Counting Euler tours?

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Euler tour

1) every edge exactly once
2) end where started
Euler tour

1) every edge exactly once
2) end where started

not an Euler tour
Basic facts

- there exists one if and only if all vertices have even degree (Eulerian graph)
- an Euler tour in an Eulerian graph can be found in linear time.

Can we find a random one?
Can we count their number? (efficiently)
Can we count their number?

→ polynomial time algorithm

→ 264
Can we count their number?

**undirected graphs**

#P-complete

(Brightwell-Winkler’05)

**directed graphs**

polynomial-time algorithm known

(using spanning trees)

eact counting in poly-time unlikely
Can we count their number? ^ approximately

(open question, listed, e.g., in Approximation algorithms (V. Vazirani))

self-reducible \Rightarrow approximate counting \Leftrightarrow approximate sampling (Jerrum, Valiant, V. Vazirani’86)

\[ 264 \pm 10\% \]
Can we find a random one?

4-regular graphs

Markov chain

pick a random vertex \(v\)
locally change the tour at \(v\)

(exactly 2 of these are valid)
Can we find a random one?

Markov chain

$X_1, X_2, X_3, \ldots, X_t, \ldots$

Can we find a random one?

4-regular graphs

Markov chain

pick a random vertex $v$
locally change the tour at $v$

(exactly 2 of these are valid)

OPEN:
is the mixing time polynomial?

$L_1$ distance to uniform distribution

$$\sum_{a} |P(X_t=a) - \frac{1}{M}|$$

mixing time =
t to get $L_1$ distance $\leq \varepsilon/2$
Can we find a random one?

Markov chain

pick a vertex \( v \)
locally change the tour at \( v \)

(exactly 2 of these are valid)
Counting A-trails in a map

vertices with “rotations”
map

vertices with “rotations”
and a graph
A-trail = euler tour without crossings

not allowed
A-trail
Can we (approximately) count their number?

for planar maps
yes (Kotzig’68)

for general maps?
Our results: A-trails in 4-reg enough approximate sampling/counting of A-trails in 4-regular maps

approximate sampling/counting of Euler tours in Eulerian graphs

(AP-reduction (Goldberg, Dyer, Greenhill, Jerrum’04))
A-trails in 4-reg enough

approximate sampling/counting
of A-trails in 4-regular maps

→

approximate sampling/counting
of Euler tours in 4-regular graphs
approximate sampling/counting of A-trails in 4-regular maps

approximate sampling/counting of Euler tours in 4-regular graphs
A-trails in 4-reg enough

approximate sampling/counting
of A-trails in 4-regular maps

→

approximate sampling/counting
of Euler tours in Eulerian graphs

1st layer

2nd layer

T layers
Theorem (Wilson’04): in $O(d^3 \ln^2 d \ln(1/\varepsilon))$ steps get $\varepsilon/2$ $L_1$ distance from uniform on permutations.
Exact: A-trails in 4-reg enough

exact counting
of A-trails in 4-regular maps

exact counting
of Euler tours in Eulerian graphs

(corollary: counting A-trails in 4-regular graphs #P-complete)
Exact: A-trails in 4-reg enough
Exact: A-trails in 4-reg enough
Our results: A-trails in 4-reg enough approximate sampling/counting of A-trails in 4-regular maps

approximate sampling/counting of Euler tours in Eulerian graphs

(AP-reduction)
(Goldberg, Dyer, Greenhill, Jerrum’04)
Questions:

- AP reduction from Euler tours in Eulerian graphs to Euler tours in 4-regular graphs?
- Approximate sampling/counting of Euler tours/A-trails?
- Which subsets of the hypercube can be sampled from?