

***Computing Dehn Twists and Geometric
Intersection Numbers in Polynomial
Time***

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outline



What is a curve?

Some history of algorithmic problems.

Representing surfaces.

Representing simple curves in surfaces.

Transforming between various representations.

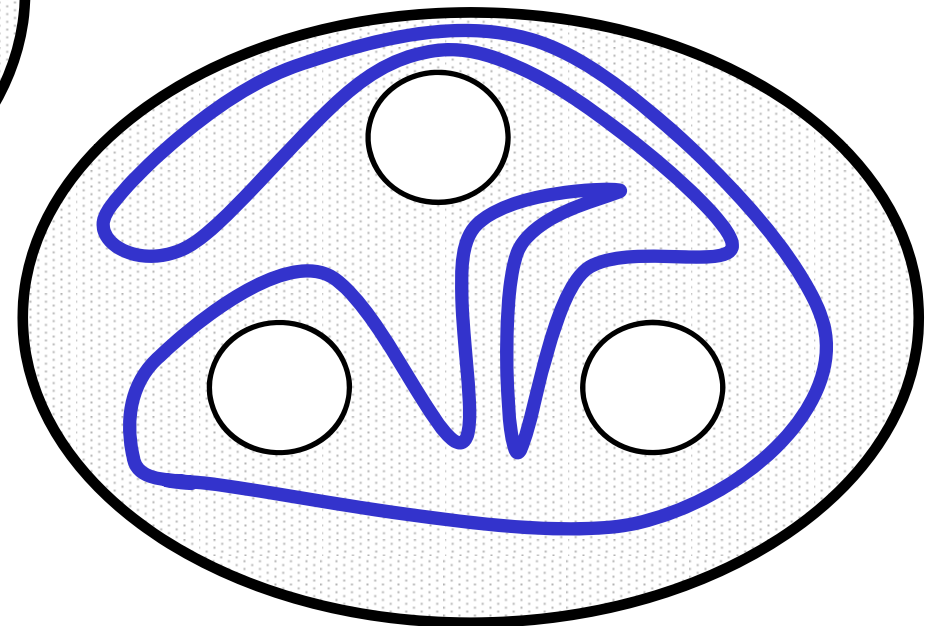
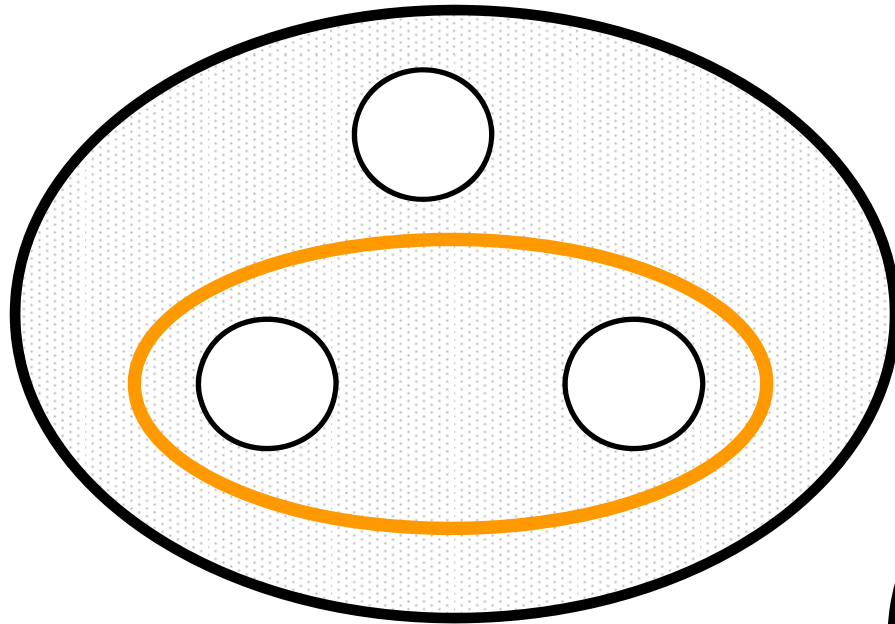
TOOL: Word equations.

What is a Dehn twist and why is it interesting?

Computing Dehn twists.

Open questions.

Curves on surfaces

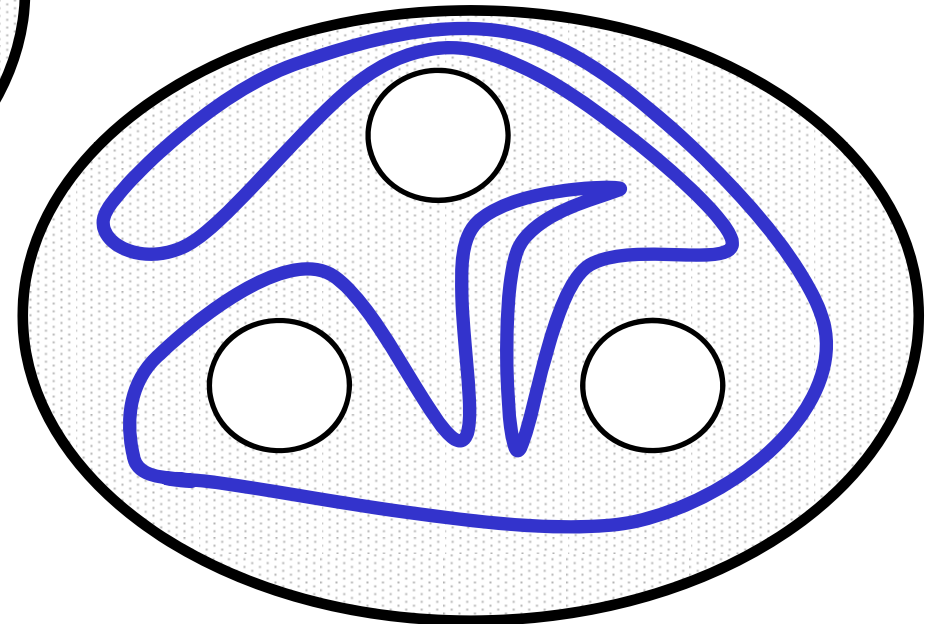
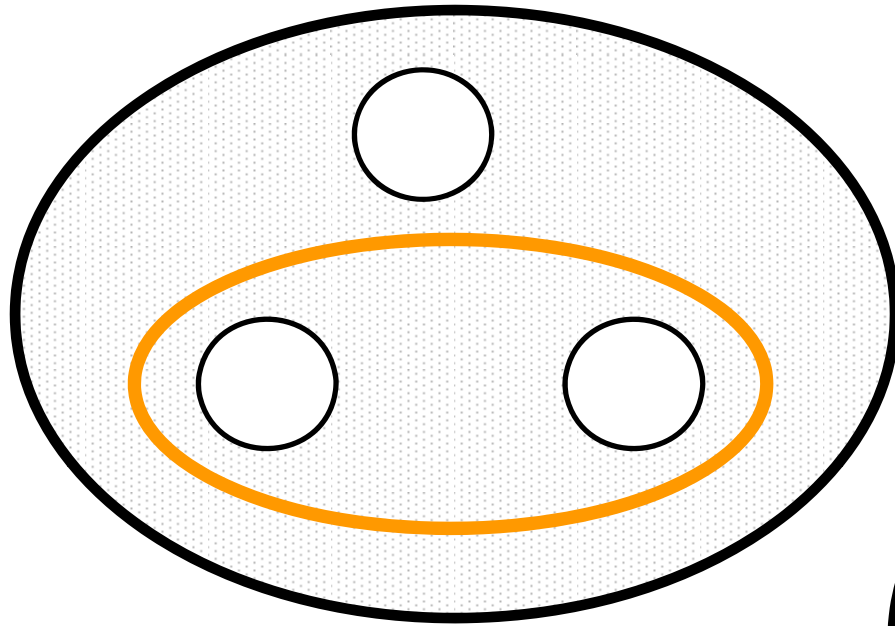


different? same?

closed curve = homeomorphic image of circle S^1

simple closed curve = γ is injective (no self-intersections)

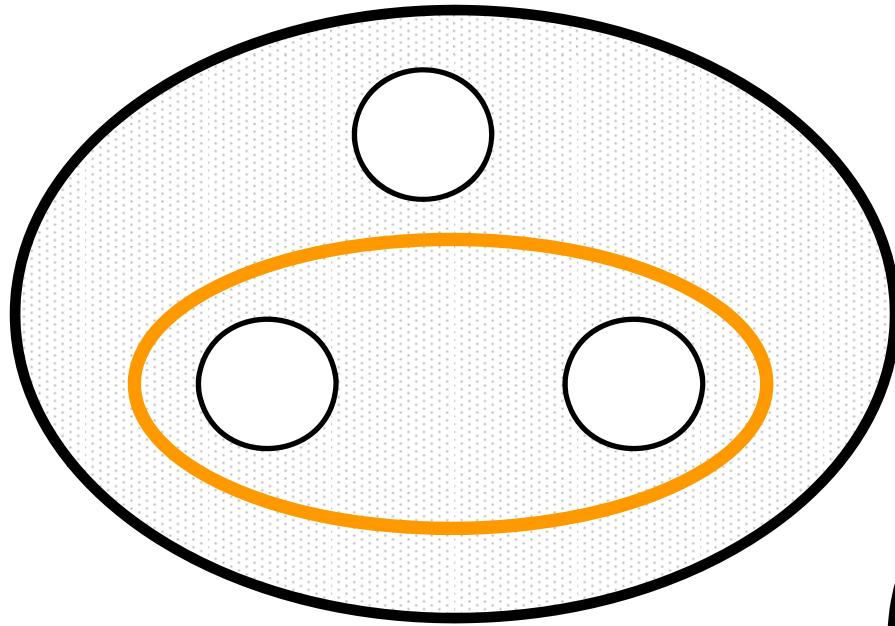
Curves on surfaces



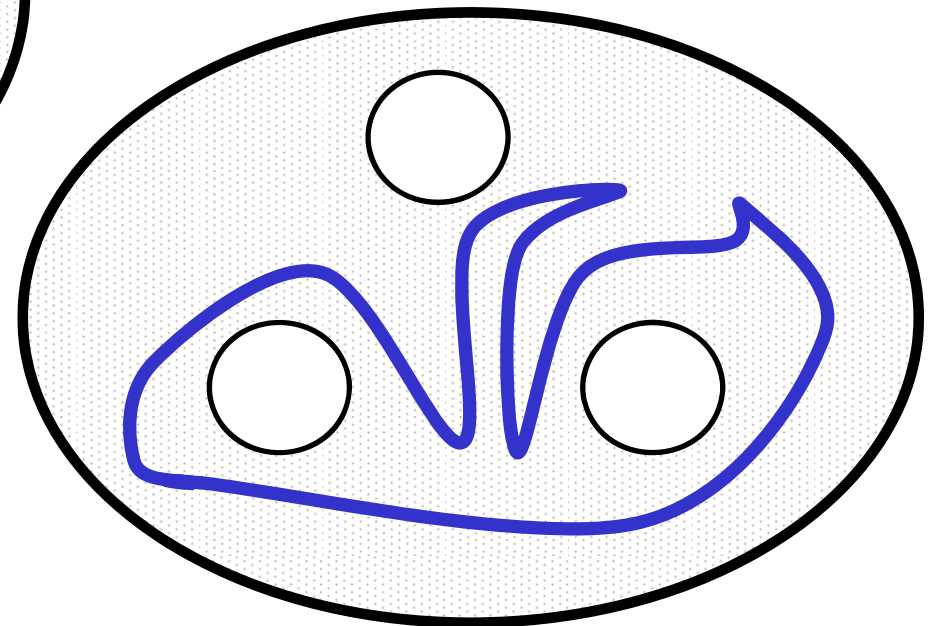
different? same?

homotopy equivalent

Curves on surfaces

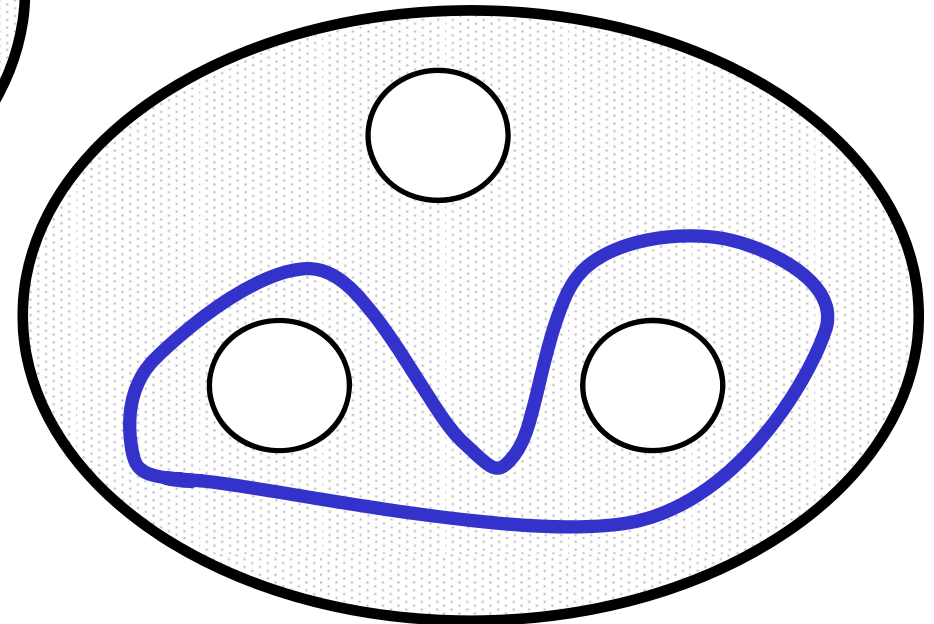
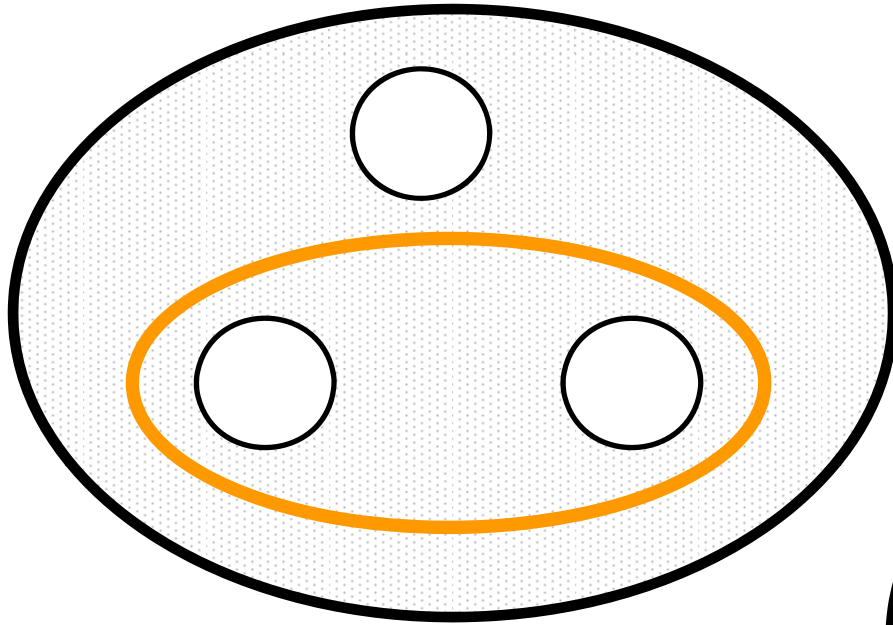


different? same?



homotopy equivalent

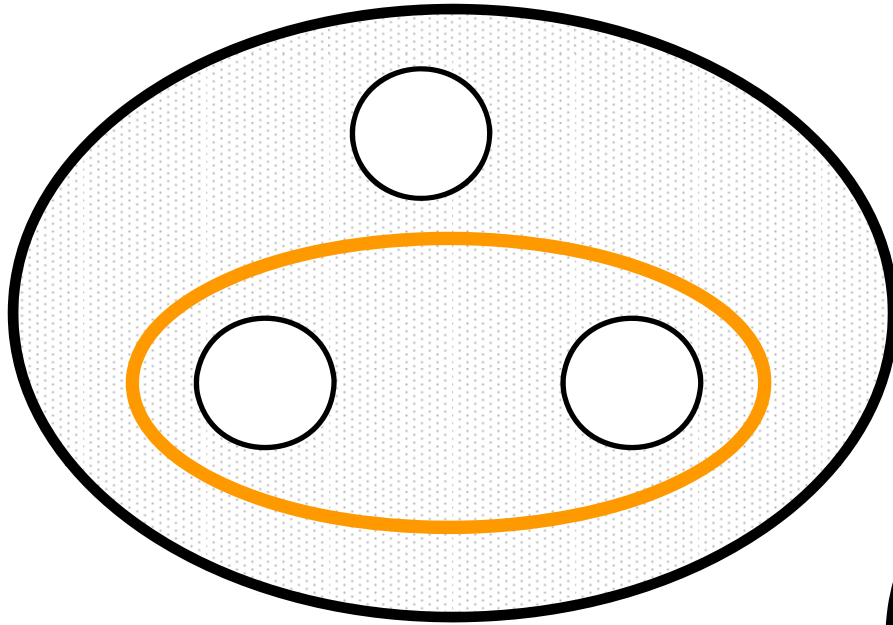
Curves on surfaces



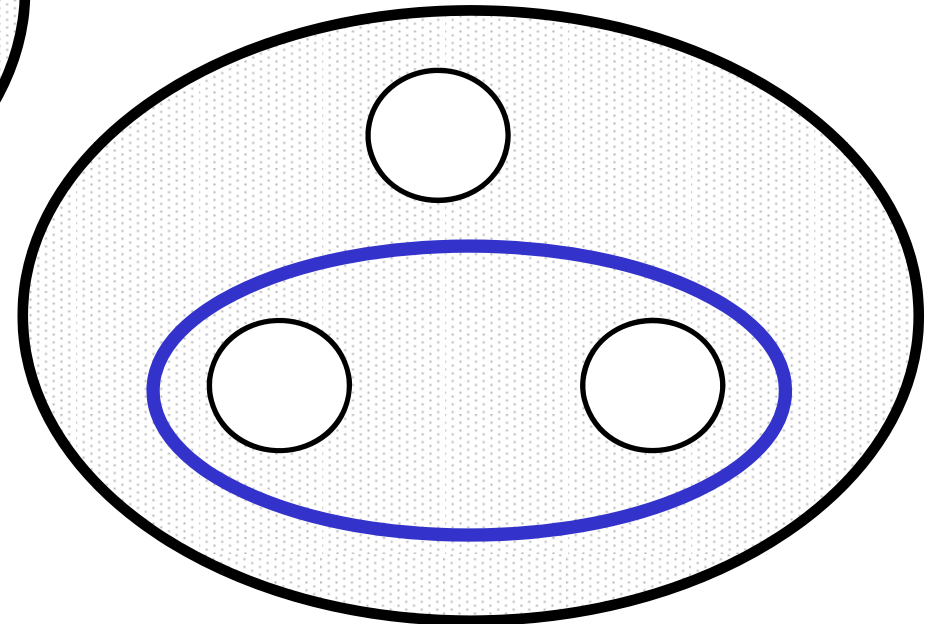
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homotopy equivalent

Curves on surfaces

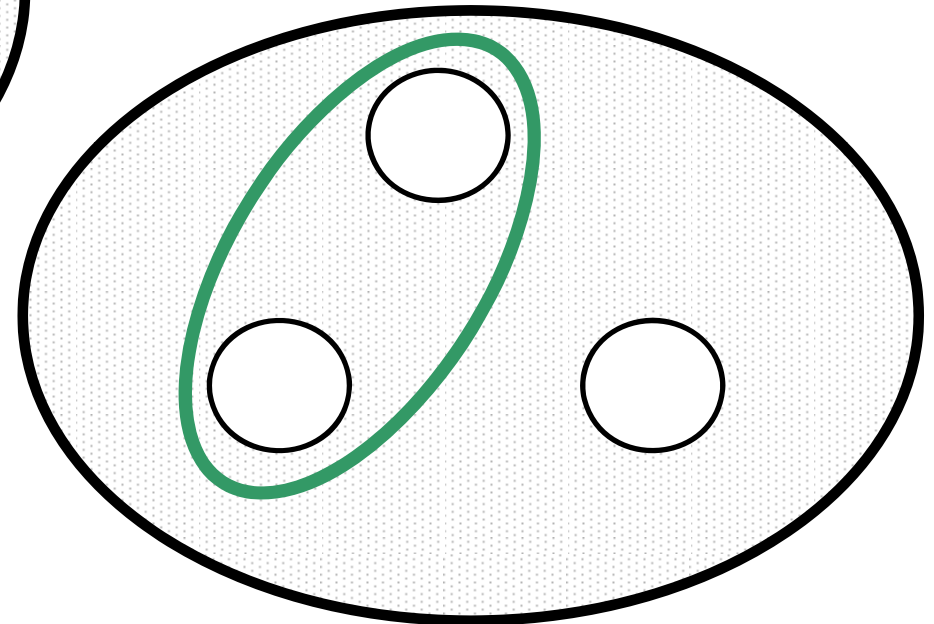
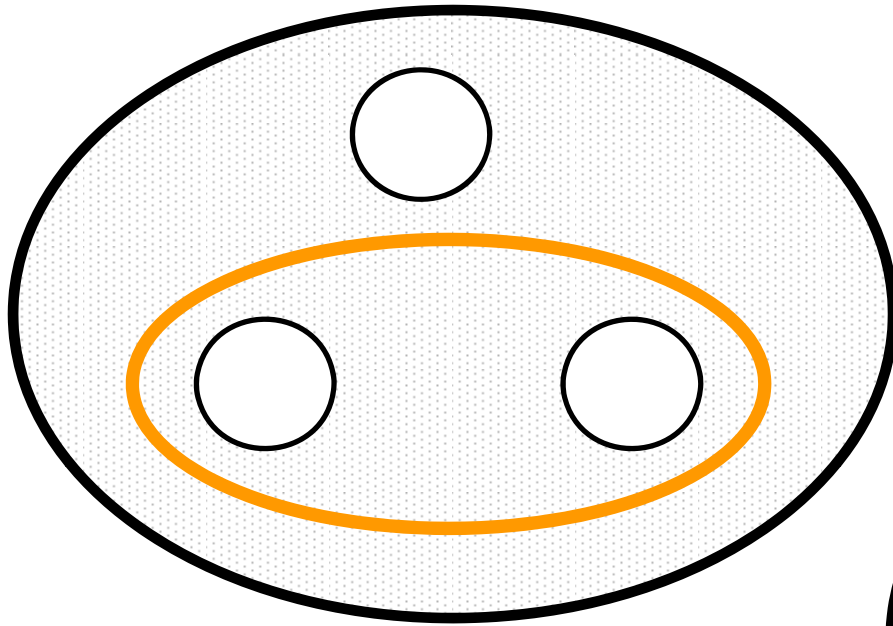


different? same?



homotopy equivalent

Curves on surfaces



not homotopy equivalent

Curves on surfaces

curves

continuous objects

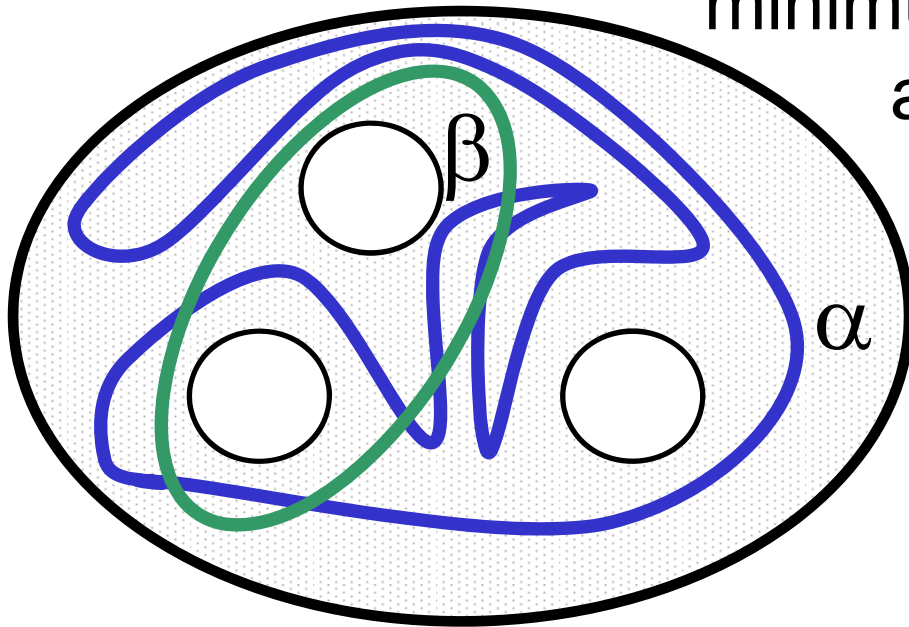
homotopy classes of curves

combinatorial objects

- 1) how to represent them?
- 2) what/how to compute?

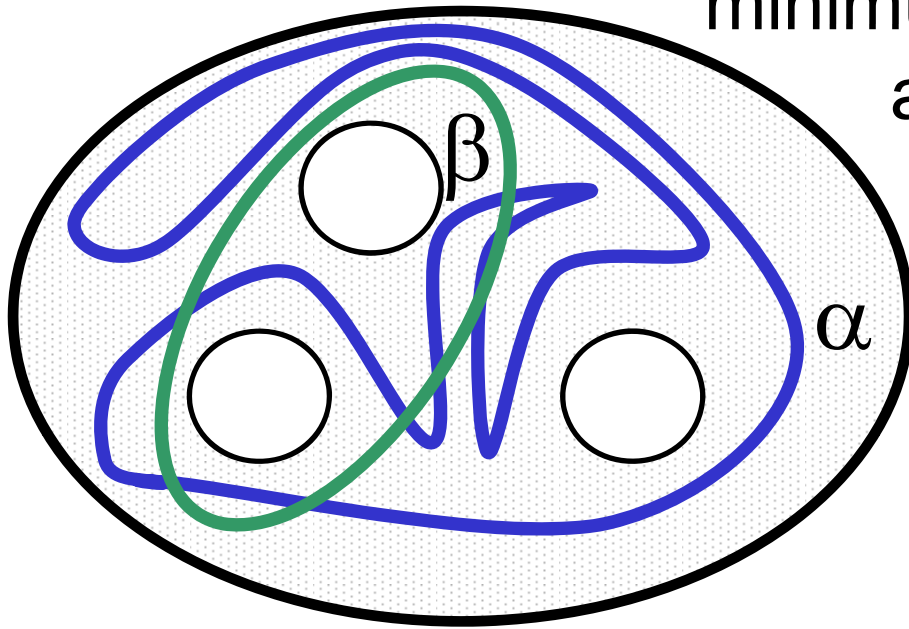
Geometric intersection number

minimum number of intersections
achievable by continuous
deformations.

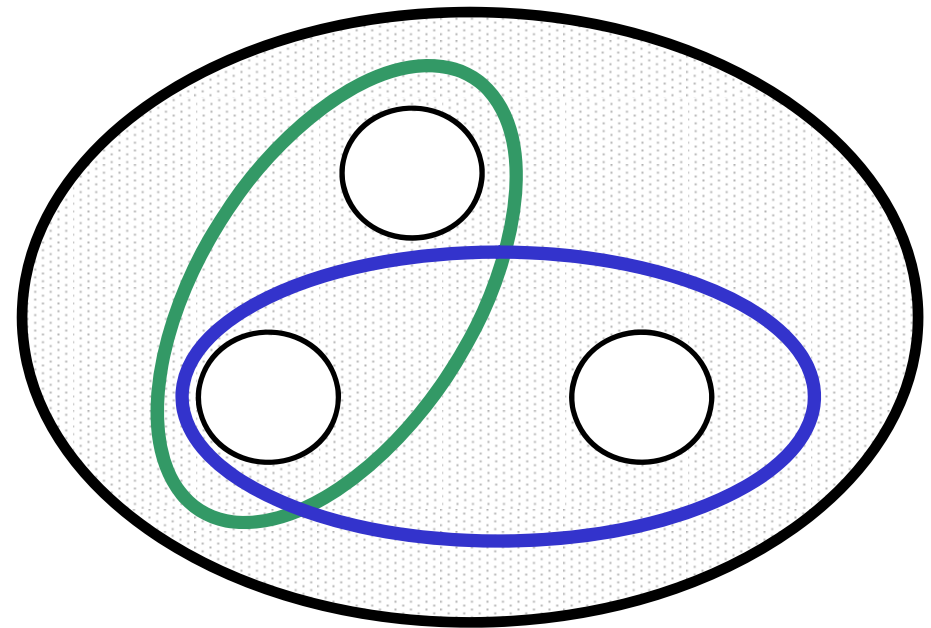


Geometric intersection number

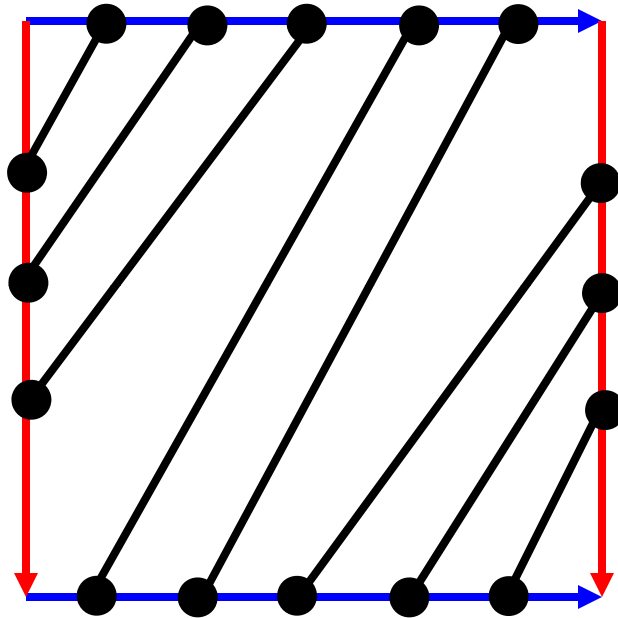
minimum number of intersections
achievable by continuous
deformations.



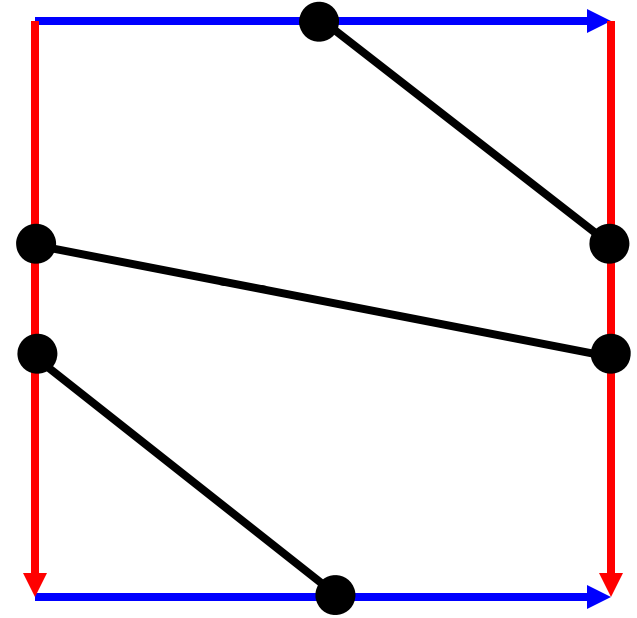
$$i(\alpha, \beta) = 2$$



EXAMPLE: Geometric intersection numbers are well understood on the torus



$(3, 5)$



$(2, -1)$

$$\det \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix} = -13$$

outline

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What is a Dehn twist and why is it interesting?

Computing Dehn twists.

Open questions.

Algorithmic problems - History

Contractibility (Dehn 1912)

can shrink curve to point?

Transformability (Dehn 1912)

are two curves homotopy equivalent?

Schipper '92; Dey '94; Schipper, Dey '95

Dey-Guha '99 (linear-time algorithm)

Simple representative (Poincaré 1895)

can avoid self-intersections?

Reinhart '62; Ziechang '65; Chillingworth '69

Birman, Series '84

Algorithmic problems - History

Geometric intersection number

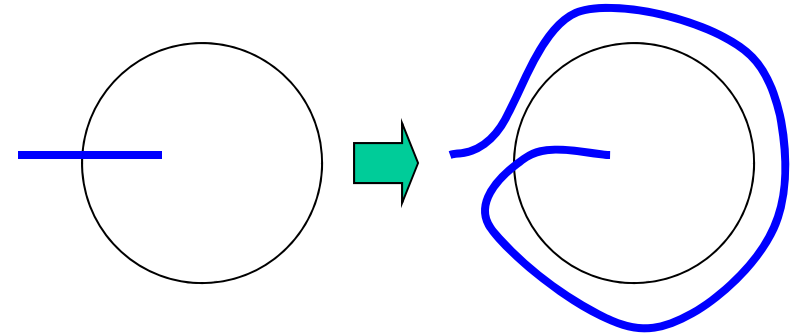
minimal number of intersections of two curves

Reinhart '62; Cohen, Lustig '87; Lustig '87;
Hamidi-Tehrani '97

polynomial only in explicit representations

Computing Dehn-twists

“wrap” curve along curve



Penner '84; Hamidi-Tehrani, Chen '96;
Hamidi-Tehrani '01

polynomial in compressed representations, but
only for fixed set of curves

Algorithmic problems – our results

Geometric intersection number

minimal number of intersections of two curves

Reinhart '62; Cohen, Lustig '87; Lustig '87;
Hamidi-Tehrani '97, [Schaefer-Sedgewick-Š '08](#)

polynomial in ~~explicit~~ [compressed](#) representations

Computing Dehn-twists

“wrap” curve along curve

Penner '84; Hamidi-Tehrani, Chen '96;
Hamidi-Tehrani '01, [Schaefer-Sedgewick-Š '08](#)

polynomial in compressed representations, for
~~fixed set of curves~~ [any pair](#) of curves

outline

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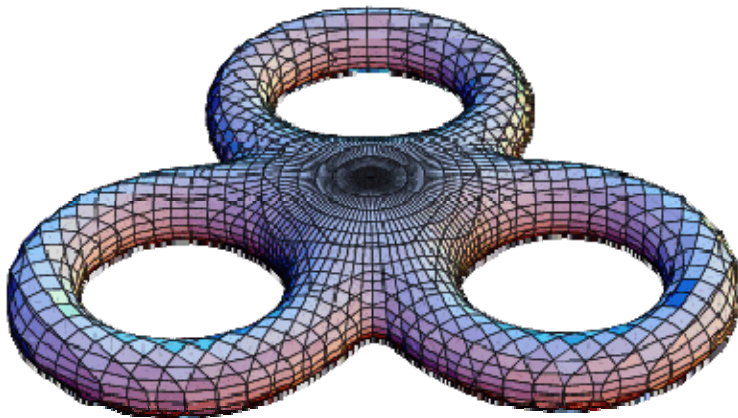
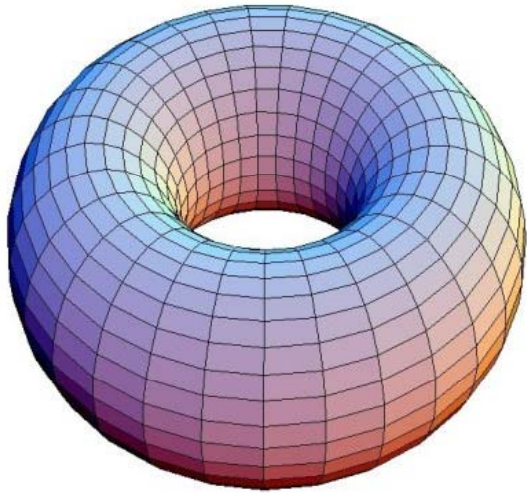
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Computing Dehn twists.

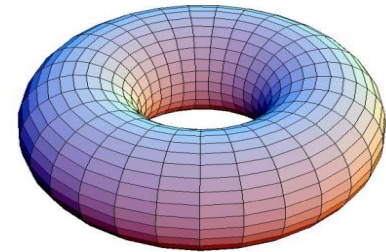
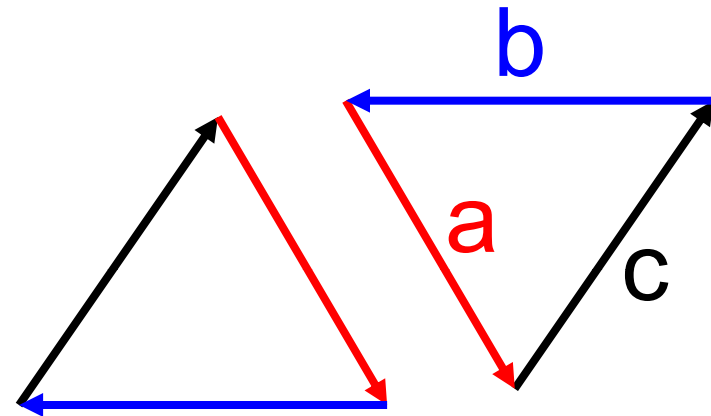
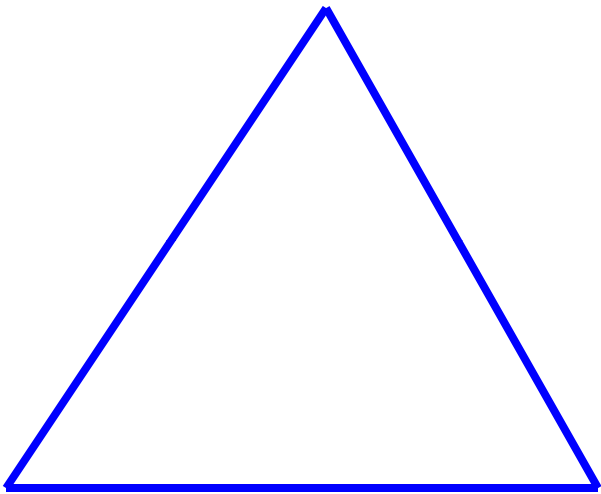
Open questions.

How to represent surfaces?



Combinatorial description of a surface

1. (pseudo) triangulation



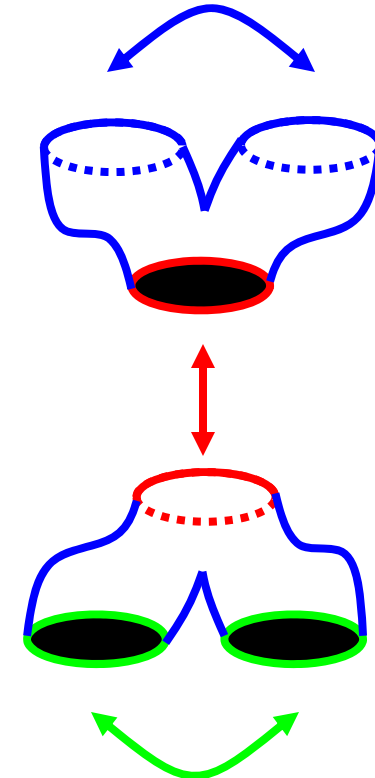
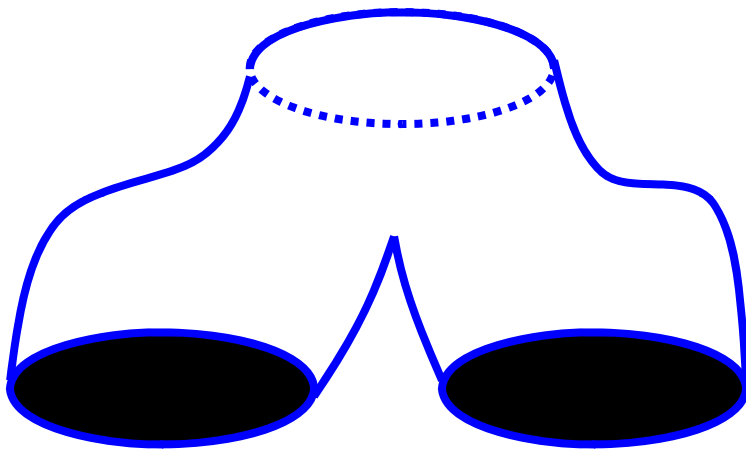
bunch of triangles

+ description of how to glue them

Combinatorial description of a surface



2. pair-of-pants decomposition



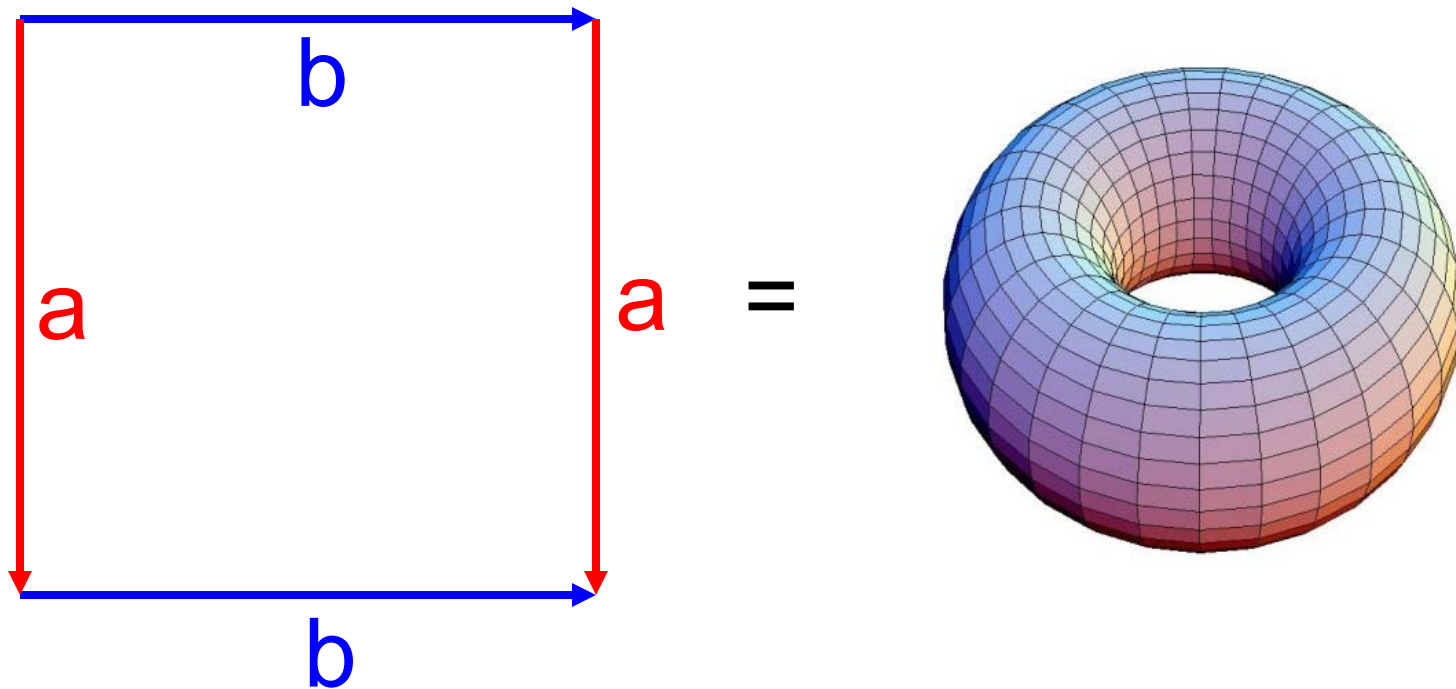
bunch of pair-of-pants

+ description of how to glue them

(cannot be used to represent: ball with ≤ 2 holes, torus)

Combinatorial description of a surface

3. polygonal schema



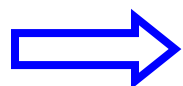
$2n$ -gon + pairing of the edges

outline

What is a curve?

Some history of algorithmic problems.

Representing surfaces.



Representing simple curves in surfaces.

Transforming between various representations.

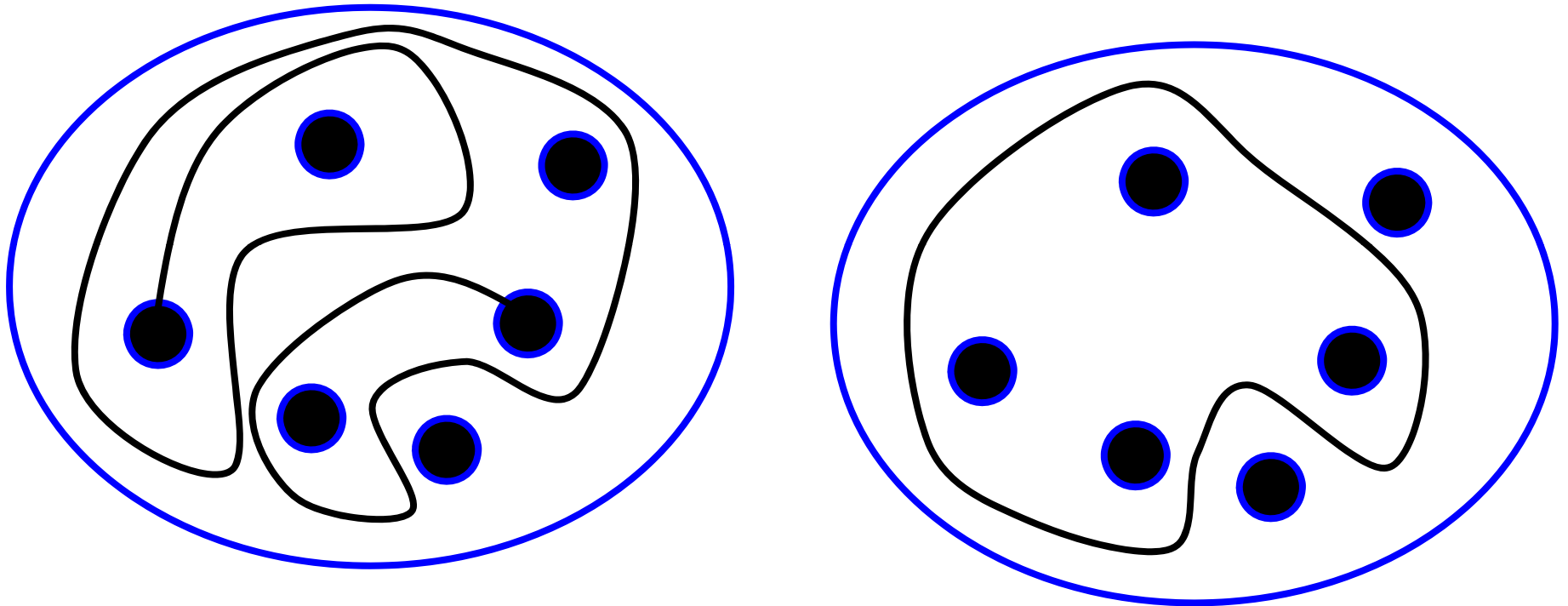
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Open questions.

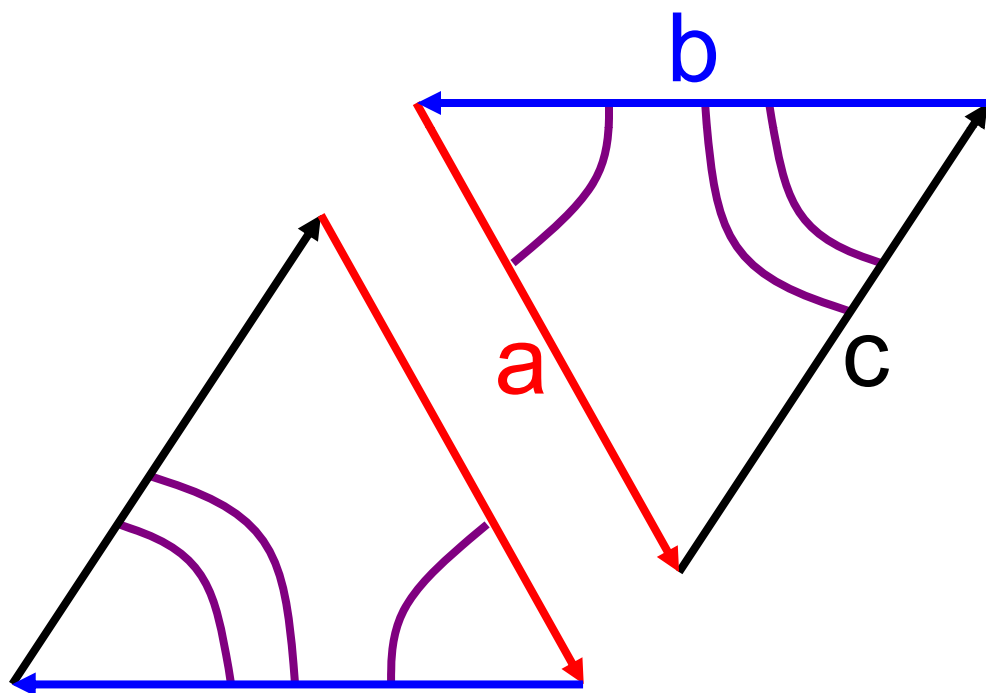
How to represent simple curves in surfaces (up to homotopy)?



Ideally the representation is “unique”
(each curve has a unique representation)

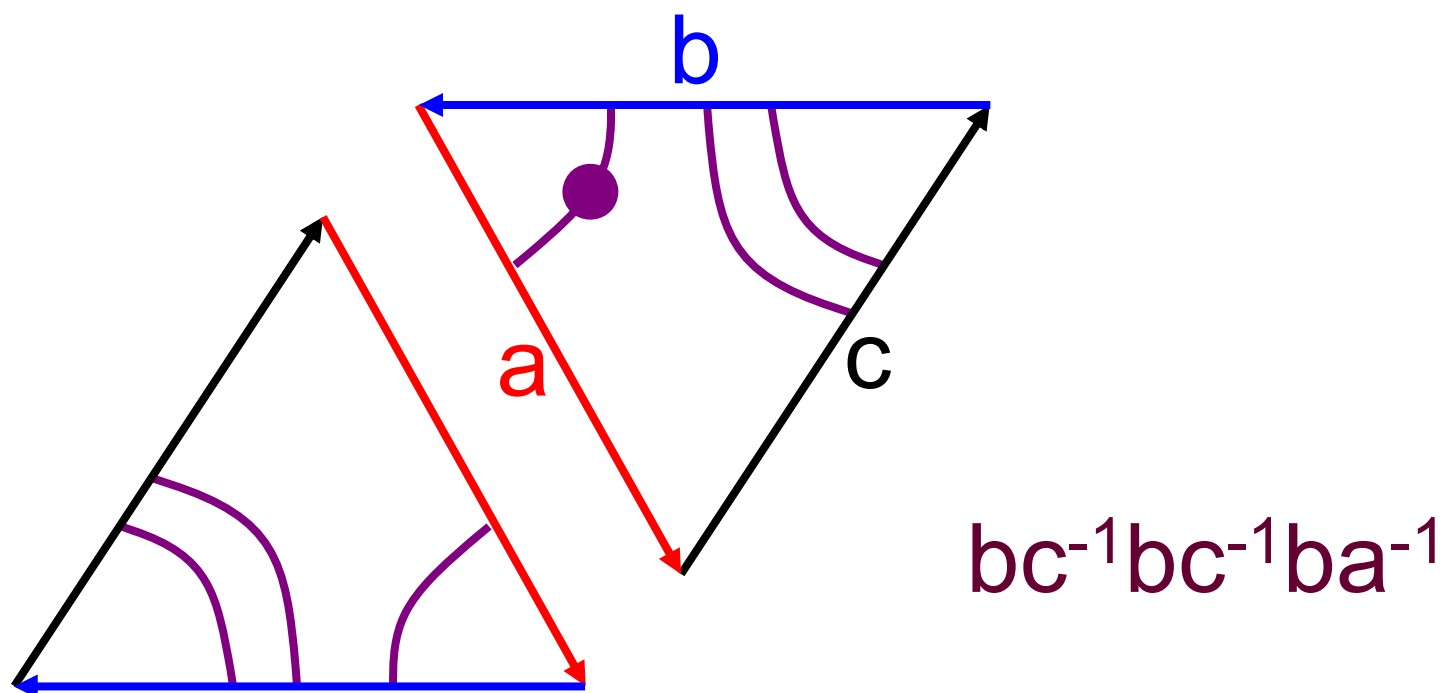
Combinatorial description of a (homotopy type of) a simple curve in a surface

1. intersection sequence with a triangulation



Combinatorial description of a (homotopy type of) a simple curve in a surface

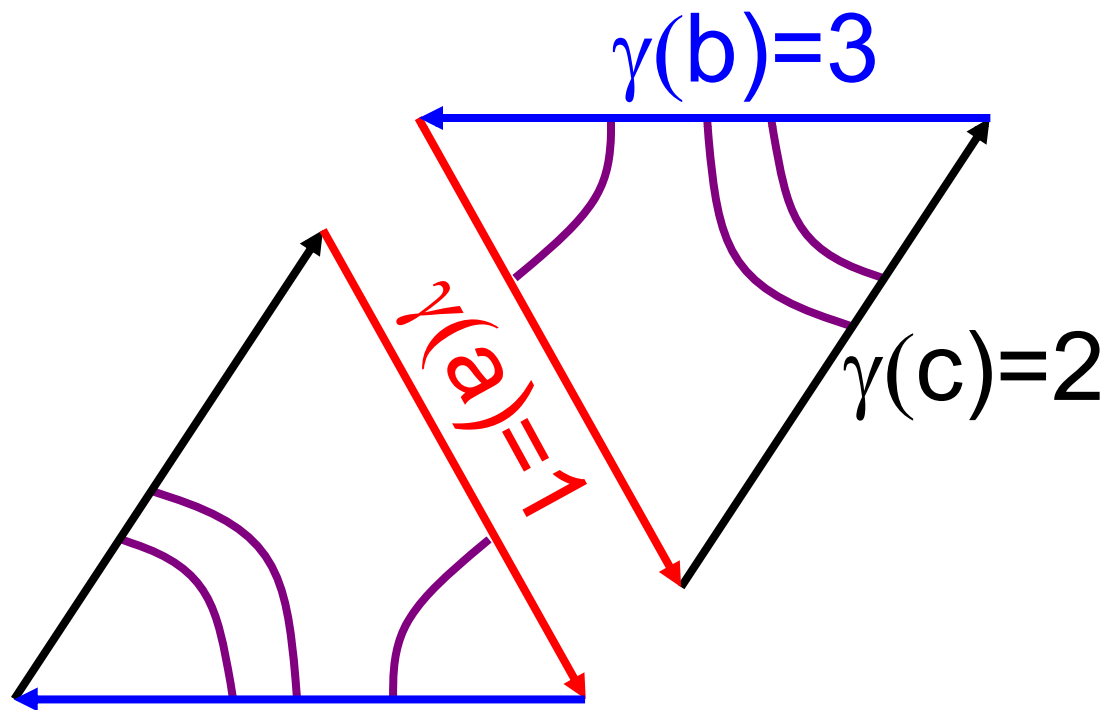
1. intersection sequence with a triangulation



almost unique if triangulation points on ∂S

Combinatorial description of a (homotopy type of) a simple curve in a surface

2. normal coordinates (w.r.t. a triangulation)

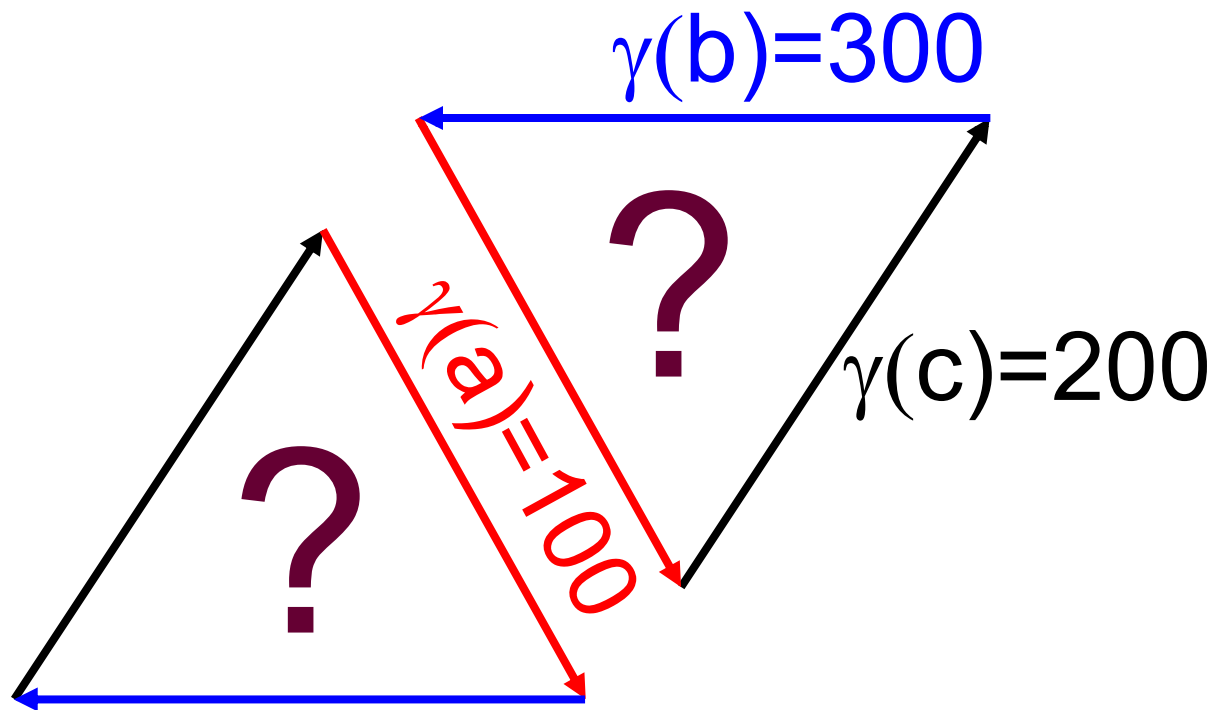


(Kneser '29)

unique if triangulation points on ∂S

Combinatorial description of a (homotopy type of) a simple curve in a surface

2. normal coordinates (w.r.t. a triangulation)



a very concise representation!

Recap:

1) how to represent them?

1. intersection sequence with a triangulation

$$bc^{-1}bc^{-1}ba^{-1}$$

2. normal coordinates (w.r.t. a triangulation)

$$\gamma(a)=1 \quad \gamma(b)=3 \quad \gamma(c)=2$$

2) what/how to compute?

geometric intersection number

outline

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TOOL: Word equations.

What is a Dehn twist and why is it interesting?

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Open questions.

STEP1: Moving between the representations

1. intersection sequence with a triangulation

$$bc^{-1}bc^{-1}ba^{-1}$$

2. normal coordinates (w.r.t. a triangulation)

$$\gamma(a)=1 \quad \gamma(b)=3 \quad \gamma(c)=2$$

Can we move between these two representations efficiently?

$$\gamma(a)=1+2^{100} \quad \gamma(b)=1+3 \cdot 2^{100} \quad \gamma(c)=2^{101}$$

STEP1: Moving between the representations

1. intersection sequence with a triangulation

$$bc^{-1}bc^{-1}ba^{-1}$$

2. normal coordinates (w.r.t. a triangulation)

$$\gamma(a)=1 \quad \gamma(b)=3 \quad \gamma(c)=2$$

Can we move between these two representations efficiently?

YES

$$\gamma(a)=1+2^{100} \quad \gamma(b)=1+3 \cdot 2^{100} \quad \gamma(c)=2^{101}$$

Theorem (SSS'08):

normal coordinates \rightarrow compressed intersection sequence
in time $O(\sum \log \gamma(e))$

compressed intersection sequence \rightarrow normal coordinates
in time $O(|T|. \text{SLP-length}(S))$

compressed = straight line program (SLP)

$X_0 := a$

$X_1 := b$

$X_2 := X_1 X_1$

$X_3 := X_0 X_2$

$X_4 := X_2 X_1$

$X_5 := X_4 X_3$

$X_5 = bbbabb$

outline

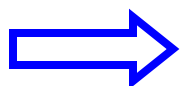
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TOOL: Word equations.

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Main tool: Word equations

$$xax = yxy$$

x, y – variables

a, b - constants

Main tool: Word equations

$$xabx = yxy$$

x, y – variables
 a, b - constants

a solution:

$$x=ab$$

$$y=ab$$

Word equations with given lengths

$$xayxb = axbxy$$

x, y – variables

a, b - constants

additional constraints: $|x|=4, |y|=1$

Word equations with given lengths

$$xayxb = axbxy$$

x, y – variables

a, b – constants

additional constraints: $|x|=4, |y|=1$

a solution:

$$x=aaaa$$

$$y=b$$

Word equations

word equations

word equations with given lengths

Word equations

word equations - **NP-hard**

decidability – Makanin 1977

PSPACE – Plandowski 1999

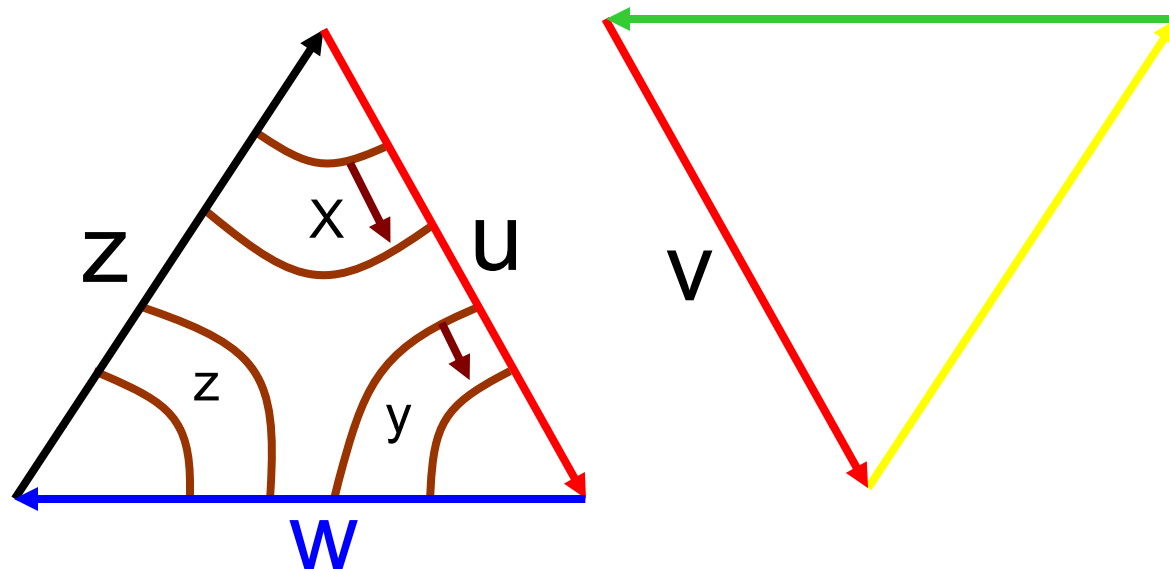
word equations with given lengths

Plandowski, Rytter '98 – **polynomial time** algorithm

Diekert, Robson '98 – **linear time** for quadratic eqns

(quadratic = each variable occurs ≤ 2 times)

Simulating curve using quadratic word equations



$$|u|=|v|=\gamma(u) \quad |x|=(|z|+|u|-|w|)/2$$

...

$$u=xy$$

...

$$v=u$$

—————→
Diekert-Robson

number of
components

Moving between the representations

1. intersection sequence with a triangulation

$$bc^{-1}bc^{-1}ba^{-1}$$

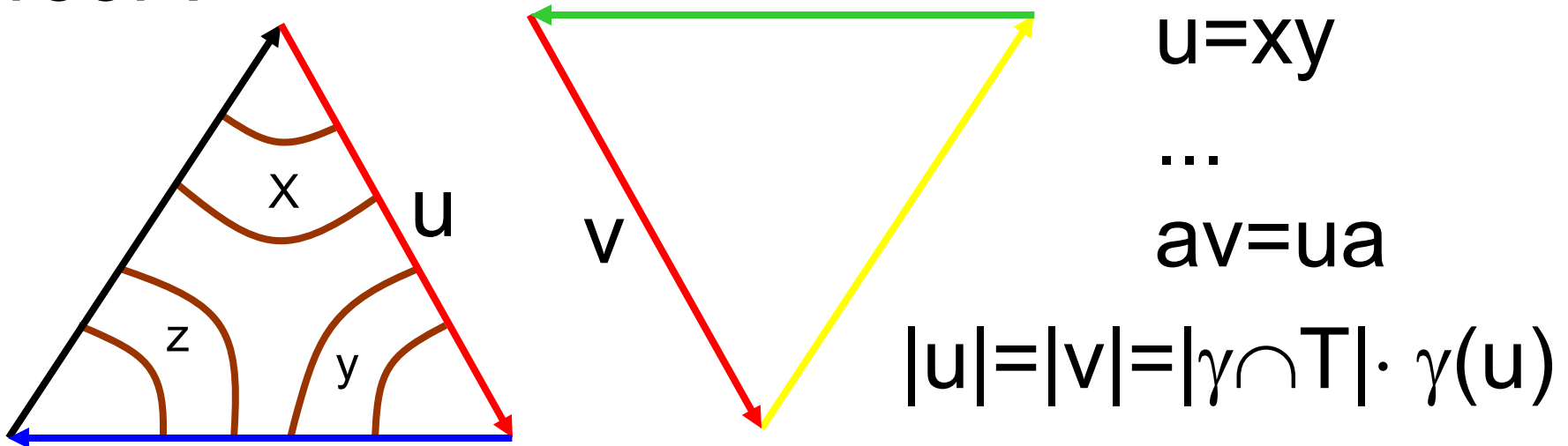
2. normal coordinates (w.r.t. a triangulation)

$$\gamma(a)=1 \quad \gamma(b)=3 \quad \gamma(c)=2$$

Theorem:

normal coordinates \rightarrow compressed intersection sequence
in time $O(\sum \log \gamma(e))$

“Proof”:



outline

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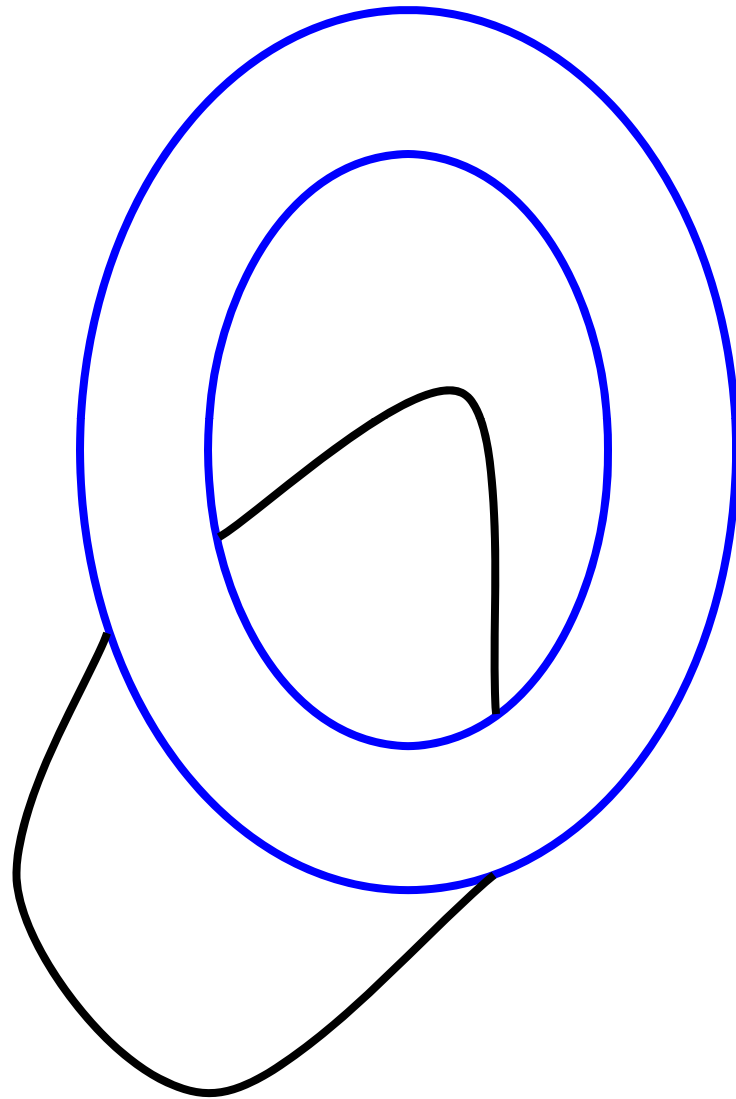
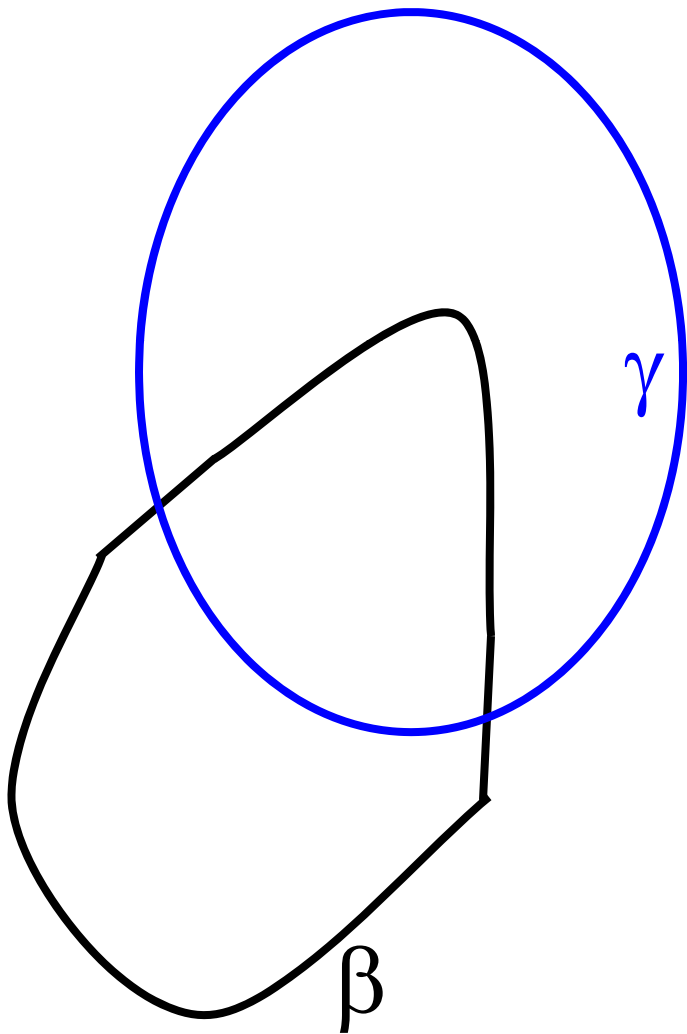


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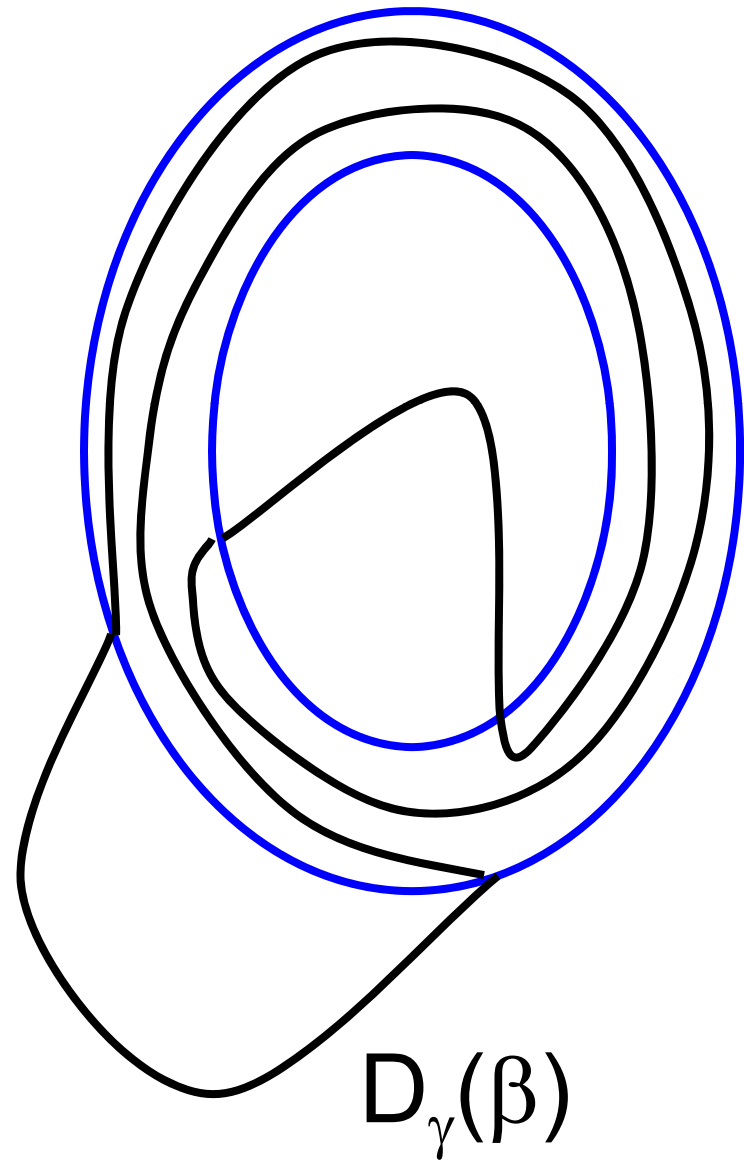
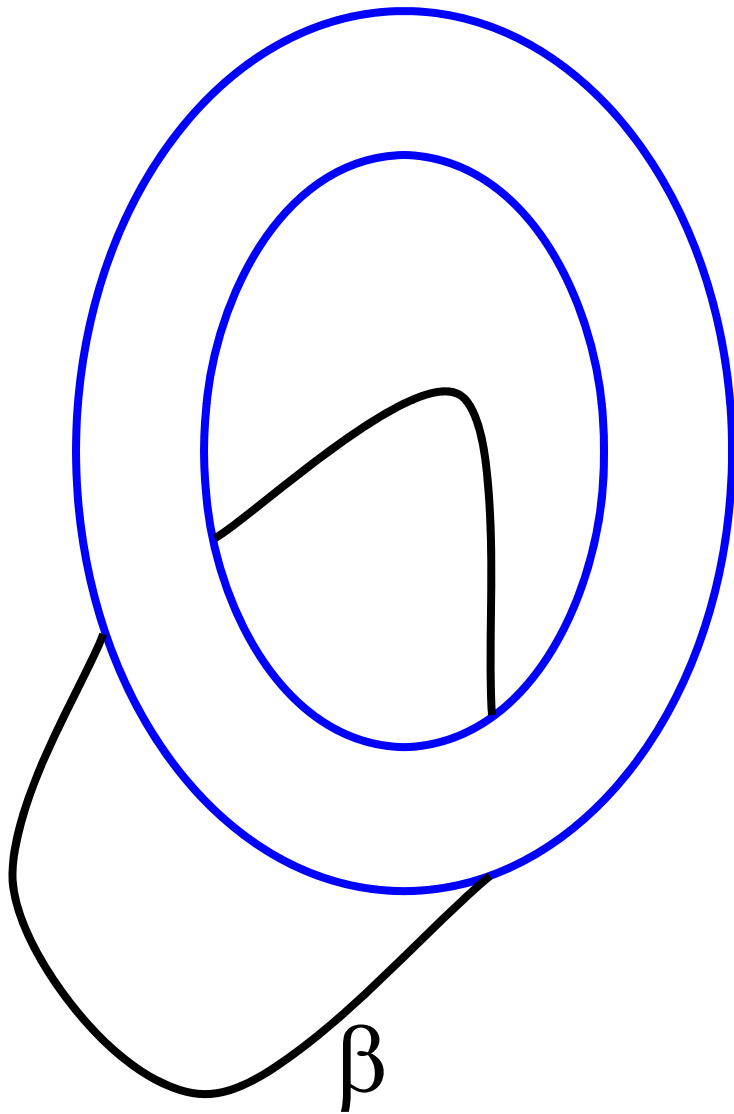
Computing Dehn twists.

Open questions.

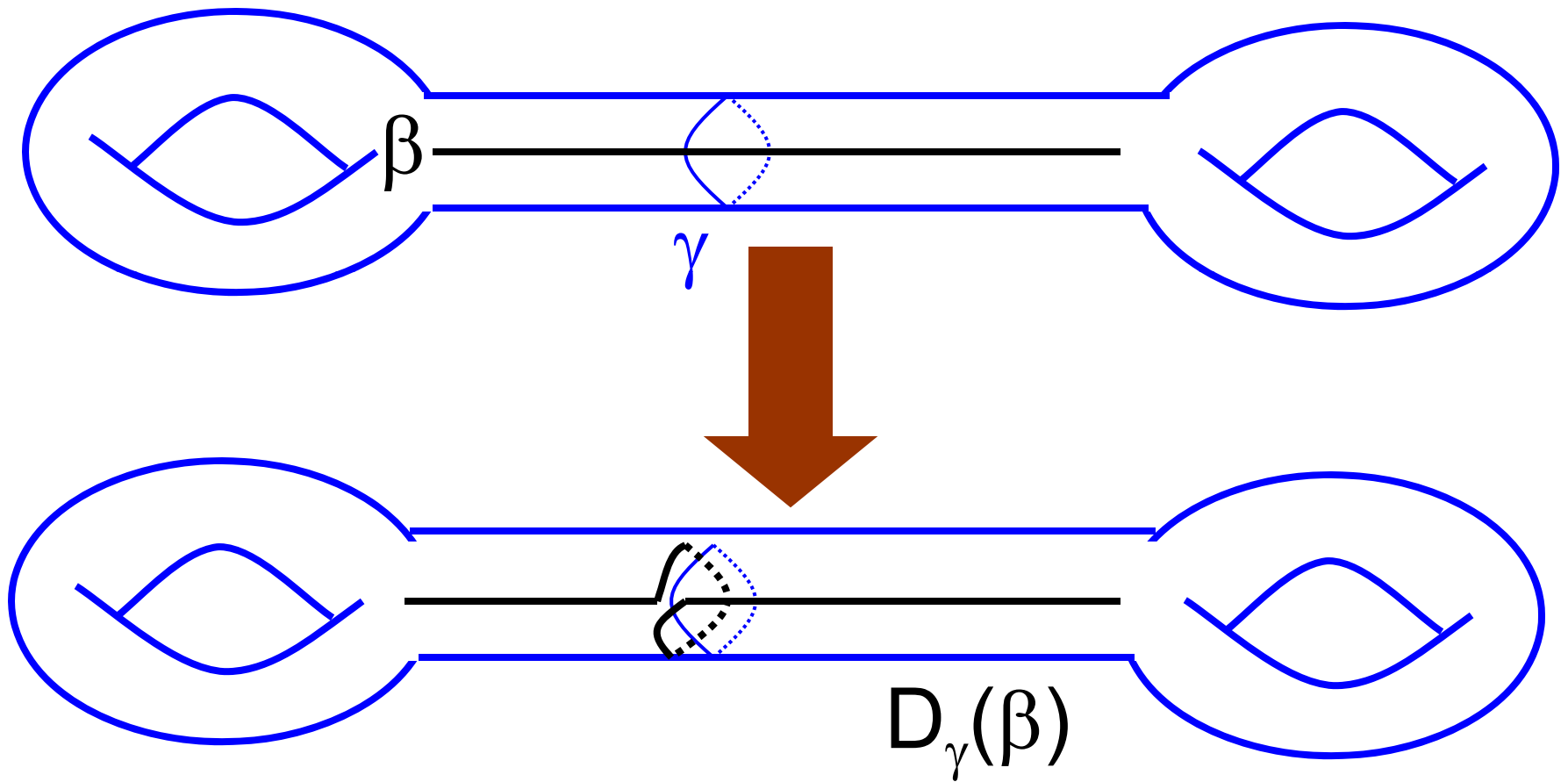
Dehn twist of β along γ



Dehn twist of β along γ

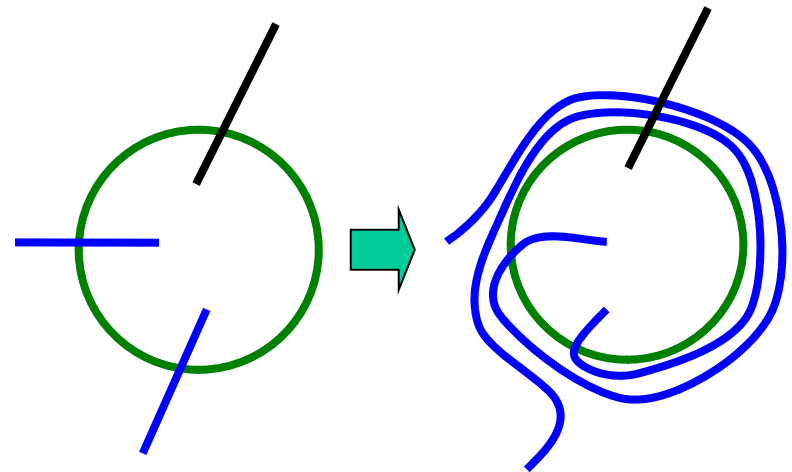
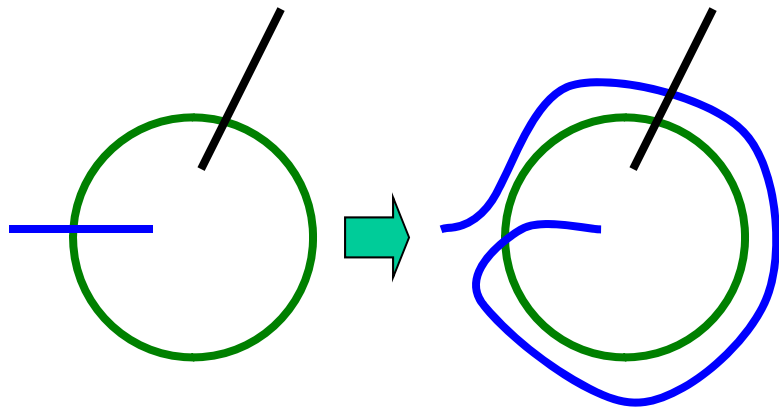


Dehn twist of β along γ



Geometric intersection numbers

$$i(\alpha, D_\gamma^n(\beta)) / i(\alpha, \gamma) \neq i(\gamma, \beta)$$



$$n \neq i(\alpha, \gamma) i(\gamma, \beta) - i(\alpha, \beta)$$

$$\leq i(\alpha, D_\gamma^n(\beta)) \leq$$

$$n \neq i(\alpha, \gamma) i(\gamma, \beta) + i(\alpha, \beta)$$

outline

What is a curve?

Some history of algorithmic problems.

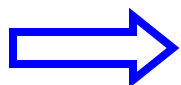
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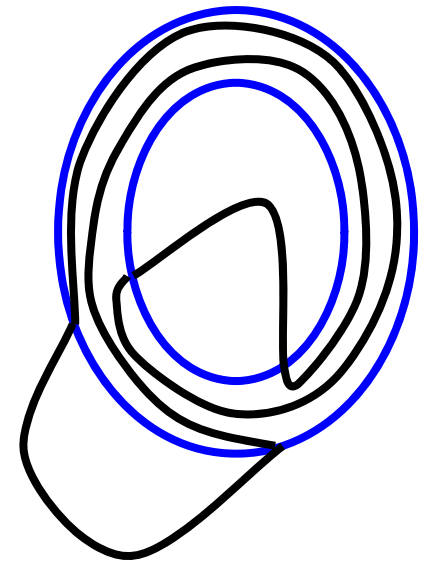


Computing Dehn twists.

Open questions.

Computing Dehn-Twists (outline)

1. normal coordinates ! word equations
with given lengths
2. solution = compressed intersection
sequence with triangulation
3. sequences ! (non-reduced) word for
Dehn-twist (substitution in SLPs)
4. Reduce the word ! normal coordinates



outline

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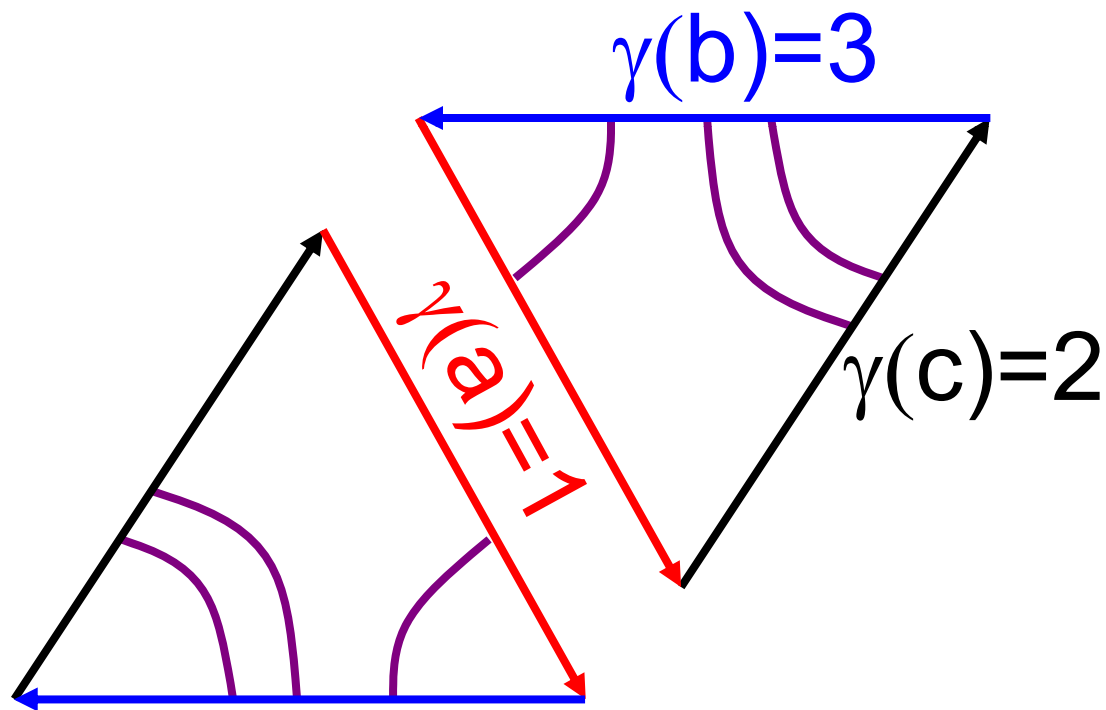
Computing Dehn twists.



Open questions.

PROBLEM #1: Minimal weight representative

2. normal coordinates (w.r.t. a triangulation)



unique if triangulation points on ∂S

PROBLEM #1: Minimal weight representative

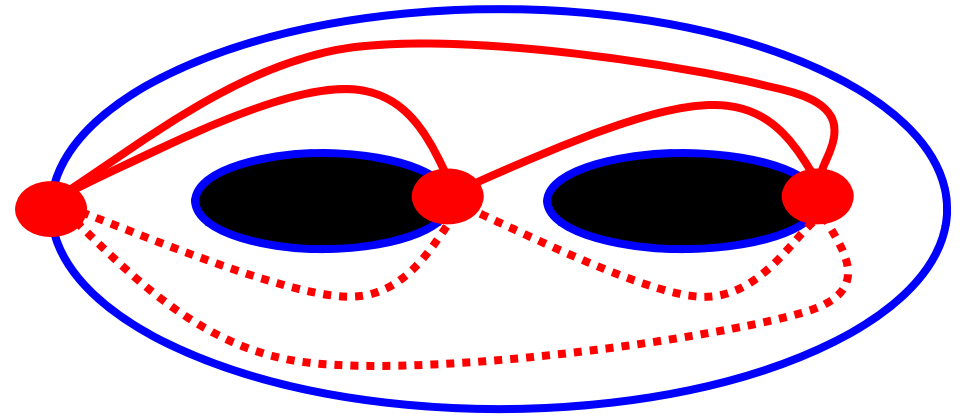
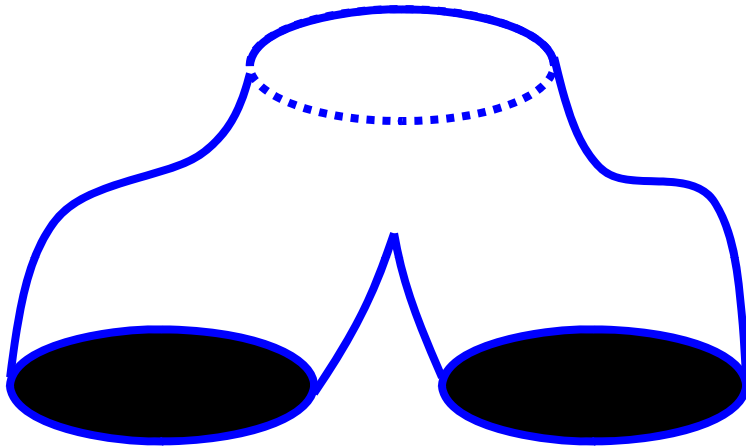
INPUT: triangulation + gluing
normal coordinates of γ
edge weights

OUTPUT: $\gamma' \sim \gamma$
minimizing $\sum_{e \in T} \gamma'(e)$

PROBLEM #2: Moving between representations

3. Dehn-Thurston coordinates

(Dehn '38, W.Thurston '76)



unique representation for closed surfaces!

PROBLEM

normal coordinates \rightarrow Dehn-Thurston coordinates

in polynomial time? linear time?

PROBLEM #3: Word equations

NP-hard

decidability – Makanin 1977

PSPACE – Plandowski 1999

PROBLEM:

are word equations in NP?

are quadratic word equations in NP?

PROBLEM #4: Computing Dehn-Twists faster?

1. normal coordinates ! word equations
with given lengths

2. solution = compressed intersection
sequence with triangulation

3. sequences ! (non-reduced) word for
Dehn-twist (substitution in SLPs)

4. Reduce the word ! normal coordinates

$O(n^3)$ randomized, $O(n^9)$ deterministic

