, when $h \to \infty$. Notice that reconstruction (defined in (1.1)) is a necessary condition for any reconstruction algorithm to be effective, since

$$\begin{aligned}
&\mathrm{E}_{\sigma\sim\nu_h}\left[\left|\mu_{h,\sigma_h}(r\in\tau)-\frac{\omega}{1+\omega}\right|\right]\\
&\geq \mathrm{E}_{\sigma\sim\nu_h}\left[\left(\mu_{h,\sigma_h}\left(r\in\tau\right)-\nu_h\left(r\in\sigma\right)\right)\mathbf{1}(A\left(\sigma\right)=1)\right]\\
&\geq r_{h,A},
\end{aligned}$$

where $\mathbf{1}()$ is the indicator function. We define the *sensitivity* of A, for the configuration $\sigma \in \Omega(T_h)$, as the fraction of vertices v such that switching the spin at v in σ changes the final result of A. More precisely, let σ^v be the configuration obtained from changing σ at v. Define the sensitivity as:

$$S_A(\sigma) = \frac{1}{n}\#\{v \in L : A(\sigma^v) \neq A(\sigma)\}.$$

The *average sensitivity* (with respect to the root being occupied) $\bar{S}_A$ is hence defined as

$$\bar{S}_A = \mathrm{E}_{\sigma\sim\nu_h}\left[S_A(\sigma)\mathbf{1}(A(\sigma)=1)\right].$$

It is fine to define the average sensitivity without the indicator function, which only affects a constant factor in the analysis. We are doing so to simplify some of the results' statements and proofs.

Typically when one proves reconstruction, it is done by presenting an effective reconstruction algorithm. Using the following theorem, by further analyzing the sensitivity of the reconstruction algorithm, one obtains a lower bound on the relaxation time or mixing time of the Glauber dynamics.

THEOREM 3.2. *Suppose that A is an effective reconstruction algorithm. Then, the relaxation time T_{relax} of the Glauber dynamics satisfies $T_{\mathrm{relax}} = \Omega\left((\bar{S}_A)^{-1}\right)$.*

REMARK 3.2. *The above theorem can be generalized to any spin system. To illustrate the usefulness of this theorem, we note that the lower bound on the mixing time of the Glauber dynamics for k-colorings in the reconstruction region proved in [31] fits this conceptually appealing framework.*

Proof. Throughout the proof let $\nu := \nu_h$. Consider the set $U = \{\sigma : A(\sigma) = 1\}$. Then,

$$\begin{aligned}
\Phi_U &= \frac{\sum_{\sigma\in U}\nu(\sigma)\sum_{w\in L}\sum_{\tau:\tau(w)\neq\sigma(w)}P(\sigma,\tau)}{\nu(U)(1-\nu(U))}\\
&\leq \frac{\sum_{\sigma\in U}\nu(\sigma)S_A(\sigma)}{\nu(U)(1-\nu(U))}.
\end{aligned}$$

From the definition of $r_{h,A}$, we have that $\nu(U) \geq \nu(A(\sigma)=\sigma(r)=1) \geq r_{h,A}$, and similarly $(1-\nu(U)) \geq \nu(A(\sigma)=\sigma(r)=0) \geq r_{h,A}$. Now, because the algorithm is effective, we have $\liminf_{h\to\infty}(r_{h,A}) = c_0 > 0$ and hence for all h big enough, $r_{h,A} > 0$. Therefore, $\Phi_U \leq (r_{h,A})^{-2}\bar{S}_A$, which concludes that

$$T_{\mathrm{relax}} = c_{gap}^{-1} \geq 1/\Phi_U = \Omega((\bar{S}_A)^{-1}).$$

To prove Theorem 3.1, we analyze the sensitivity of the reconstruction algorithm by Brightwell and Winkler [6, Section 5] which yields the best known upper bounds on the reconstruction threshold. Our goal is to show that the average sensitivity of this algorithm is small. The analysis of the sensitivity of the Brightwell-Winkler (BW) algorithm, which then proves Theorem 3.1, is presented in Section 4.

Our main objective remains of constructing a sequence of "bad" boundary conditions under which the Glauber dynamics for the hard-core model slows down in the reconstruction region. An initial approach is that if we can find a complete tree T' with some boundary condition such that the marginal of the root being occupied exactly equals $\omega/(1+\omega)$, then by attaching the same tree T' with the corresponding boundary conditions to all of the leaves of a complete tree T, we are able to simulate the nonuniform hard-core model on T, (i.e., the resulting measure projected onto T is the same as the one in the broadcasting model) and hence we can do the same approach to upper-bound the conductance of the dynamics on this new tree. However, from a cardinality argument, not for every ω there exists a complete tree of finite height with some boundary condition such that the marginal probability of the root being occupied equals $\omega/(1+\omega)$. Alternatively, we give a constructive way to find boundary conditions that approximate the desired marginal probability relatively accurately. This is done in Section 5.

Finally, at the end of Section 5 we argue that since the error is shrinking very fast from the bottom level under our construction of boundary conditions, we can again analyze the sensitivity of the Brightwell-Winkler algorithm starting from just a few levels above the leaves. This approach yields the lower bound stated in Part 2b of Theorem 1.1.

4 Lower Bound for Broadcasting: Proof of Theorem 3.1

Throughout this section we work on the broadcasting model. To prove Theorem 3.1 we analyze the average sensitivity of the reconstruction algorithm used by Brightwell and Winkler [6], which we refer to as the BW algorithm. For any configuration σ as the input, the algorithm works in a bottom up manner labeling each

vertex from the leaves: a parent is labeled to occupied if all of its children are labeled to unoccupied; otherwise, it is labeled to unoccupied. The algorithm will output the labeling of the root as the final result. Formally, it can be described by the following deterministic recursion deciding the labeling of every vertex:

$$R_\sigma(v) = \begin{cases} \sigma(v) & \text{if } v \in L \\ 1 - \max\{R_\sigma(w_1), \ldots, R_\sigma(w_b)\} & \text{otherwise} \end{cases}$$

where $w_1, \ldots, w_b$ are the children of v. Finally, let $\mathrm{BW}(\sigma) = \mathrm{BW}(\sigma_h) = R_\sigma(r)$. Note that, $\mathrm{BW}(\sigma)$ only depends on the configuration σ_h on the leaves. The algorithm is proved to be effective in [6] when $\delta > 0$. Therefore, it can be used in our case to lower bound the relaxation time. In this algorithm, by definition we have

$$n\bar{S}_{\mathrm{BW}} = O\big(\mathrm{E}_\sigma\left[\#\{v \in L : \mathrm{BW}(\sigma) = 1 \wedge \mathrm{BW}(\sigma^v) = 0\}\right]\big), \tag{4.2}$$

where the expectation is over the probability measure ν_h. Due to the symmetry of the function $R_\sigma(v)$ and the measure ν_h, the expectation can be further simplified as

$$\begin{aligned} &\mathrm{E}_\sigma\left[\#\{v \in L : \mathrm{BW}(\sigma) = 1 \wedge \mathrm{BW}(\sigma^v) = 0\}\right] \\ &\qquad = b^h \nu_h(\mathrm{BW}(\sigma) = 1 \wedge \mathrm{BW}(\sigma^{\hat{u}}) = 0), \end{aligned} \tag{4.3}$$

where $\hat{u}$ is now a fixed leaf. To bound the right hand side of Eq.(4.3), let $\kappa \in \Omega(T_h)$ be a fixed configuration such that $\mathrm{BW}(\kappa) = 1$. Let the path $\mathcal{P}$ from $\hat{u}$ to the root r be $u_0 = \hat{u} \rightsquigarrow u_1 \rightsquigarrow \cdots \rightsquigarrow u_h = r$, and for any $i > 0$, let $w_{i,j}$ be the children of u_i so that the labeling is such that for $j = 1$, $w_{i,1} = u_{i-1}$ and for $j \neq 1$, $w_{i,j}$ is not on the path $\mathcal{P}$. An important observation is that, in order to make $\mathrm{BW}(\kappa)$ change to 0 by changing only the configuration at $\hat{u}$ of κ, a necessary condition for κ is $R_\kappa(u_i) = 1 - R_\kappa(u_{i-1})$ for all $i \geq 1$. Then for all $i \geq 1$ and $j \in \{2, \ldots, b\}$, we have $R_\kappa(w_{i,j}) = 0$. To calculate the probability that a random $\kappa \sim \nu_h$ satisfies such conditions, it would be easier if we expose the configurations along the path $\mathcal{P}$. Since then, conditioning on the configurations on the path, the events $R_\kappa(w_{i,j}) = 0$ are independent for all i, j. And if $\kappa(u_i) = 0$, we have for all $j > 1$, the conditional probability of $R_\kappa(w_{i,j}) = 0$ equals $\Pr_{\eta \sim \nu_{i-1}}[\mathrm{BW}(\eta) = 0]$, the probability BW algorithm outputs a 0 over a random configuration η of the leaves of the complete tree T_{i-1} with height $i-1$. The analysis above leads to the following lemma, which bounds the probability $\nu_h(\mathrm{BW}(\sigma) = 1 \text{ and } \mathrm{BW}(\sigma^{\hat{u}}) = 0)$.

LEMMA 4.1. *For every $i > 0$, let $\eta \in \Omega(T_{i-1})$ be a configuration chosen randomly according to measure ν_{i-1}, then*

$$\begin{aligned} &\nu_h(BW(\sigma) = 1 \text{ and } BW(\sigma^{\hat{u}}) = 0) \leq \\ &\qquad \mathrm{E}_{\kappa \sim \nu_h}\left[\prod_{i>0:\kappa(u_i)=0} \Pr_{\eta \sim \nu_{i-1}}[BW(\eta) = 0]^{(b-1)}\right]. \end{aligned}$$

Complete proofs of lemmas in this section are deferred to Section 4.1. To use Lemma 4.1, we derive the following uniform bound on the probability $\Pr_{\eta \sim \nu_i}[\mathrm{BW}(\eta) = 0]$, for all i. Here and through out the paper, $b_0(\delta)$ is a function explicitly defined in Lemma 4.3 in Section 4.2. This function is of order $\exp(\delta^{-1}\ln(\delta^{-1}))$ as $\delta \to 0$ and remains bounded as $\delta \to \infty$.

LEMMA 4.2. *Let $\delta > 0$, and let $\omega = (1+\delta)\ln b/b$. For all $b \geq b_0(\delta)$ and $i \geq 1$,*

$$\Pr_{\eta \sim \nu_i}[BW(\eta) = 0] \leq \frac{(1.01)^{1/b}}{1+\omega}.$$

Combining Equations (4.2), (4.3), Lemma 4.1 and Lemma 4.2, we are able to upper bound the average sensitivity of the BW algorithm:

$$\begin{aligned} \bar{S}_{\mathrm{BW}} &= O\big(\nu_h(\mathrm{BW}(\sigma) = 1 \text{ and } \mathrm{BW}(\sigma^{\hat{u}}) = 0)\big) \\ &= O\left(\mathrm{E}_{\kappa \sim \nu_h}\left[\left(\frac{1.01\omega(1+\omega)}{\lambda}\right)^{\#\{i:\kappa(u_i)=0\}}\right]\right). \end{aligned}$$

In this expectation, the number of unoccupied vertices in the path $\mathcal{P}$ can be trivially lower bounded by $h/2$, since it is impossible that there exists $i > 0$, $\kappa(u_i) = \kappa(u_{i-1}) = 1$. Therefore, the above expectation can be easily bounded by $O^*(n^{-(1+\delta)/2})$. This is not good enough in our case. We sharpen the bound using Lemma 4.4 in Section 4.2, leading to the following theorem, whose complete proof is contained in Section 4.1.

THEOREM 4.1. *Let $\delta > 0$, and let $\omega = (1+\delta)\ln b/b$. For all $b \geq b_0(\delta)$,*

$$T_{\mathrm{relax}} = \Omega(n^d), \quad \textit{where } d = \left(1 + \frac{\ln\left(\lambda/(1.01\omega b)^2\right)}{2\ln b}\right).$$

Theorem 3.1 is a simple corollary of Theorem 4.1 by noticing that $d = 1 + \delta/2 - O\left(\frac{\ln\ln b}{\ln b}\right)$. Furthermore, we can "hide" the fact that $b \geq b_0(\delta)$ in this residual term, by using the trivial lower bound $\Omega(n)$ for all $b < b_0(\delta)$ and $b_0(\delta) \approx \exp(\delta^{-1}\ln(\delta^{-1}))$ as $\delta \to 0$.

4.1 Proofs. Note that throughout this paper, we will use the following notations for the relationships

between two functions $f(x)$ and $g(x)$ for simplicity. If $\lim_{x\to\infty} f(x)/g(x) = 1$, we write $f(x) \approx g(x)$; if $f(x) = O(g(x))$, we write $f(x) \lesssim g(x)$ and if $f(x) = \Omega(g(x))$, we write $f(x) \gtrsim g(x)$.

Proof. [Proof of Lemma 4.1] Let $\mathbf{x} = \{0,1\}^h$ be a valid configuration on the path $\mathcal{P}$. Conditioning on $\kappa(u_i) = \mathbf{x}(i)$ for all i, we know that the events $R_\kappa(w_{i,j}) = 0$ are independent for all i and j. Given $\kappa(u_i) = 0$, the probability of the event $R_\kappa(w_{i,j}) = 0$ equals $\Pr_{\eta\sim\nu_{i-1}}[\mathrm{BW}(\eta) = 0]$; and given $\kappa(u_i) = 1$, the probability of this event can be trivially upper bounded by 1 (this bound is "safe", in the sense that the actual quantity is close to 1 for big λ). By this, we can conclude that

$$\begin{aligned}
&\nu_h(\mathrm{BW}(\kappa) = 1 \text{ and } \mathrm{BW}(\kappa^{\hat{u}}) = 0)\\
&\quad\le \sum_{\mathbf{x}} \nu_h(\kappa : \forall i, \kappa(u_i) = \mathbf{x}(u_i))\\
&\qquad\cdot \prod_{i>0:\kappa(u_i)=0} \Pr_{\eta\sim\nu_{i-1}}[\mathrm{BW}(\eta) = 0]^{(b-1)}\\
&\quad= \mathrm{E}_\kappa\left[\prod_{i>0:\kappa(u_i)=0} \Pr_{\eta\sim\nu_{i-1}}[\mathrm{BW}(\eta) = 0]^{(b-1)}\right].
\end{aligned}$$

Proof. [Proof of Lemma 4.2] In the proof, we will use the fact that $\exp\left(\frac{2(1.01)(\omega b)^2}{\lambda}\right) \le 1.01$, whenever $b \ge b_0(\delta)$ (Lemma 4.3). Now, for simplicity, denote $f_i = \Pr_{\eta\sim\nu_{i-1}}[A(\eta) = 0]$. First of all, notice the recurrences

$$\begin{aligned}
f_{i+1} &= \frac{\omega}{1+\omega}\left(1-\left(1-f_{i-1}^b\right)^b\right) + \frac{1}{1+\omega}\left(1-f_i^b\right),\\
f_1 &= \frac{1}{1+\omega} \quad , \quad f_2 = \frac{1}{1+\omega}\left(1-\left(\frac{1}{1+\omega}\right)^b\right).
\end{aligned}$$

The result follows by an easy induction: For $h = 1, 2$, the result is clear. On the other hand, from the previous recurrences, if it is the case that $f_i \le \frac{(1.01)^{1/b}}{1+\omega}$, then

$$\begin{aligned}
f_{i+1}^b &\le \left[\frac{\omega}{1+\omega}\left(1-\left(1-f_{i-1}^b\right)^b\right) + \frac{1}{1+\omega}\right]^b\\
&\le \left[\frac{\omega}{1+\omega}\left(1-\left(1-\frac{1.01\omega}{\lambda}\right)^b\right) + \frac{1}{1+\omega}\right]^b\\
&\le \left(\frac{1+\frac{1.01\omega^2 b}{\lambda}}{1+\omega}\right)^b\\
&\le \frac{\exp(1.01(\omega b)^2/\lambda)}{(1+\omega)^b}\\
&\le \frac{1.01}{(1+\omega)^b},
\end{aligned}$$

where the third inequality follows from the fact that $(1-u)^b \ge 1-ub$ for $u < 1$, the fourth inequality follows from $(1+u) \le e^u$, and the last inequality follows from the fact that $\exp\left(\frac{2(1.01)(\omega b)^2}{\lambda}\right) \le 1.01$ for $b \ge b_0(\delta)$.

Proof. [Proof of Theorem 4.1] From Lemma 4.4 in Section 4.2, we have that

$$\begin{aligned}
&\mathrm{E}\left[\left(\frac{1.01\omega(1+\omega)}{\lambda}\right)^{\#\{i:\sigma(u_i)=0\}}\right] \approx\\
&\quad\left(1+\frac{1-\epsilon}{2\epsilon(1+\omega)}\right)\frac{(1+\epsilon)}{2}\left(\frac{1.01\omega}{2\lambda}\left[1+\sqrt{1+4\lambda/1.01}\right]\right)^h
\end{aligned}$$

where $\epsilon = \left[\sqrt{1+4\lambda/1.01}\right]^{-1}$. The previous term is asymptotically dominated by $\left(1.01\frac{\omega}{\lambda^{1/2}}\right)^h$. Therefore,

$$\bar{S}_A = O\left(\left[\frac{1.01\omega}{\lambda^{1/2}}\right]^h\right) = O\left(n^{-\left[1+\frac{\ln\left(\lambda/(1.01\omega b)^2\right)}{2\ln b}\right]}\right).$$

Now, from [6, Section 5], it is known that the BW algorithm is effective for $\omega > (1+\delta)\ln b/b$ and $b > b_0(\delta)$, therefore Theorem 3.2 applies. The conclusion follows.

4.2 Some Technical Lemmas. The proofs of the following lemmas will be included in the full version of this paper.

LEMMA 4.3. *Define*

$$b_0(\delta) = \min\{b_0 : \exp\left(\frac{2(1.01)(\omega b)^2}{\lambda}\right) \le 1.01, \forall b \ge b_0\},$$

then $b_0(\delta)$ is a continuous function such that

1. *$b_0(\delta) < \infty$ for all $\delta > 0$ (that is, it is well defined).*
2. *$b_0(\delta) \approx \exp\left((1+o(1))\delta^{-1}\ln(\delta^{-1})\right)$ as $\delta \to 0$.*
3. *$b_0(\delta) \approx b_0(\infty)$ as $\delta \to \infty$, where $b_0(\infty)$ is a fixed constant ≤ 2.*

LEMMA 4.4. *Let $\zeta_0, \zeta_1, \ldots$ be a Markov process with state space $\{0,1\}$, such that $\zeta_0 = 0$ and with transition rates $p_{0\to0} = p$, $p_{0\to1} = q$, $p_{1\to0} = 1$, $p_{1\to1} = 0$. Let $N_h = \#\{1 \le i \le h : \zeta_i = 0\}$, then*

$$\begin{aligned}
\mathrm{E}\left[a^{N_h}\right] \approx &\left(1+\frac{p(1-\epsilon)}{2\epsilon}\right)\frac{(1+\epsilon)}{2}\\
&\cdot\left(\frac{pa}{2}\left[1+\sqrt{1+4q/(ap^2)}\right]\right)^h.
\end{aligned}$$

where $\epsilon = \frac{1}{\sqrt{1+4q/(ap^2)}}$. Moreover, if the transition rate $p_{0\to0}$ is inhomogeneous but such that $\left|p - p_{0\to0}^{(i)}\right| \le \delta$,

then

$$\mathrm{E}\left[a^{N_h}\right] \lesssim \left(1+\frac{(p+\delta)(1-\bar{\epsilon})}{2\bar{\epsilon}}\right)\frac{(1+\bar{\epsilon})}{2} \cdot \left(\frac{(p+\delta)a}{2}\left[1+\sqrt{1+4(q+\delta)/\left(a(p+\delta)^2\right)}\right]\right)^h,$$

where $\bar{\epsilon} = \frac{1}{\sqrt{1+4(q+\delta)/\left(a(p+\delta)^2\right)}}$.

5 "Bad" Boundary Conditions: Proof of Theorem 1.1.2b

First, we will show that for any ω, there exists a sequence of boundary conditions, denoted as $\Gamma_\omega := \{\Gamma_i\}_{i>0}$, one for each complete tree of height $i > 0$, such that if $i \to \infty$, the probability of the root being occupied converges to $\frac{\omega}{1+\omega}$. Later in this section we will exploit such a construction to attain in full the conclusion of Part 2b of Theorem 1.1.

As a first observation, note that, the Gibbs measure for the hard-core model on T_i with boundary condition Γ is the same as the Gibbs measure for the hard-core model (with the same activity λ) on the tree T obtained from T_i by deleting all of the leaves as well as the parent of each (occupied) leaf $v \in \Gamma$. It will be convenient to work directly with such "trimmed" trees, rather than the complete tree with boundary condition. Having this in mind, our construction will be inductive in the following way. We will define a sequence of (trimmed) trees $\{(L_i, U_i)\}_{i\geq 0}$ such that L_{i+1} is comprised of s_{i+1} copies of U_i and $b - s_{i+1}$ copies of L_i with $\{s_i\}_{i\geq 1}$ properly chosen. Similarly, U_{i+1} is comprised of t_{i+1} copies of U_i and $b - t_{i+1}$ copies of L_i, with $\{t_i\}_{i\geq 1}$ properly chosen.

We will show that, for either $T_i^* = L_i$, or $T_i^* = U_i$, it is the case that the 'Q'-value, defined as:

$$Q(T_i^*) = \frac{\mu_{T_i^*}(\sigma(r)=1)}{\omega\mu_{T_i^*}(\sigma(r)=0)},$$

where $\mu_{T_i^*}(\cdot)$ is the hard-core measure on the trimmed tree T_i^*, satisfies $Q(T_i^*) \to 1$. Note that if $Q(T_i^*) = 1$, then the probability of the root being occupied is $\omega/(1+\omega)$ as desired. To attain this, we will construct L_i and U_i in such a way that $Q(U_i) \geq 1$ and $Q(L_i) \leq 1$.

The recursion for $Q(L_{i+1})$ can be derived easily as

$$Q(L_{i+1}) = \frac{(1+\omega)^b}{(1+\omega Q(U_i))^{s_{i+1}}(1+\omega Q(L_i))^{b-s_{i+1}}},$$

and a similar equation holds for $Q(U_{i+1})$ by replacing s_{i+1} with t_{i+1}.

To keep the construction simple, we inductively define the appropriate t_i and s_i, so that once L_i and U_i are given, we let t_{i+1} be the minimum choice so that the resulting Q-value is ≥ 1, more precisely, we let:

$$t_{i+1} = \arg\min_\ell\{Q = \frac{(1+\omega)^b}{(1+Q(U_i))^\ell(1+\omega Q(L_i))^{b-\ell}} : Q \geq 1\}.$$

And similarly, we let

$$s_{i+1} = \arg\max_\ell\{Q = \frac{(1+\omega)^b}{(1+Q(U_i))^\ell(1+Q(L_i))^{b-\ell}} : Q \leq 1\}.$$

The recursion starts with U_1 being the graph of a single node and L_1 being the empty set, so that $Q(U_1) = \lambda/\omega$ and $Q(L_1) = 0$. Observe that, by definition, $s_{i+1} \in \{t_{i+1}, t_{i+1}+1\}$ and that the construction guarantees that the values $Q(L_i)$ are at most 1, and the values $Q(U_i)$ are at least 1. The following simple lemma justifies the correctness of our construction.

LEMMA 5.1.

$$\lim_{i\to\infty} Q(U_i)/Q(L_i) = 1.$$

Proof. It is easy to see that either $t_i = s_i$ (meaning that $Q(L_i) = Q(U_i) = 1$), or $t_i = s_i - 1$, which implies that

$$\frac{Q(U_i)}{Q(L_i)} = \frac{1+\omega Q(U_{i-1})}{1+\omega Q(L_{i-1})} < \frac{Q(U_{i-1})}{Q(L_{i-1})}.$$

So the ratio is shrinking. Suppose the limit is not 1 but some value $q > 1$. Then,

$$\frac{Q(U_{i-1})}{Q(L_{i-1})} - \frac{Q(U_i)}{Q(L_i)} = \frac{Q(U_{i-1}) - Q(L_{i-1})}{(1+\omega Q(L_{i-1}))Q(L_{i-1})}.$$

Since $Q(U_i)/Q(L_i) > q$ and $Q(L_i) \leq 1$, we have

$$\frac{Q(U_{i-1}) - Q(L_{i-1})}{(1+\omega Q(L_{i-1}))Q(L_{i-1})} \geq \frac{(q-1)Q(L_{i-1})}{Q(L_{i-1})(1+\omega)} = \frac{q-1}{1+\omega},$$

which is a constant.

Therefore as long as $q > 1$, we show that the difference between the ratios for each step i is at least some constant which is impossible. Hence the assumption is false.

By this lemma, it is easy to check that if we let T_i^* to be equal to either U_i or L_i, then $Q(T_i^*) \to 1$. Indeed, we can show that the additive error decreases exponentially fast. The following lemma indicates that this is the case for $\omega < 1$ (although a similar result holds for any ω).

LEMMA 5.2. *Let* ϵ_i^+ *be the value of* $Q(U_i) - 1$ *and let* ϵ_i^- *be the value of* $1 - Q(L_i)$*, then*

$$\epsilon_{i+1}^+ + \epsilon_{i+1}^- \leq \omega(\epsilon_i^+ + \epsilon_i^-).$$

Proof. We can rewrite the expression

$$(1+\omega)^b/(1+\omega Q(U_i))^j(1+\omega Q(L_i))^{b-j}$$

as

$$\frac{1}{(1+\frac{\omega}{1+\omega}\epsilon_i^+)^j(1-\frac{\omega}{1+\omega}\epsilon_i^-)^{b-j}}.$$

Now, let k be the biggest index over $[b]$ such that the denominator of the previous expression is less than 1 (thus, $k+1$ will be the least index such that the denominator is greater than 1). Then,

$$\begin{aligned}
\epsilon_{i+1}^+ + \epsilon_{i+1}^- &= \frac{1}{(1+\frac{\omega}{1+\omega}\epsilon_i^+)^k(1-\frac{\omega}{1+\omega}\epsilon_i^-)^{b-k}} \\
&\quad - \frac{1}{(1+\frac{\omega}{1+\omega}\epsilon_i^+)^{k+1}(1-\frac{\omega}{1+\omega}\epsilon_i^-)^{b-k-1}} \\
&= \frac{\frac{\omega}{1+\omega}(\epsilon_i^+ + \epsilon_i^-)}{(1+\frac{\omega}{1+\omega}\epsilon_i^+)^{k+1}(1-\frac{\omega}{1+\omega}\epsilon_i^-)^{b-k}} \\
&\le \frac{\frac{\omega}{1+\omega}(\epsilon_i^+ + \epsilon_i^-)}{1-\frac{\omega}{1+\omega}\epsilon_i^-} \\
&\le \omega(\epsilon_i^+ + \epsilon_i^-).
\end{aligned}$$

Coming back to the original tree-boundary notation, let Γ_h^1 be the boundary corresponding to the trimming of the tree U_h and let Γ_h^2 be the boundary corresponding to the trimming of the tree L_h. By our construction, for any vertex v on the tree of height h, the measure from μ_{h,Γ_h^1} (or μ_{h,Γ_h^2}) projected onto the space of the independent sets of the subtree rooted at v with the boundary condition corresponding to the correct part of Γ and the parent of v being unoccupied is either μ_{i,Γ_i^1} or μ_{i,Γ_i^2}, where i is the distance of v away from the leaves on T_h. Conditioning on the parent of v being unoccupied, in the broadcast process defined in the Introduction, we would occupy v with probability $\omega/(1+\omega)$. Therefore, in the above construction, the probability v is occupied (or rather unoccupied) is close to the desired probability, and the error will decay exponentially fast with the distance from the leaves. This is formally stated in the following corollary of Lemma 5.2.

COROLLARY 5.1. *Given any $\omega < 1$ and the complete tree of height i, for Γ equal to Γ_i^1 or Γ_i^2 inductively constructed above, we have*

$$\left|\mu_{i,\Gamma}(\sigma(r)=0) - \frac{1}{1+\omega}\right| \le \omega^{i-1}\lambda/b.$$

Throughout the rest of this section it is assumed that we are dealing with the boundary conditions $\{\Gamma_h^1\}_{h\in\mathbf{N}}$ and $\{\Gamma_h^2\}_{h\in\mathbf{N}}$ constructed above. We will then show that for every $\omega = (1+\delta)\ln b/b$ under these two boundary conditions, the Glauber dynamics on the hard-core model slows down, whenever $\delta > 0$. As we know from Corollary 5.1, the error of the marginal goes down very fast, so that roughly we can think of the marginal distribution of the configurations on the tree from the root to the vertices a few levels above the leaves as being close to the broadcasting measure. In fact, by following the same proof outline as we did in Section 4, we are able to prove the same lower bound in the hard-core model for these boundaries. To do that we need a slight generalization of the reconstruction algorithm and extensions of the corresponding lemmas used in that section to handle the errors in the marginal probabilities.

To generalize the notion of a reconstruction algorithm to the case of a boundary condition we need to add an extra parameter ℓ depending only on ω and b. We will essentially ignore the bottom ℓ levels in the analysis, and we will use that for the top $h-\ell$ levels the marginal probabilities are close to those on the broadcasting tree. We define a *reconstruction algorithm with a parameter ℓ* for the tree T_h with boundary condition Γ as a function $A_\ell : \Omega(L_{h-\ell}) \to \{0,1\}$. The algorithm A_ℓ takes the configurations of the vertices at height $h-\ell$ as the input and tries to compute the configuration at the root. For any $\sigma \in \Omega(T_{h,\Gamma})$, the sensitivity is defined as: $S_{\ell,A}(\sigma) = \frac{1}{n}\#\left\{v \in L_{h-\ell} : A_\ell(\sigma^v_{h-\ell}) \neq A_\ell(\sigma_{h-\ell})\right\}$. The average sensitivity of the algorithm at height $h-\ell$ with respect to the boundary Γ is defined as: $\bar{S}^\Gamma_{\ell,A} = \mathrm{E}_\sigma\left[S_{\ell,A}(\sigma)\mathbf{1}(A_\ell(\sigma_{h-\ell})=1)\right]$. And the effectiveness is defined as:

$$\begin{aligned}
r^\Gamma_{\ell,A} = \min_{x\in\{0,1\}}[&\mu_{h,\Gamma}(A_\ell(\sigma_{h-\ell}) = x \text{ and } \sigma(r) = x) - \\
&\mu_{h,\Gamma}(A_\ell(\sigma_{h-\ell}) = x)\mu_{h,\Gamma}(\sigma(r) = x)].
\end{aligned}$$

We can show the analog of Theorem 3.2 in this setting.

THEOREM 5.1. *Suppose that A_ℓ is an effective reconstruction algorithm. Then, it is the case that the spectral gap c_{gap} of the Glauber dynamics for the hard-core model on the tree of height h with boundary condition Γ, satisfies $c_{gap} = O(\bar{S}^\Gamma_{\ell,A})$, and hence the relaxation time of this Glauber dynamics satisfies $T_{\text{relax}} = \Omega(1/\bar{S}^\Gamma_{\ell,A})$.*

To bound the average sensitivity for the boundary conditions Γ_h^1 and Γ_h^2 constructed above, we again use the same BW algorithm as we analyzed for the broadcasting tree. As in Eq.(4.2) and (4.3), it is again enough to bound the probability

$$P^\Gamma_{\ell,\mathrm{BW}} := \mu_{h,\Gamma_h}(\mathrm{BW}_\ell(\sigma_{h-\ell}) = 1 \text{ and } \mathrm{BW}_\ell(\sigma^{\hat{u}}_{h-\ell}) = 0)$$

for a fixed vertex $\hat{u}$ at a distance ℓ from the leaves, although in this case, this probability will not be the same for all $\hat{u}$. Let the path $\mathcal{P}$ from $\hat{u}$ to the root r be $u_0 = \hat{u} \rightsquigarrow u_1 \rightsquigarrow \cdots \rightsquigarrow u_{h-\ell} = r$, and for each $i > 0$ and $j \in \{1, \ldots, b\}$, let $w_{i,j}$ be defined similarly as in Section 4. Further, let $\Gamma_{i,j}$ be the boundary condition Γ_h restricted to the subtree $T_{w_{i,j}}$ of T_h rooted at the vertex $w_{i,j}$. These subtrees are of height $i + \ell - 1$ for each i. Note that, by our construction of the boundary conditions, for each fixed i, $\Gamma_{i,j} = \Gamma^1_{i+\ell-1}$ or $\Gamma_{i,j} = \Gamma^2_{i+\ell-1}$. The probability $P^{\Gamma}_{\ell,\mathrm{BW}}$ can be calculated by the following lemma, which is the analog of Lemma 4.1 for the broadcasting tree.

LEMMA 5.3.

$$P^{\Gamma}_{\ell,BW} \leq \mathrm{E}_{\sigma}\left[\prod_{i>0:\sigma(u_i)=0}\prod_{j=2}^{b} \Pr_{\eta\sim\mu_{i+\ell-1,\Gamma_{i,j}}}[A_\ell(\eta)=0]\right],$$

where the expectation is over the measure μ_{h,Γ_h}, and for each i, j, η is a random configuration on the subtree rooted at $w_{i,j}$ with the probability measure $\mu_{i+\ell-1,\Gamma_{i,j}}$.

The proofs of Theorem 5.1 and Lemma 5.3 use the same proof approach as for Theorem 3.2 and Lemma 4.1 respectively. However, to bound $\Pr_{\eta}[A_\ell(\eta) = 0]$ for every $i > 0$, in spite of going along the lines of Lemma 4.2, the proof does require extra care to deal with the errors in the marginal probabilities which were bounded in Corollary 5.1. In particular, we will establish the following lemma to upper bound $\Pr_{\eta\sim\mu_{i+\ell-1,\Gamma_{i,j}}}[A_\ell(\eta) = 0]$ for each $i > 0$. Here and throughout the text, we define $\ell(\lambda, b)$ to be the minimum ℓ such that for all $i \geq \ell$,

$$\Pr_{\eta\sim\mu_{i,\Gamma^2_i}}[\eta(r) = 0] \leq \frac{1}{1+\omega}\exp\left(\frac{1.01(\omega b)^2}{\lambda}\right).$$

The existence of such constant $\ell(\lambda, b)$ is guaranteed by Lemma 5.1, also from Corollary 5.1 we can deduce a explicit value for $\ell(\lambda, b)$, provided that $\omega < 1$.

LEMMA 5.4. *Given any $\delta > 0$, and $i \geq \ell(\lambda, b) = \ell$, then both $\Pr_{\eta\sim\mu_{i,\Gamma^1_i}}[A_\ell(\eta) = 0]$ and $\Pr_{\eta\sim\mu_{i,\Gamma^2_i}}[A_\ell(\eta) = 0]$ are upper bounded by $\frac{1.01^{1/b}}{1+\omega}$ for any $b \geq b_0(\delta)$.*

And also it is not hard to show that the BW algorithm under this setting is effective.

PROPOSITION 5.1. *The BW reconstruction algorithm is effective to recover the configuration at the root from the configurations at distance $\ell(\lambda, b)$ from the leaves.*

Then, we are able to again bound $\bar{S}^{\Gamma}_{\ell,\mathrm{BW}}$ for $\Gamma = \Gamma^1_h$ or Γ^2_h, proving the following theorem, which completes the proof of Part 2b in Theorem 1.1. Interested readers can look up the full version for the complete proofs.

THEOREM 5.2. *Let $\delta > 0$, and let $\omega = (1+\delta)\ln b/b$. For all $b \geq b_0(\delta)$, it is the case that*

$$T_{\mathrm{relax}} = \Omega(n^d), \quad \textit{where } d = \left(1 + \frac{\ln\left(\lambda/(1.01\omega b)^2\right)}{2\ln b}\right).$$

6 Upper Bounds of the Relaxation Time via the Coupling Method

We will use the coupling technique to prove our upper bounds on the relaxation time. Coupling implies an upper bound on the mixing time. The mixing time T_{mix} for the Glauber dynamics is defined as the number of steps, from the worst initial state, to reach within variation distance $\leq 1/2\mathrm{e}$ of the stationary distribution. It is an elementary fact that the mixing time gives a good upper bound on the relaxation time (see, e.g., [19] for the following bound), we will use this fact in our upper bound proofs:

$$T_{\mathrm{relax}} \leq T_{\mathrm{mix}} + 1. \tag{6.4}$$

Given two copies (X_t) and (Y_t) of the Glauber dynamics, a coupling is a joint process (X_t, Y_t) such that the evolution of each component viewed in isolation is identical to the Glauber dynamics (c.f., [19] for an introduction to the coupling technique). The Coupling Lemma [2] (c.f., [19, Theorem 5.2]) guarantees that if, there is a coupling and time $t > 0$, so that for every pair (X_0, Y_0) of initial states, $\Pr[X_t \neq Y_t \mid X_0, Y_0] \leq 1/2\mathrm{e}$ under the coupling, then $T_{\mathrm{mix}} \leq t$.

Before we show the main idea for our upper bound proofs, we first introduce some notation we use in this section. For a b dimensional vector ρ where $0 \leq \rho_i \leq 1$ for every $1 \leq i \leq b$, let τ_ρ be the relaxation time of the following Glauber dynamics of the hard-core model on the star graph $G^\star$ with $b + 1$ vertices. The dynamics on the star graph $G^\star$ is defined as follows. From an independent set X_t,

1. Choose a random vertex v.
2. If v is the root of G, then set
$$X' = \begin{cases} X_t \cup \{v\} & \text{with probability } \lambda/(1+\lambda) \\ X_t \setminus \{v\} & \text{with probability } 1/(1+\lambda). \end{cases}$$
3. If v is a leaf of G, then set
$$X' = \begin{cases} X_t \cup \{v\} & \text{with probability } \rho_i \\ X_t \setminus \{v\} & \text{with probability } 1-\rho_i. \end{cases}$$
4. If X' is an independent set, then set $X_{t+1} = X'$, otherwise set $X_{t+1} = X_t$.

Let $\tau^\star := \max_\rho\{\tau_\rho\}$ be defined as the worst case relaxation time over all possible choices of ρ. Using the block dynamics approach of Martinelli [21], as used in Section 2.3 of [3] (see also [20] and [31] for similar results), it is not hard to show that the relaxation time of the above Glauber dynamics is exactly the same as that of the natural block dynamics which updates the configurations of a whole subtree of the root in one step, and hence the following lemma holds.

LEMMA 6.1. *The relaxation time T_{relax} of the Glauber dynamics of the hard-core model on the complete tree of height H is upper bounded by $(\tau^\star)^H$ for any boundary condition on the leaves.*

Note that, the relaxation time on the complete tree is quite sensitive to the boundary conditions. For example, as mentioned in the introduction, Martinelli et al [23] show that when the boundary condition is all even (or similarly for all odd), i.e., all of the leaves are occupied when the height is even (respectively, odd) and all of the leaves are unoccupied when the height is odd (even), then the mixing time is $O(n \ln n)$ for all λ. In this paper we are dealing we are considering all boundary conditions, and in our lower bound, we show there are boundary conditions that slow down the Glauber dynamics. The lower bound on the relaxation time for the Glauber dynamics under those boundary conditions which we show suffer the slow-down roughly matches up with the upper bound we prove here.

We need to bound the relaxation time of the Glauber dynamics on the star graph for different cases with respect to ρ. To do this, we first use a simple coupling to bound the mixing time and then by Equation (6.4) we get an upper bound on the relaxation time. The coupling (X_t, Y_t) of the two chains is done by choosing the same random vertex v for changing the states at step t and maximizing the probability of the two chains choosing the same update for the state of v. Thus, if all of the neighbors of v are unoccupied in both X_t and Y_t, then v will be occupied in both chains or unoccupied in both chains at time $t+1$ with the correct marginal probability. In all other cases, the update choices for $X_{t+1}(v)$ and $Y_{t+1}(v)$ are coupled arbitrarily. Let $\preceq$ be a partial order on the state space Ω such that:

- $\sigma = \{r\}$ is the maximum element.
- if $\sigma_1 \subseteq \sigma_2$ and $r \notin \sigma_1, \sigma_2$, then $\sigma_2 \preceq \sigma_1$.

Then the hard-core model on the star graph is a monotone system (See, e.g., [19] for more details about monotone systems.), and the coupling we define has the following monotonicity property:

PROPOSITION 6.1. (MONOTONICITY) *If at time t we have $X_t \preceq Y_t$, then we have $X_{t+1} \preceq Y_{t+1}$.*

We will analyze this simple coupling by partitioning the analysis into several cases based on ρ.

LEMMA 6.2. *If $\sum \rho_i \leq 4 \ln \ln b$, and for all $1 \leq i \leq b$, $\rho_i \leq 1 - 1/\ln b$, then $\tau_\rho = O(b^{1+o(1)})$, where $o(1)$ is a term that goes to zero as b goes to infinity.*

LEMMA 6.3. *If $\sum \rho_i \geq 4 \ln \ln b$, then $\tau_\rho = O((\lambda+1) b \ln b)$.*

LEMMA 6.4. *If there exists an $1 \leq i \leq b$ such that $\rho_i > 1 - 1/\ln b$, then $\tau_\rho = O((\lambda+1) b \ln b)$.*

The intuition behind the above lemmas is that the typical behavior of the Glauber dynamics changes with respect to the marginal probabilities ρ of the leaves. When ρ is tiny for all leaves then Lemma 6.2 applies, and in this case the stationary distribution of the chain has a large probability that all the leaves are unoccupied, and hence X_t and Y_t can be coupled at the state where the root is occupied. When the sum of the marginal probabilities for the leaves is large then Lemmas 6.3 and 6.4 apply. In this case, it is likely that at least one of the leaves is occupied, hence the coupling analysis waits for the root to get unoccupied in both chains and then attempts to get the leaves to agree in the two chains before the root is added in to either chain.

We defer the proofs of these lemmas to the full version of this paper. By plugging in the results of Lemmas 6.2, 6.3 and 6.4 into Lemma 6.1, we get that:

$$\begin{aligned} T_{\text{relax}} &\leq (\tau^\star)^{\log_b n} \\ &= O\left(\left(\max\{(\lambda+1) b \ln b, b^{1+o(1)}\}\right)^{\log_b n}\right) \\ &= O\left(n^{1+\ln(\lambda+1)/\ln b + o(1)}\right) \end{aligned}$$

for any $\lambda > 0$. Recall that the relationship between ω and λ is $\lambda = \omega(1+\omega)^b$, and in this paper we are mainly interested in the cases when $\omega = (1+\delta)\ln b / b$ for any constant $\delta > -1$. Hence, in terms of ω, if $-1 < \delta \leq 0$ the relaxation time is upper bounded by $n^{1+o(1)}$, and if $\delta > 0$ the relaxation time is upper bounded by $n^{1+\delta+o(1)}$. This proves Theorem 1.1.

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