

# **Odd Crossing Number is NOT Crossing Number**

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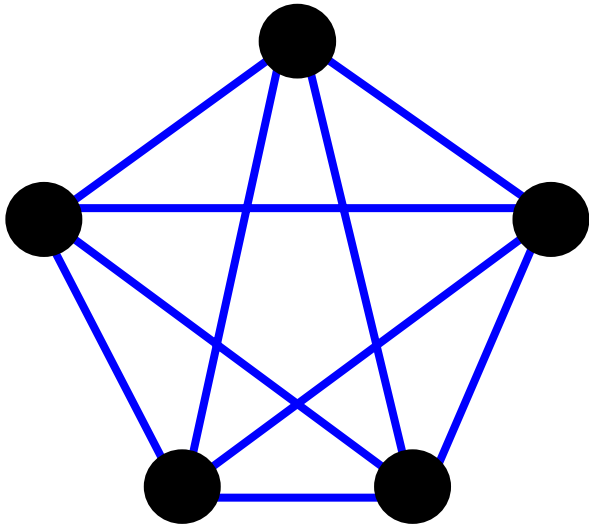
DePaul University (Chicago)

Daniel Štefankovič

University of Rochester

# Crossing number

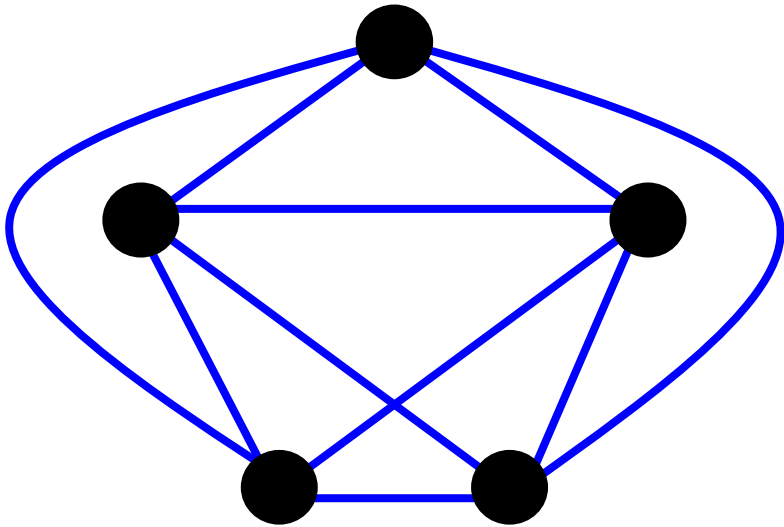
$\text{cr}(G)$  = minimum number of crossings  
in a **planar** drawing of  $G$



$$\text{cr}(K_5) = ?$$

# Crossing number

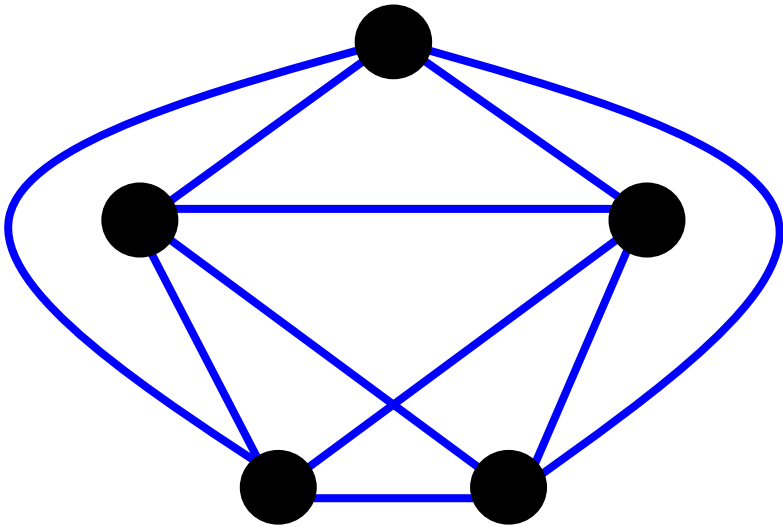
$\text{cr}(G)$  = minimum number of crossings  
in a **planar** drawing of  $G$



$$\text{cr}(K_5) = 1$$

# Rectilinear crossing number

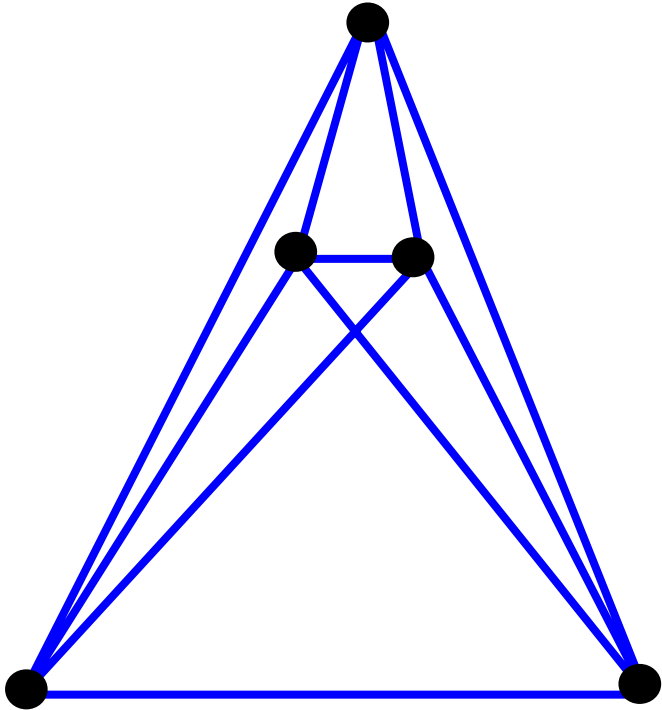
$\text{rcr}(G)$  = minimum number of crossings  
in a **planar straight-line**  
drawing of  $G$



$\text{rcr}(K_5) = ?$

# Rectilinear crossing number

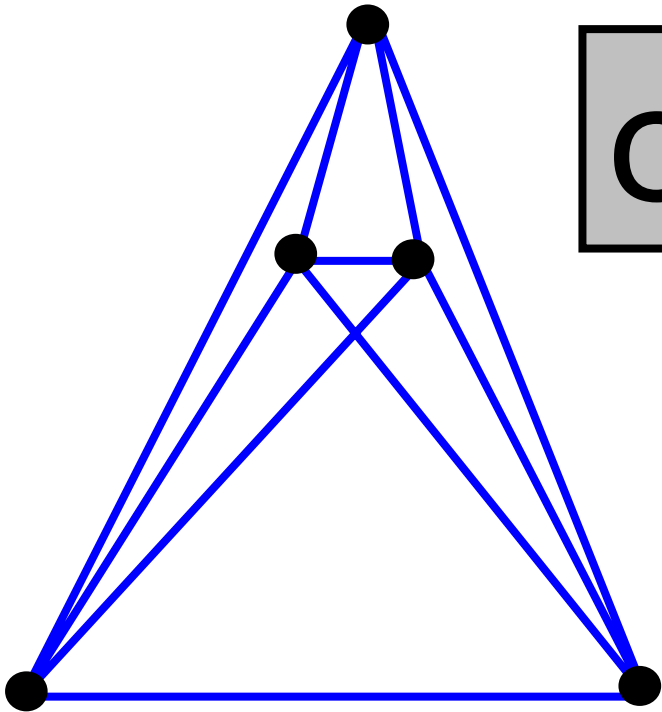
$\text{rcr}(G)$  = minimum number of crossings  
in a **planar straight-line**  
drawing of  $G$



$$\text{rcr}(K_5) = 1$$

# Rectilinear crossing number

$\text{rcr}(G)$  = minimum number of crossings  
in a **planar straight-line**  
drawing of  $G$



$$\text{cr}(G) \leq \text{rcr}(G)$$

$$\text{cr}(G)=0 \Leftrightarrow \text{rcr}(G)=0$$

THEOREM [SR34,W36,F48,S51]:

Every planar graph has a  
straight-line planar drawing.

Steinitz, Rademacher 1934

Wagner 1936

Fary 1948

Stein 1951

# Are they equal?

$$\text{cr}(G) = 0 \iff \text{rcr}(G) = 0$$

$$\text{cr}(G) = 1 \iff \text{rcr}(G) = 1$$

$$\text{cr}(G) = 2 \iff \text{rcr}(G) = 2$$

$$\text{cr}(G) = 3 \iff \text{rcr}(G) = 3$$

$$\text{cr}(G) \stackrel{?}{=} \text{rcr}(G)$$



$$\text{cr}(G) \neq \text{rcr}(G)$$

THEOREM [Guy' 69]:

$$\text{cr}(K_8) = 18$$

$$\text{rcr}(K_8) = 19$$

$$\text{cr}(G) \neq \text{rcr}(G)$$

$$\text{cr}(G) \neq \text{rcr}(G)$$

THEOREM [Guy' 69]:

$$\text{cr}(K_8) = 18$$

$$\text{rcr}(K_8) = 19$$

THEOREM [Bienstock, Dean '93]:

$$(\forall k)(\exists G)$$

$$\text{cr}(G) = 4$$

$$\text{rcr}(G) = k$$

# Crossing numbers

$cr(G)$  = minimum number of crossings  
in a **planar** drawing of  $G$

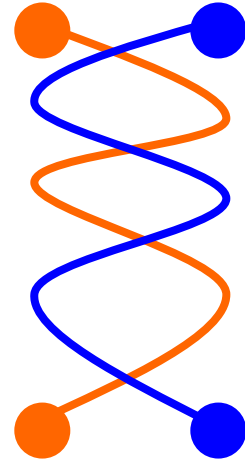
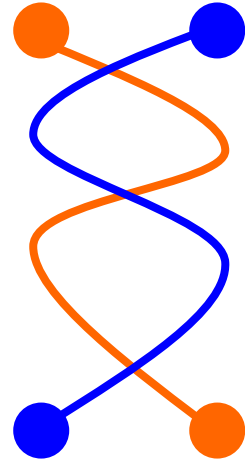
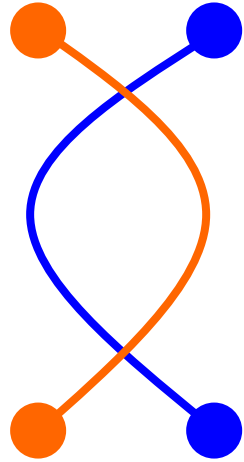
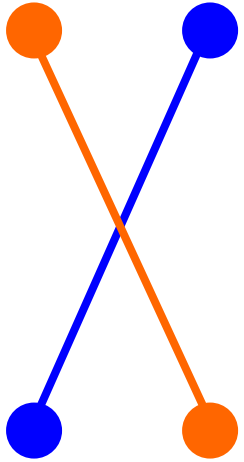
$rcr(G)$  = minimum number of crossings  
in a **planar straight-line**  
drawing of  $G$

$$cr(G) \leq rcr(G)$$

$$(\exists G) cr(G) \neq rcr(G)$$

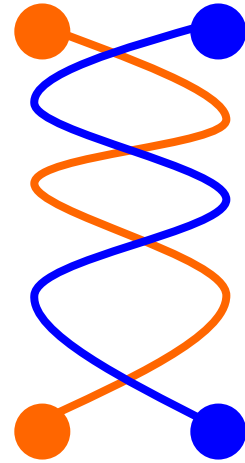
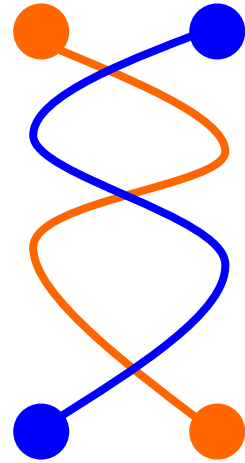
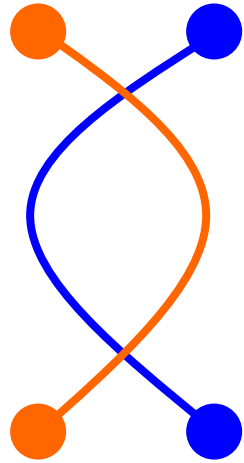
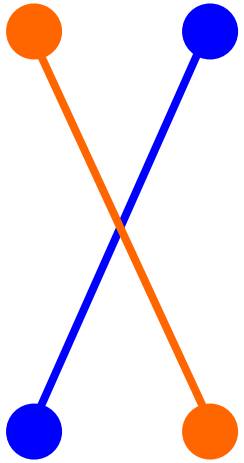
# Odd crossing number

$\text{ocr}(G)$  = minimum number of pairs of edges crossing odd number of times



# Odd crossing number

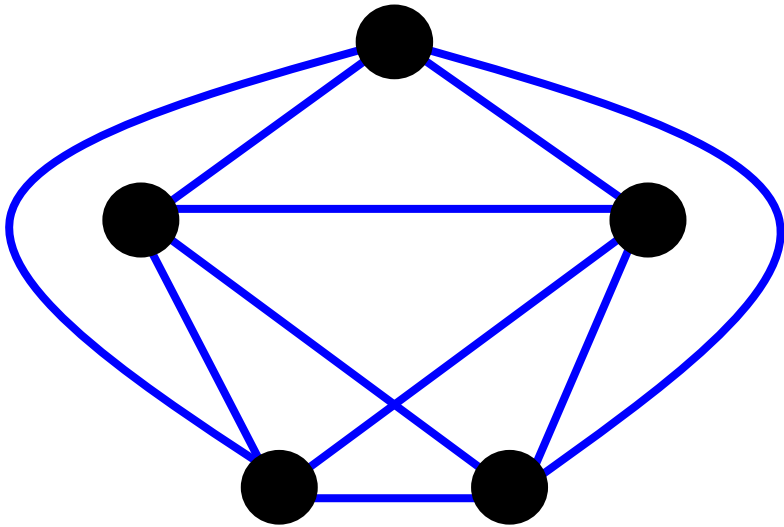
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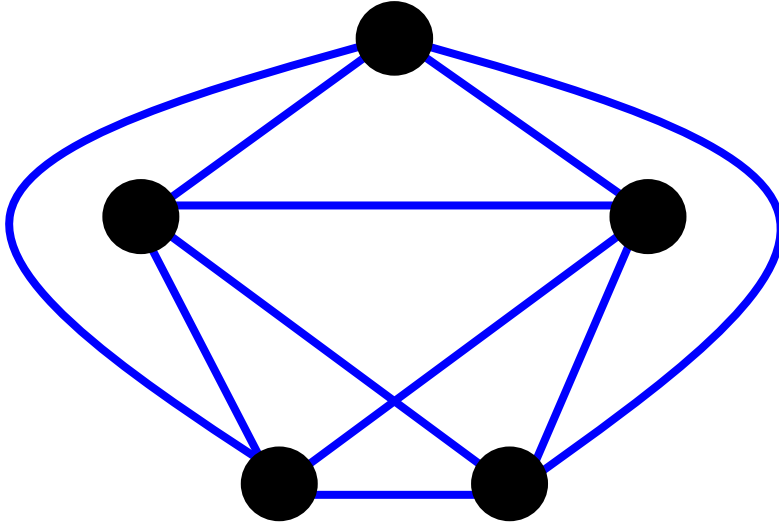
# Odd crossing number

$\text{ocr}(G)$  = minimum number of pairs of edges crossing odd number of times



$$\text{ocr}(K_5) = ?$$

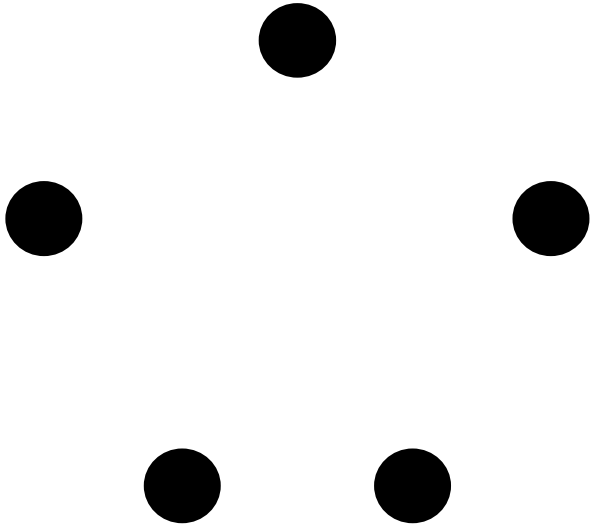
# Proof (Tutte'70): $\text{ocr}(K_5) = 1$



INVARIANT:

How many pairs of non-adjacent edges intersect **(mod 2)** ?

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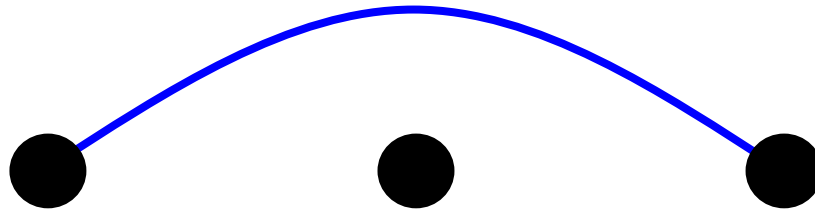




# Proof: $\text{ocr}(K_5) = 1$

How many pairs of non-adjacent edges intersect **(mod 2)** ?

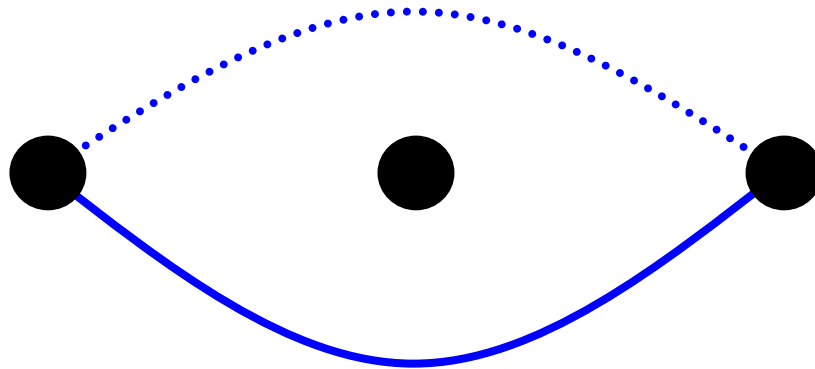
steps which change isotopy:



# Proof: $\text{ocr}(K_5) = 1$

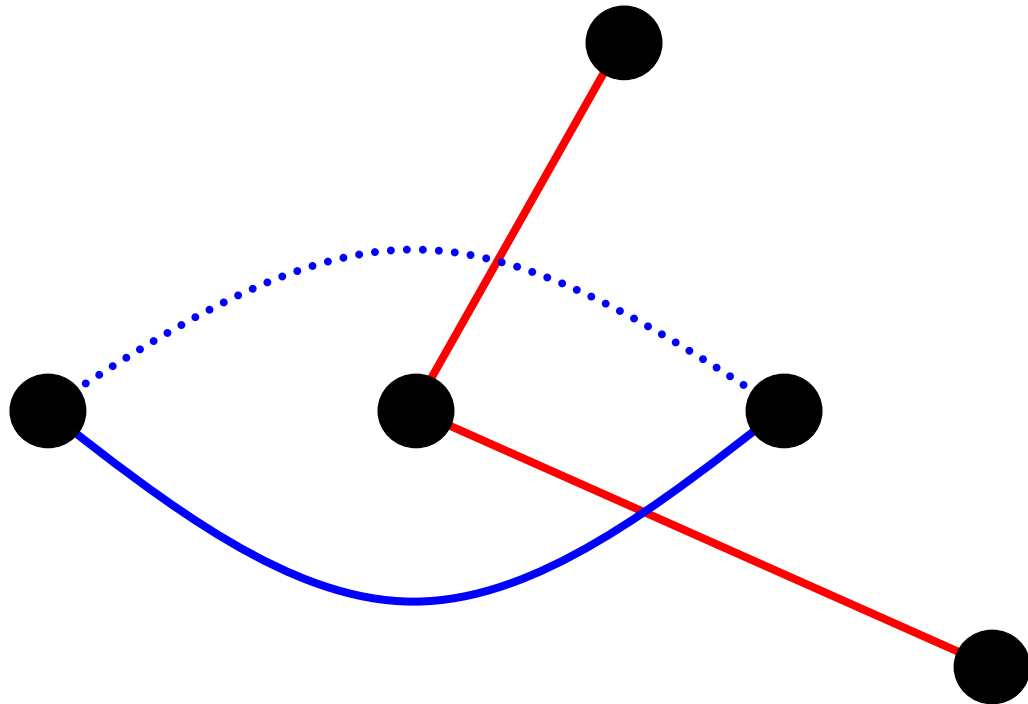
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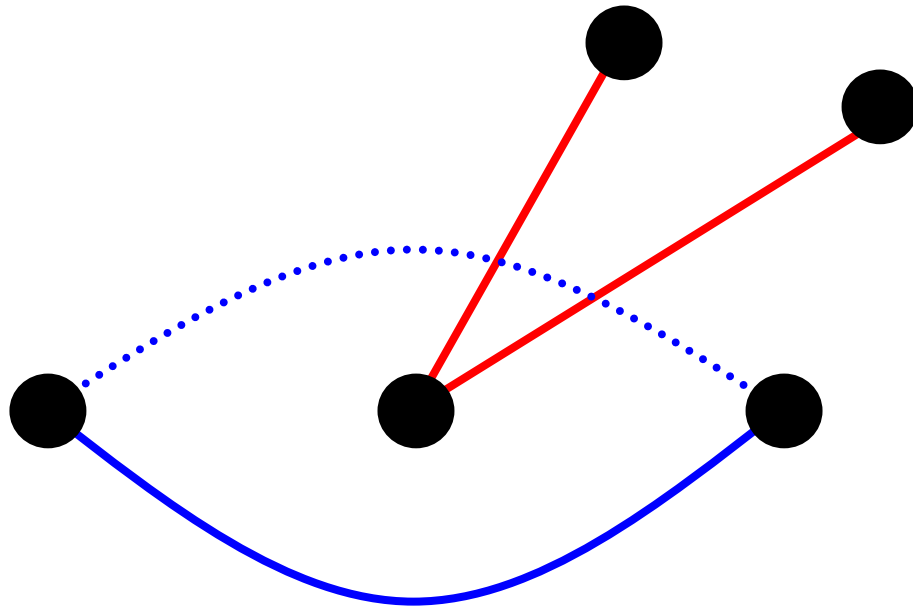
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How many pairs of non-adjacent edges intersect **(mod 2)** ?



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How many pairs of non-adjacent edges intersect **(mod 2)** ?



QED

# Hanani'34, Tutte'70:

$$\text{ocr}(G)=0 \iff \text{cr}(G)=0$$

If  $G$  has drawing in which all pairs of edges cross even # times  
 $\Rightarrow$  graph is planar!

# Are they equal?

$$\text{ocr}(G)=0 \iff \text{cr}(G)=0$$

QUESTION [Pach-Tóth'00]:

$$\text{ocr}(G) \stackrel{?}{=} \text{cr}(G)$$

# Are they equal?

$$\text{ocr}(G)=0 \Leftrightarrow \text{cr}(G)=0$$

$$\text{ocr}(G) \stackrel{?}{=} \text{cr}(G)$$

Pach-Tóth'00:

$$\text{cr}(G) \leq 2\text{ocr}(G)^2$$

# Main result

THEOREM [Pelsmajer, Schaefer, Š '05]

$$\mathbf{ocr(G) \neq cr(G)}$$



# How to prove it?

THEOREM [Pelsmajer, Schaefer, Š '05]

$$\mathbf{ocr(G) \neq cr(G)}$$

1. Find  $G$ .
2. Draw  $G$  to witness small  $ocr(G)$ .
3. Prove  $cr(G) > ocr(G)$ .

# How to prove it?

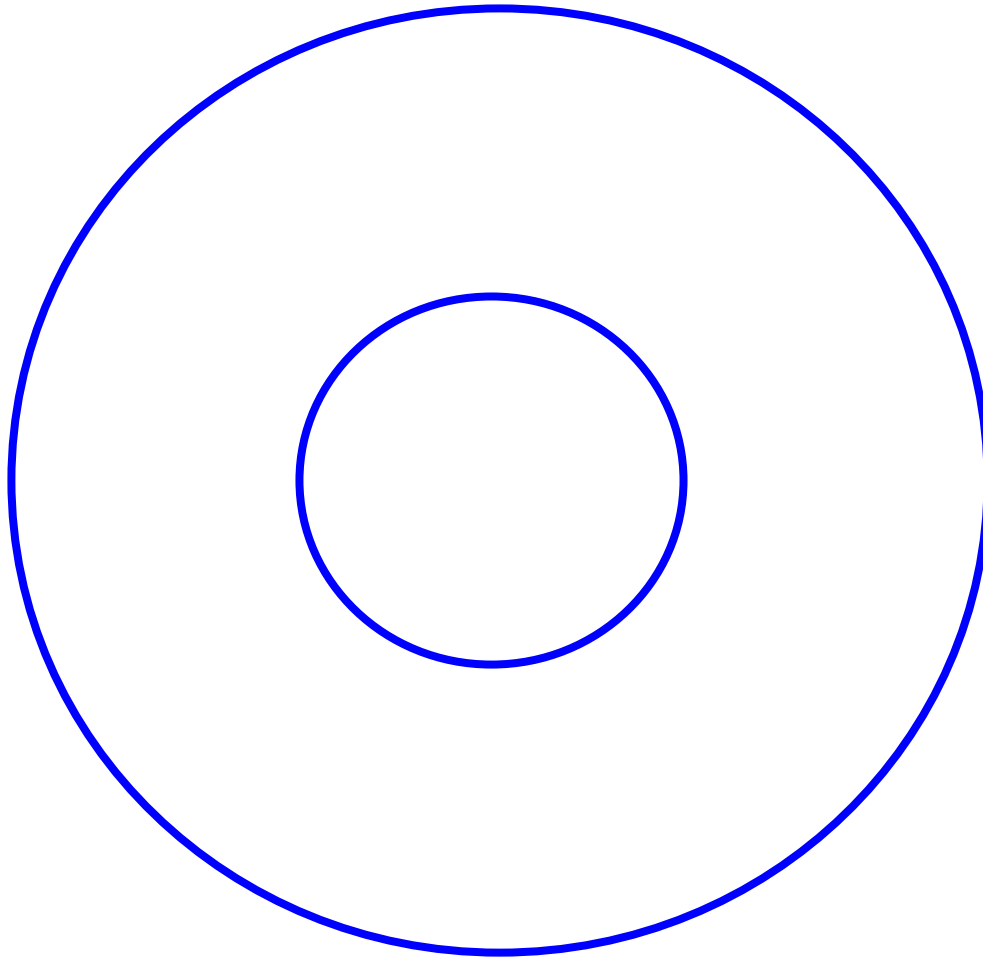
THEOREM [Pelsmajer, Schaefer, Š '05]

$$\text{ocr}(\mathbf{G}) \neq \text{cr}(\mathbf{G})$$

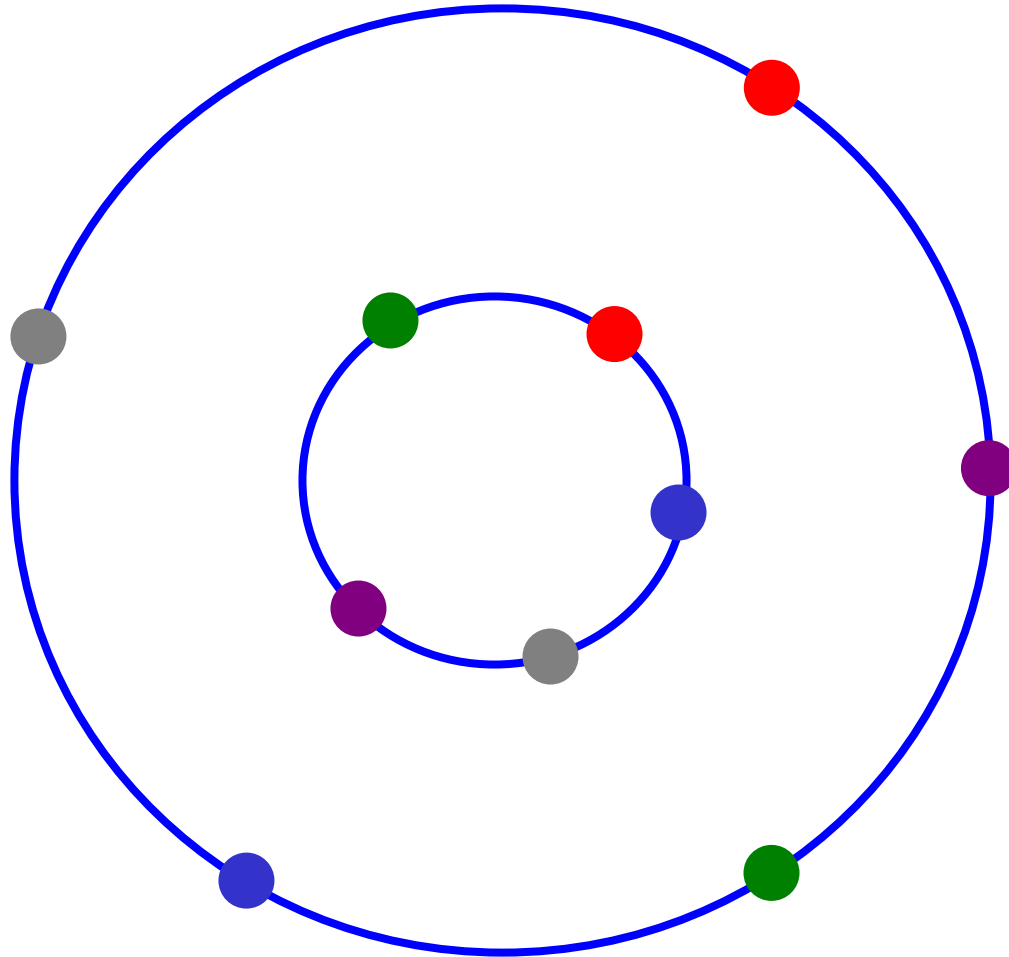
1. Find  $G$ .
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**Obstacle:**  $\text{cr}(G) > x$  is co-NP-hard!

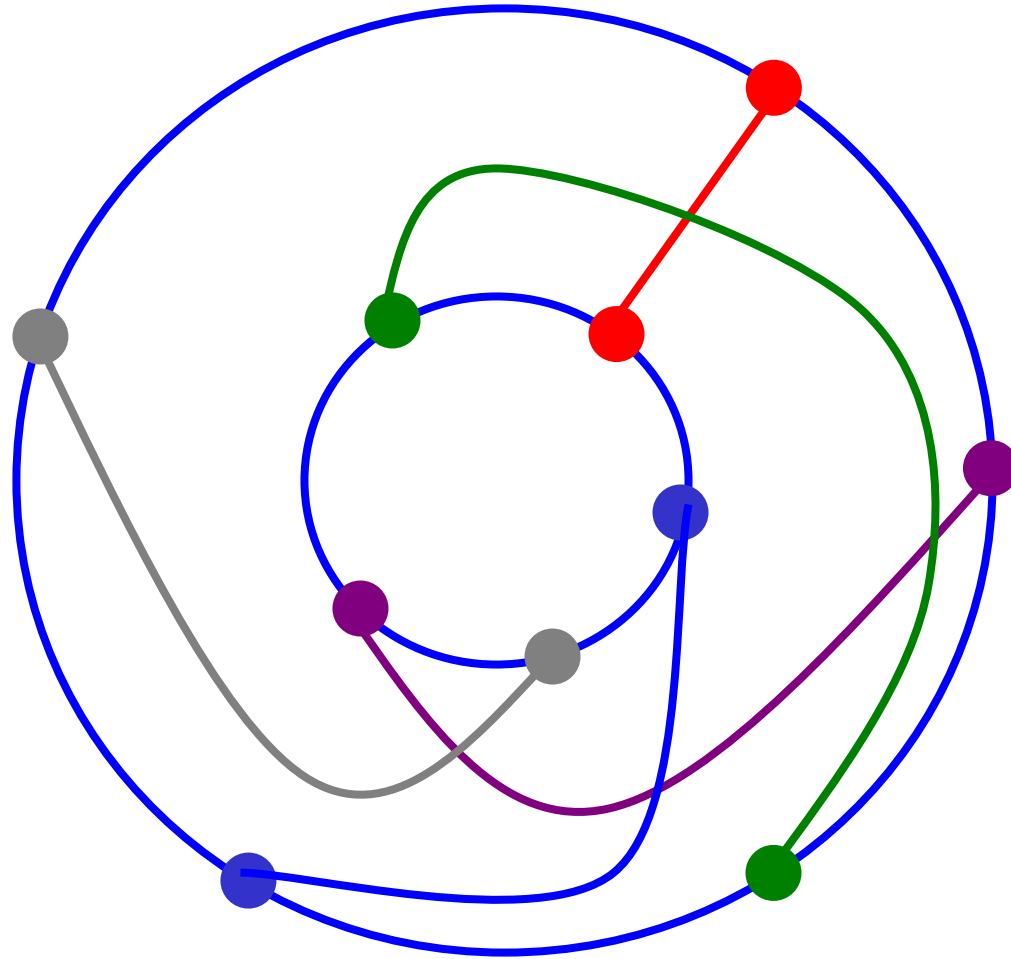
# Crossing numbers for "maps"



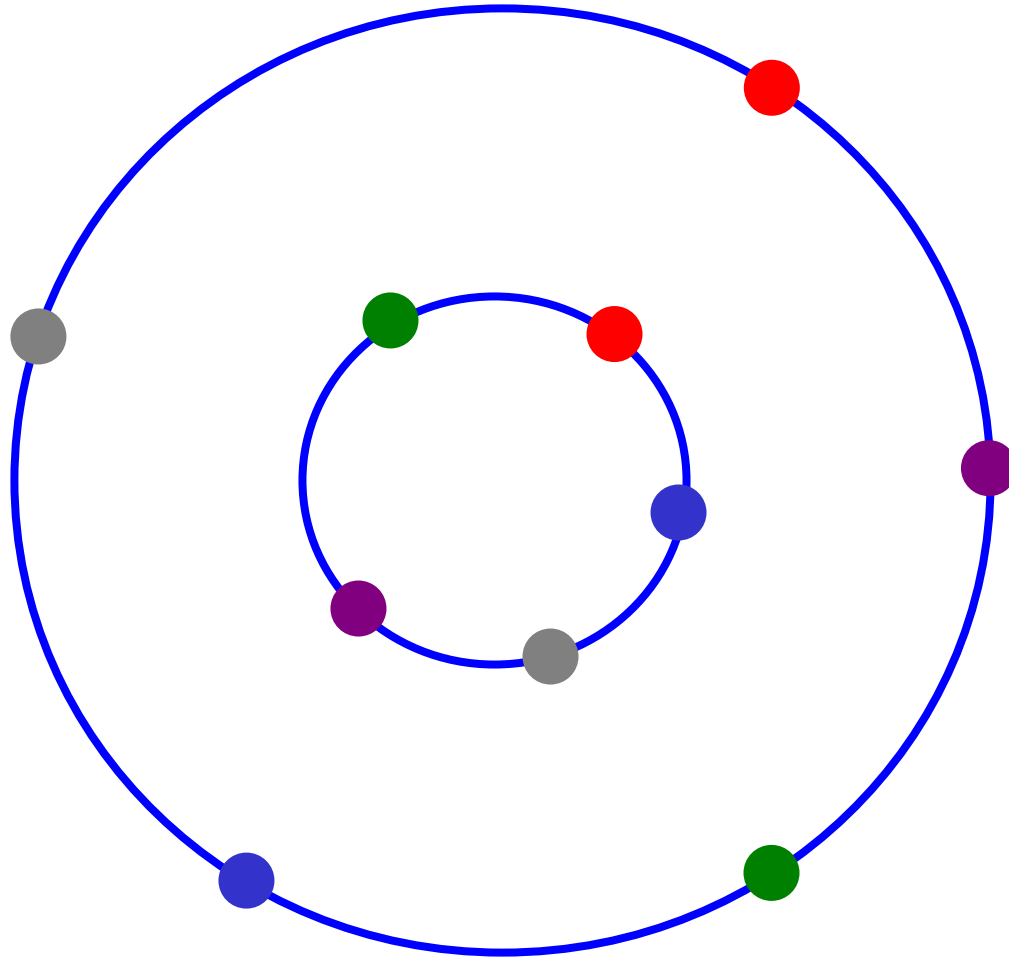
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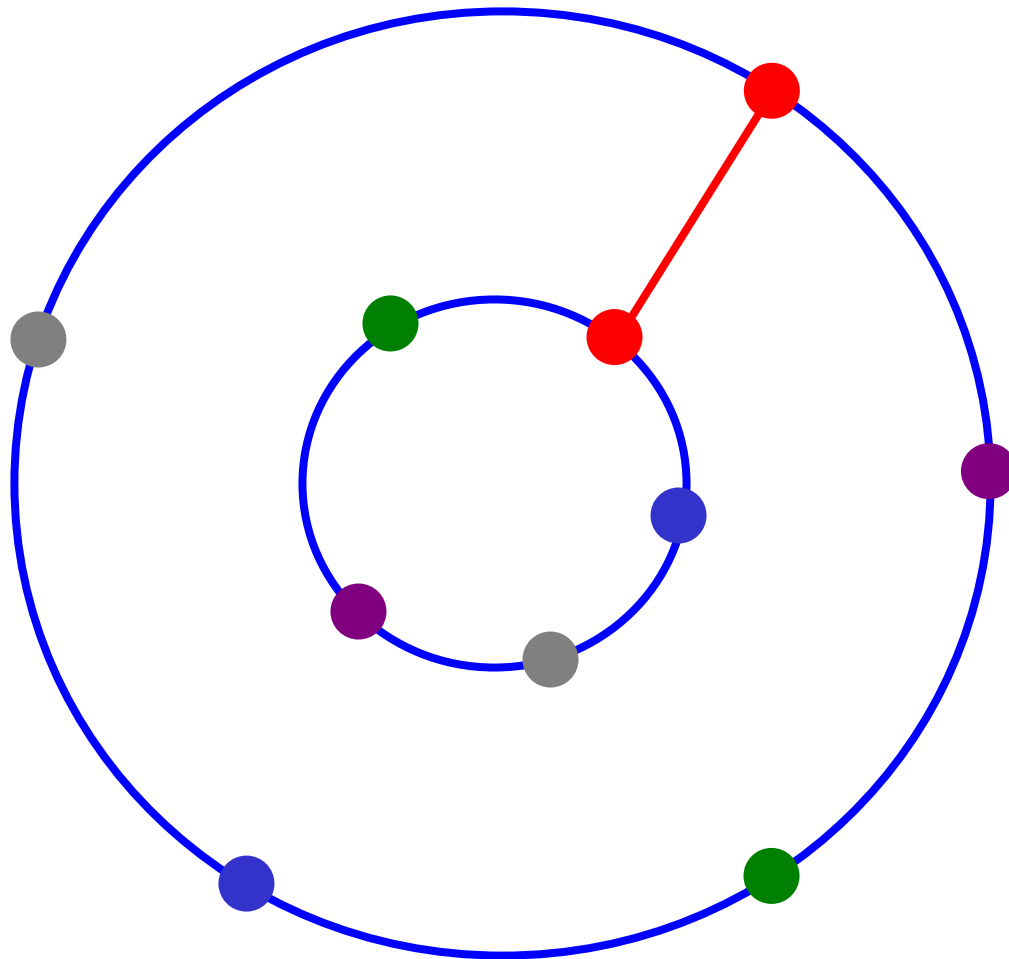
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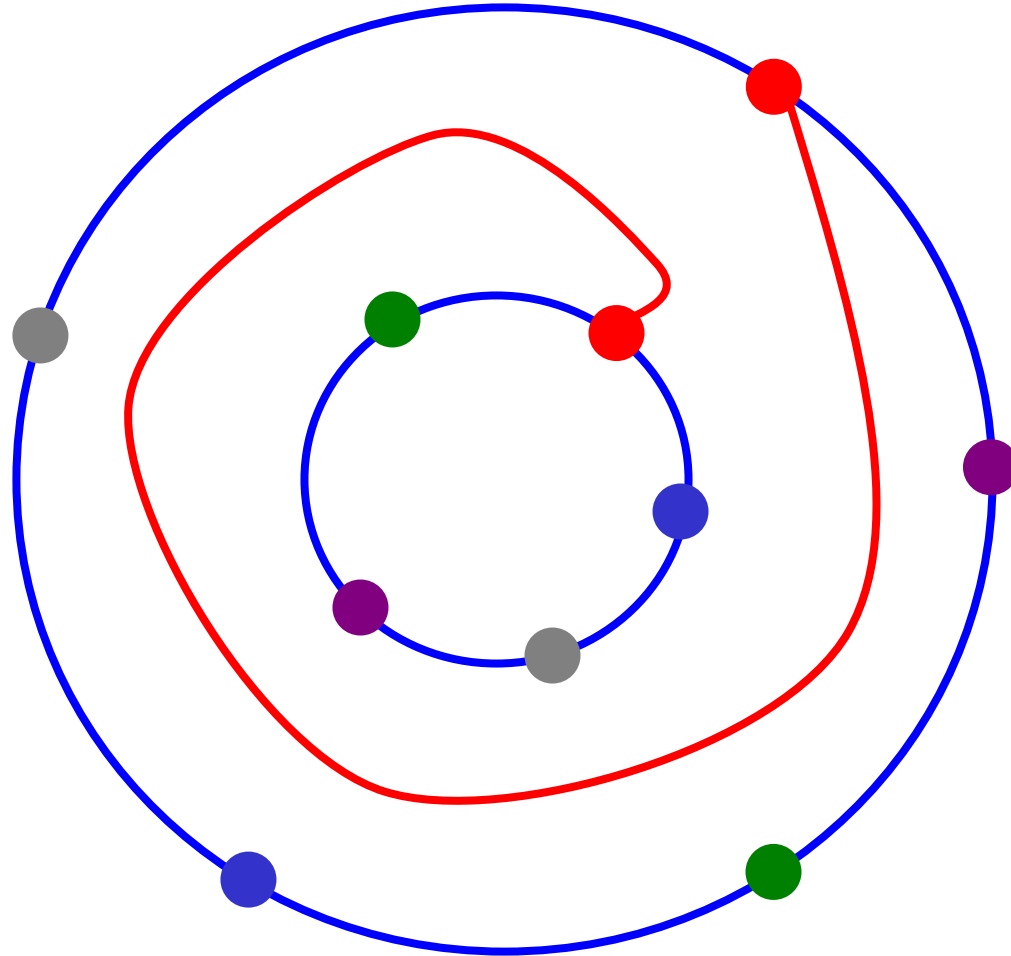
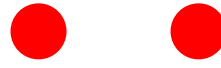
# Ways to connect



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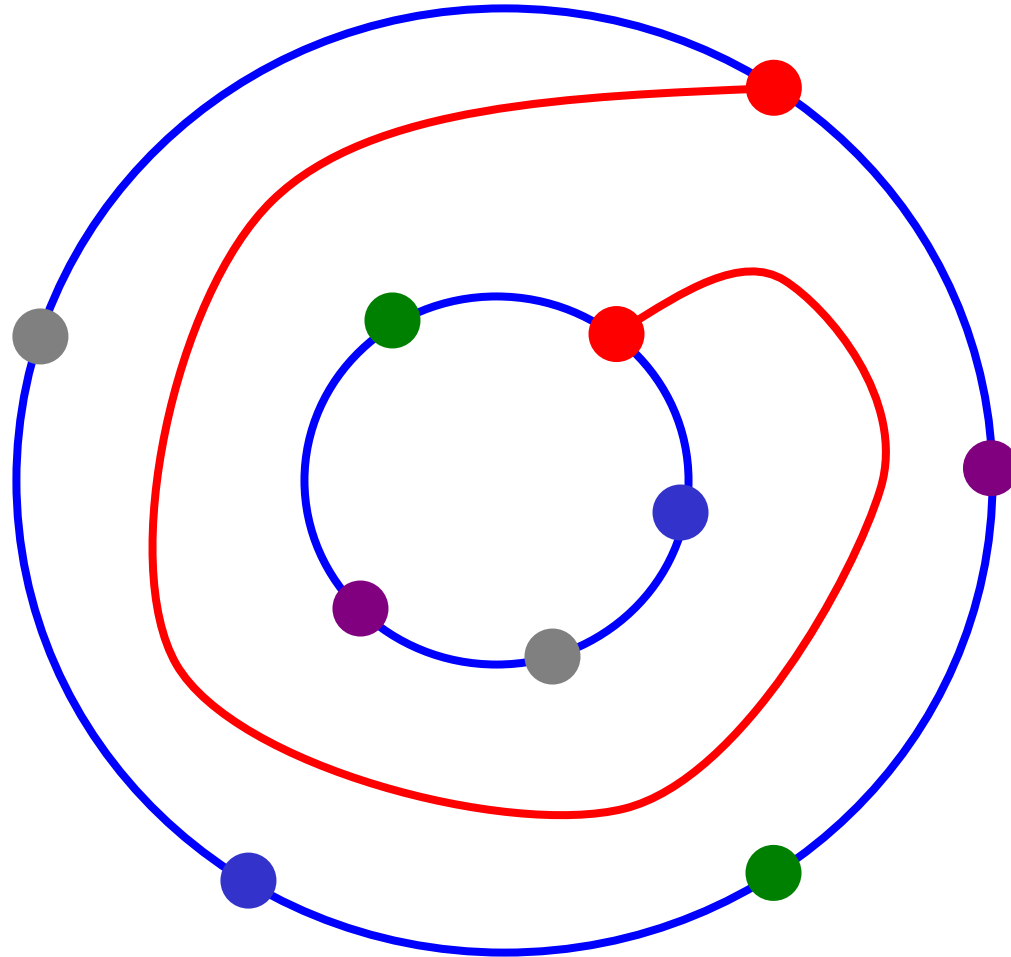
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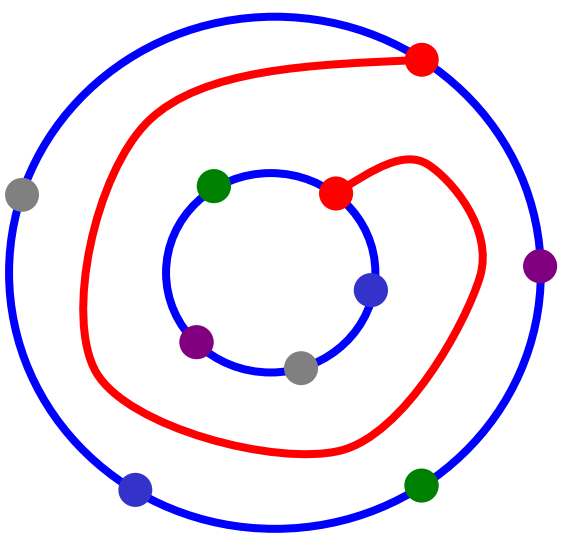


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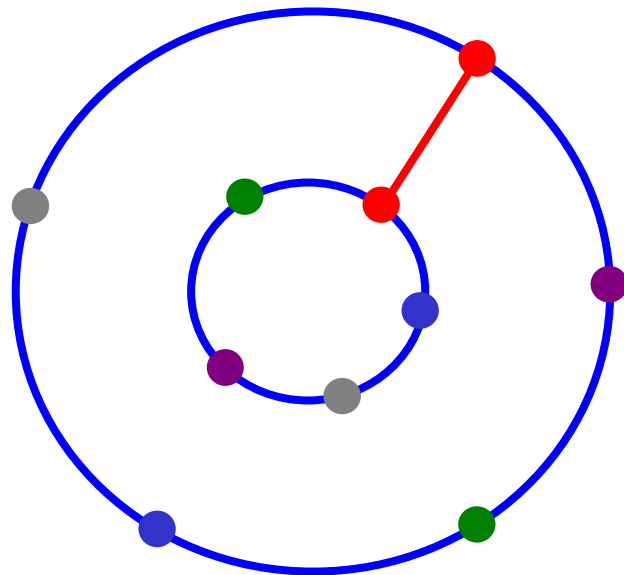


# Ways to connect ● ●

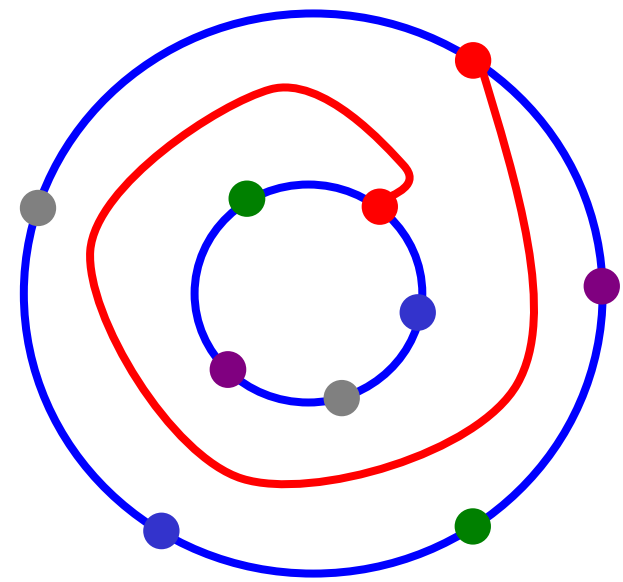
number of "Dehn twists"



-1



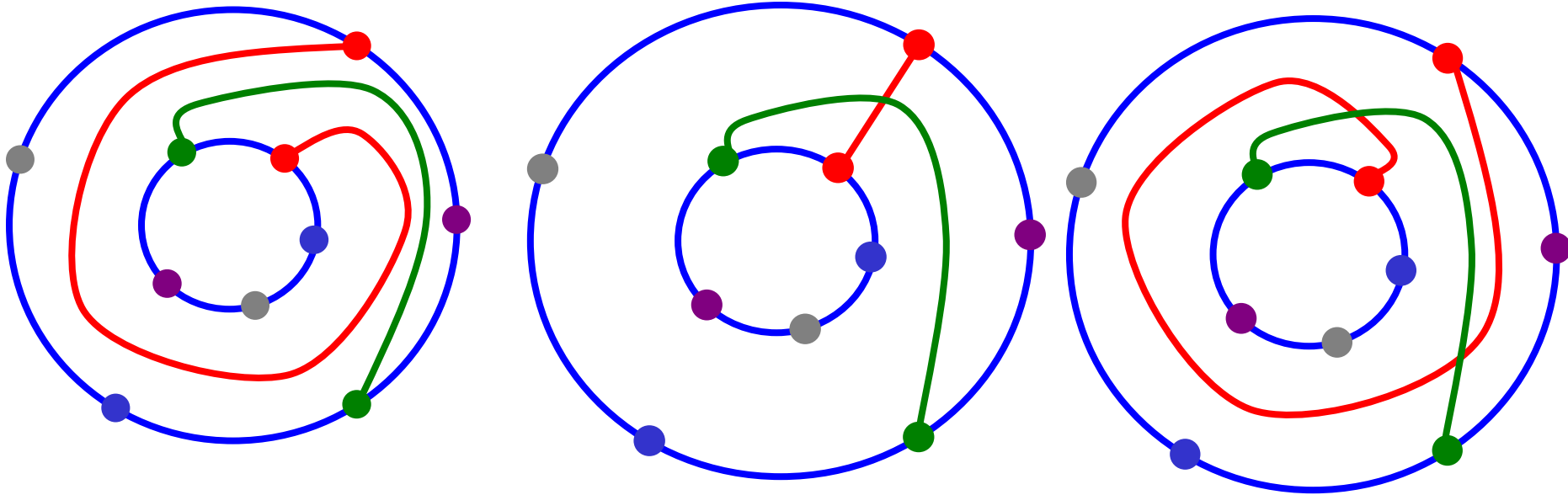
0



+1

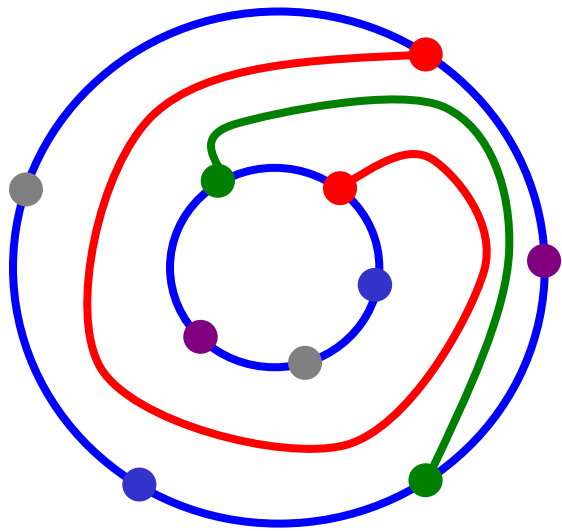
# Ways to connect ● ●

How to compute # intersections ?

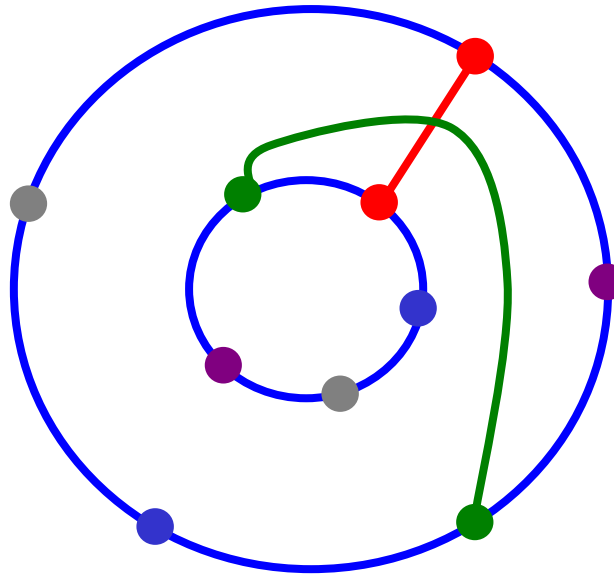


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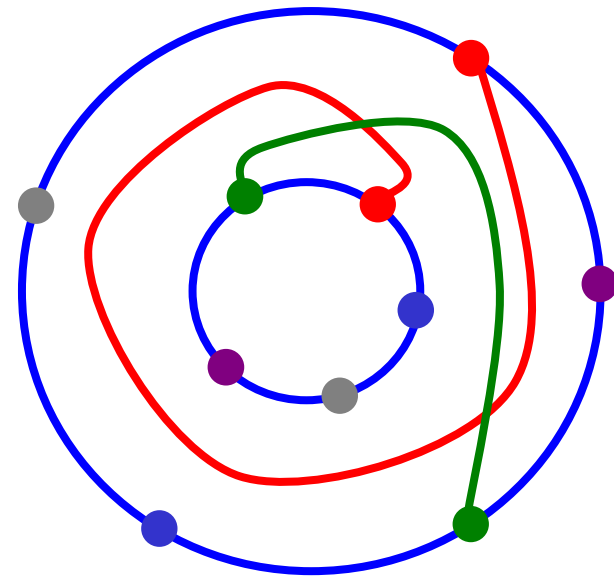
How to compute # intersections ?



0



1

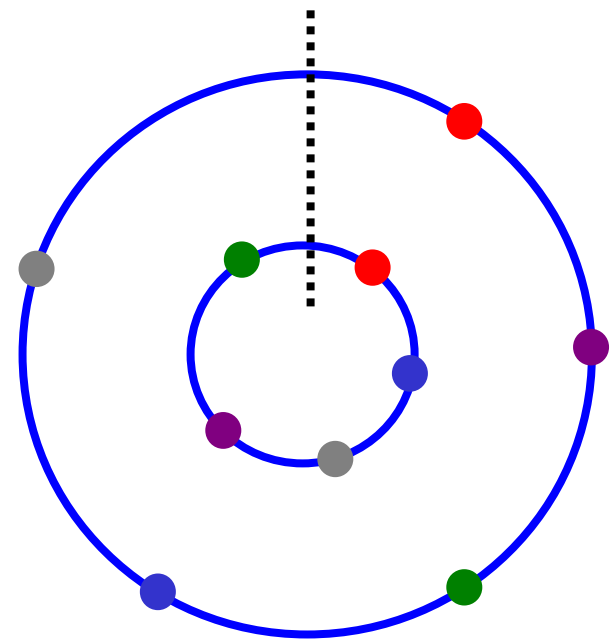


2

# Crossing number

$$\min \sum_{i < j} |x_i - x_j + (\pi_i > \pi_j)|$$

$$x_i \in \mathbb{Z}$$



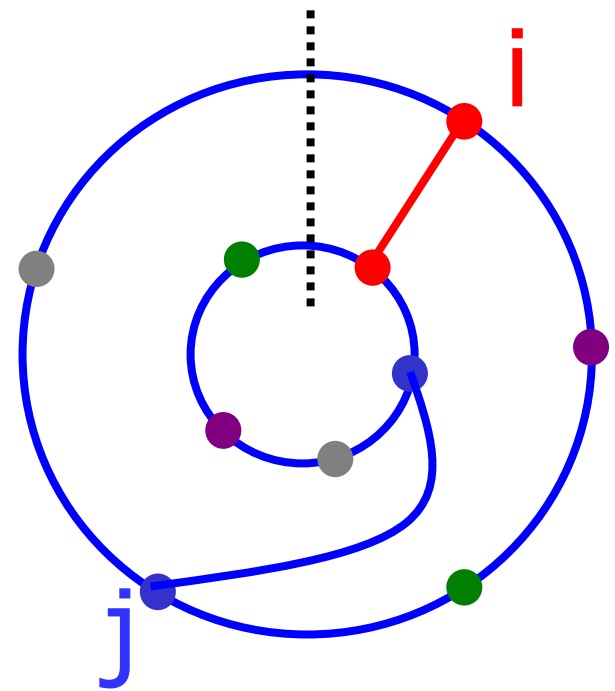
do arcs  $i, j$  intersect in the initial drawing?

the number of twists of arc  $i$

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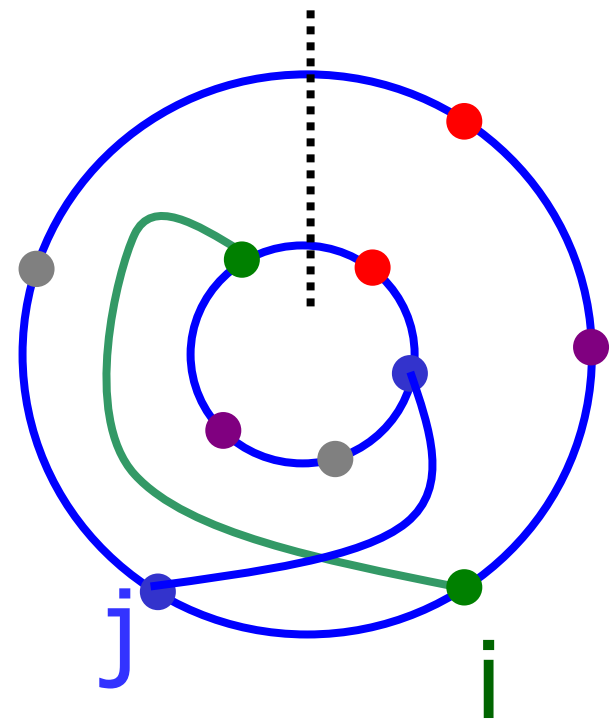
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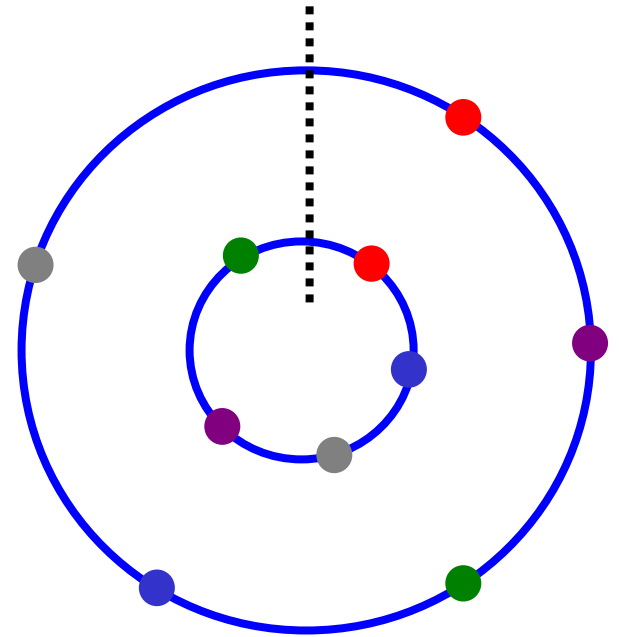


# Crossing number

$$\min \sum_{i < j} |x_i - x_j + (\pi_i > \pi_j)|$$

$$x_i \in \mathbb{Z} \longrightarrow \text{OPT}$$

$$x_i \in \mathbb{R} \longrightarrow \text{OPT}^*$$

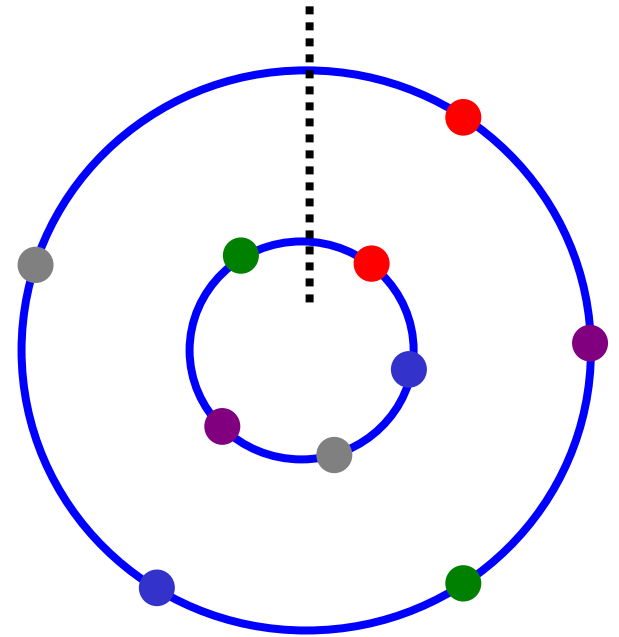


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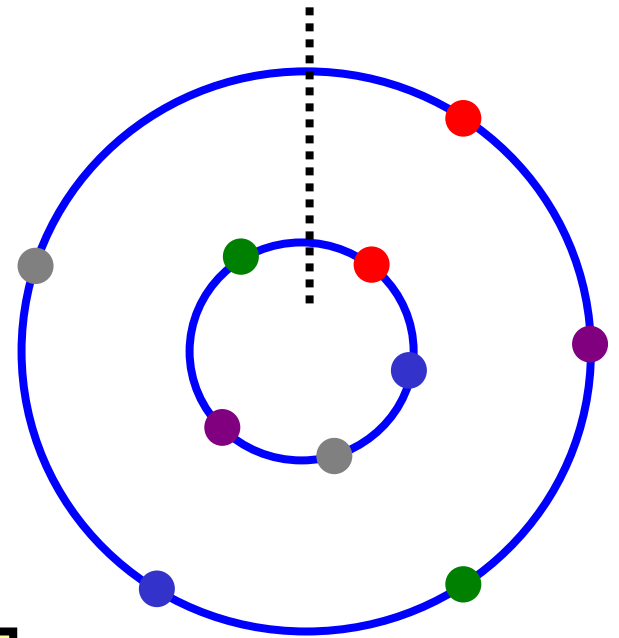


Lemma:

$$\text{OPT}^* = \text{OPT}.$$

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Lemma:

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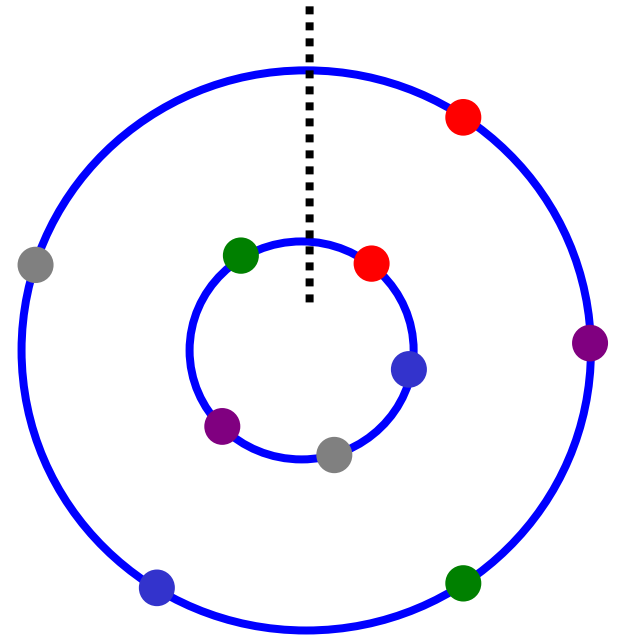
~~Obstacle:  $\text{cr}(G) > x$  is co-NP-hard!~~

# Crossing number

$$\min \sum_{i < j} |x_i - x_j + (\pi_i > \pi_j)|$$

$$y_{ij} \geq x_i - x_j + (\pi_i > \pi_j)$$

$$y_{ij} \geq -x_i + x_j - (\pi_i > \pi_j)$$



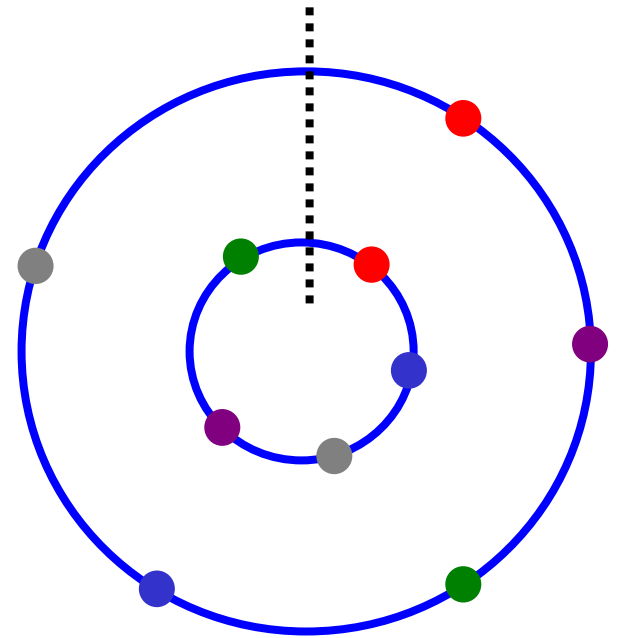
~~**Obstacle:**  $\text{cr}(G) \geq x$  is co-NP-hard!~~

# Crossing number

$$\min \sum_{i < j} \gamma_{ij}$$

$$\gamma_{ij} \geq x_i - x_j + (\pi_i > \pi_j)$$

$$\gamma_{ij} \geq -x_i + x_j - (\pi_i > \pi_j)$$



~~**Obstacle:**  $\text{cr}(G) \geq x$  is co-NP-hard!~~

# Crossing number

Dual linear program

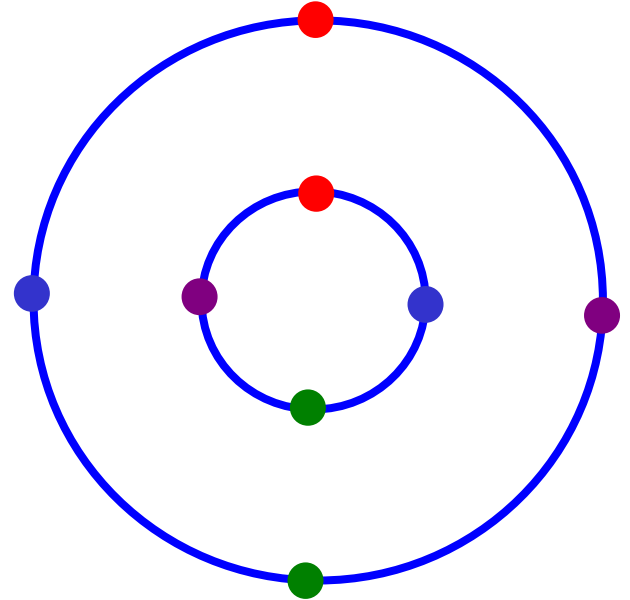
$$\max \sum_{i < j} Q_{ij} (\pi_i > \pi_j)$$

$$Q^T = -Q$$

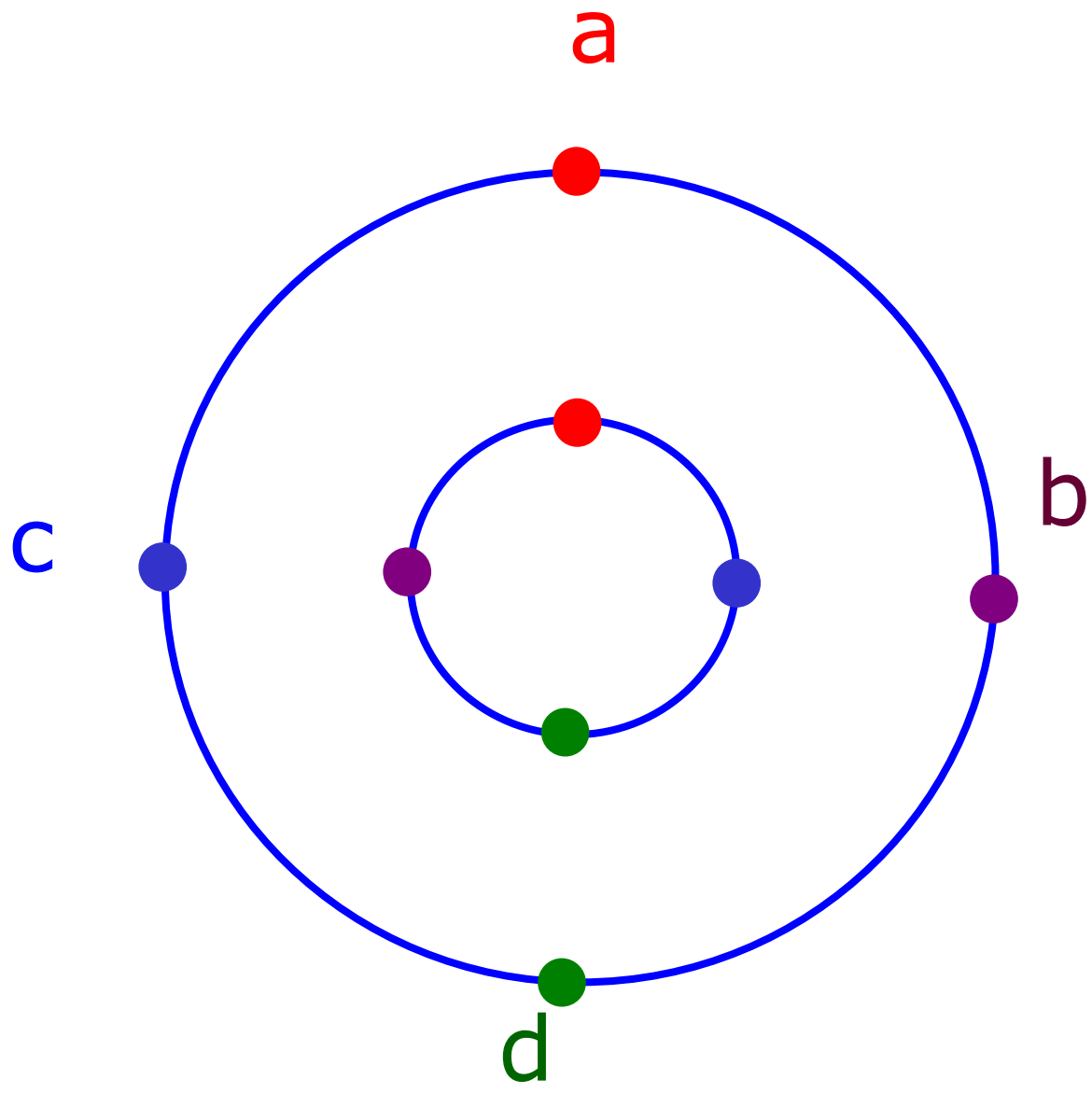
$$Q\mathbf{1} = 0$$

$$-1 \leq Q_{ij} \leq 1$$

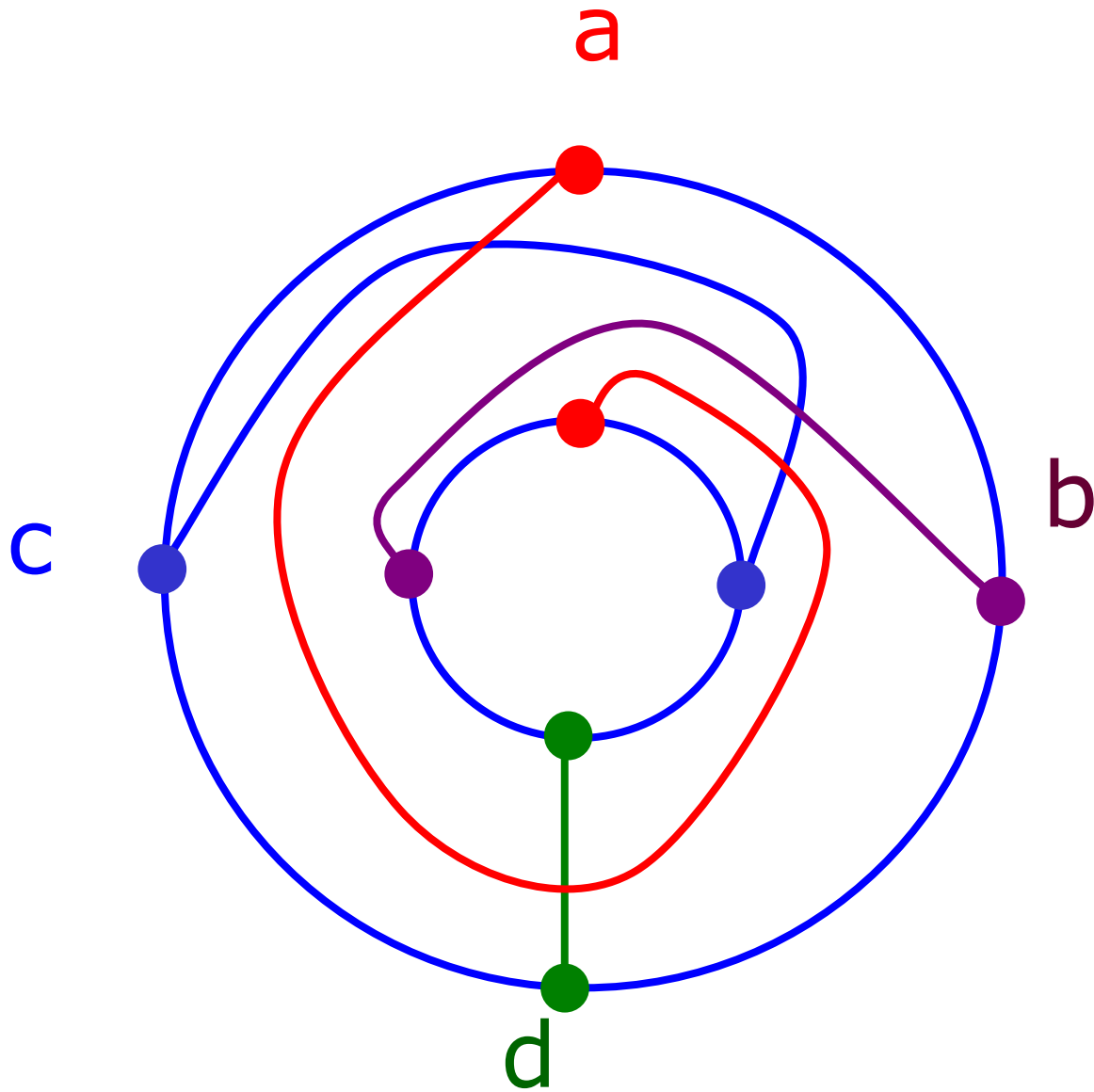
$Q$  is an  $n \times n$  matrix



# EXAMPLE:



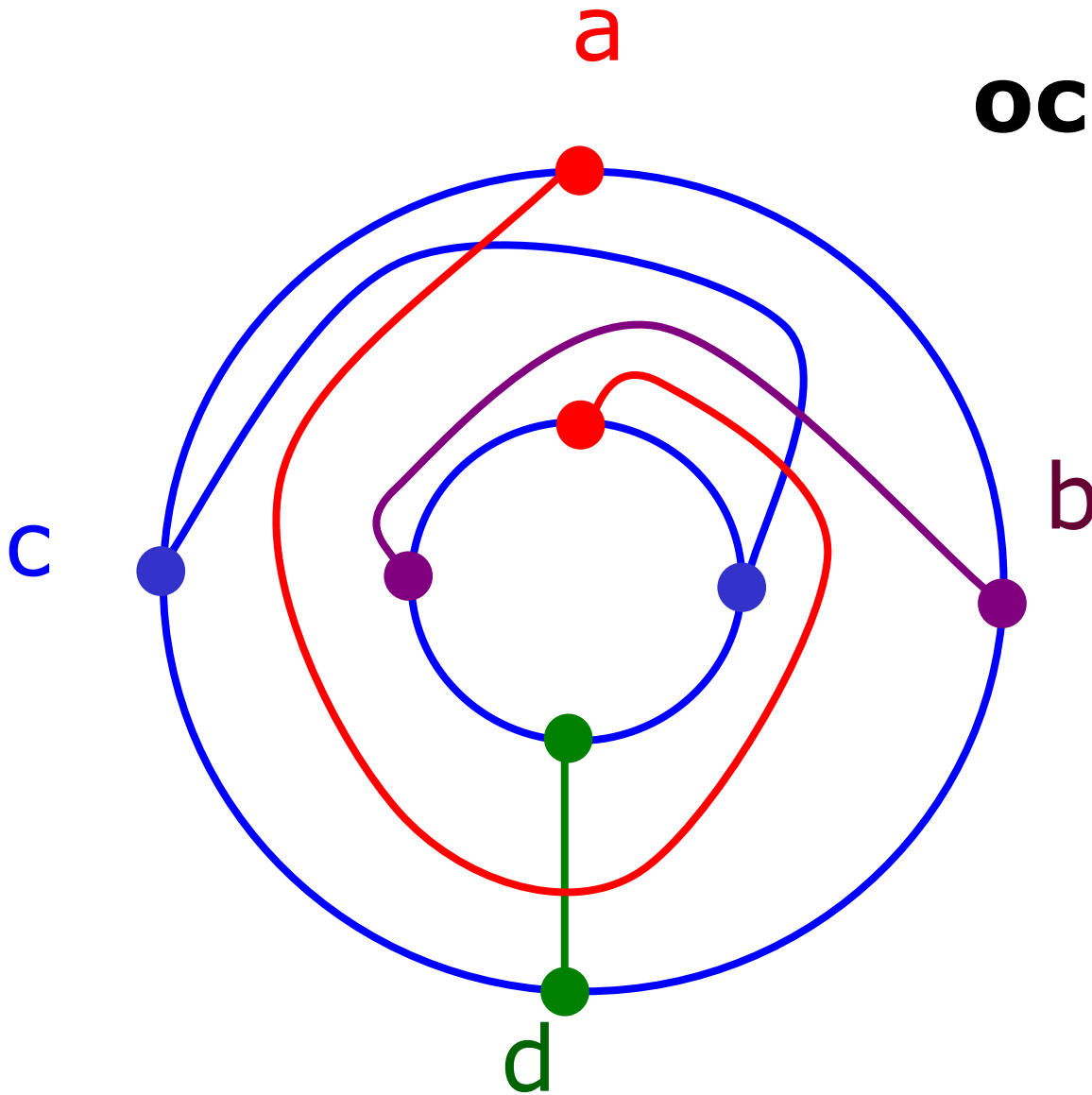
# Odd crossing number ?





# Odd crossing number

$$\text{ocr} \leq \text{ad} + \text{bc}$$



# Crossing number ?

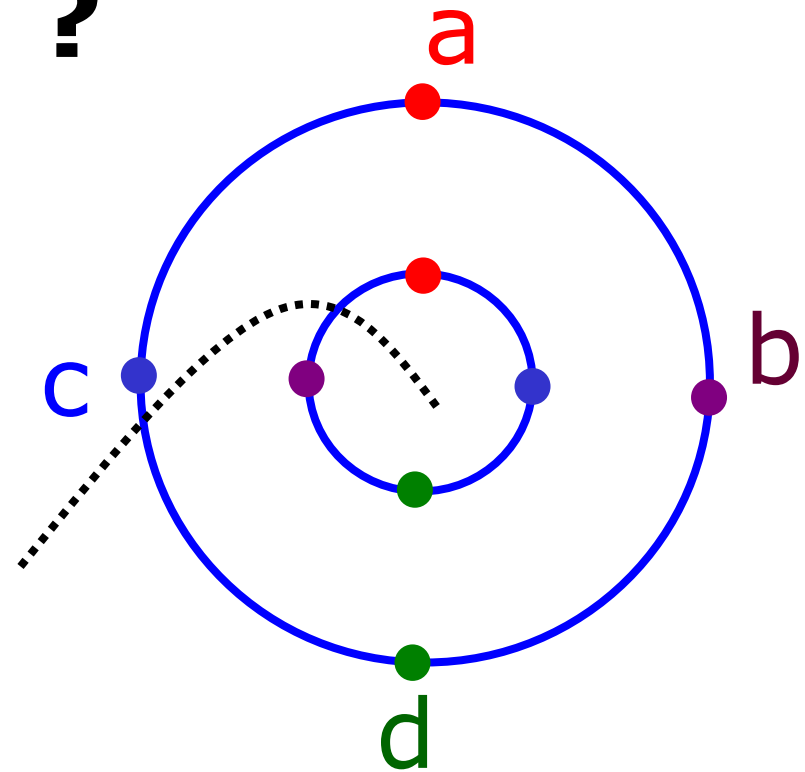
$$\max \sum_{i < j} Q_{ij} (\pi_i > \pi_j)$$

$$Q^T = -Q$$

$$Q \mathbf{1} = 0$$

$$-1 \leq Q_{ij} \leq 1$$

$$\pi = (2, 1, 4, 3)$$



$$a \leq b \leq c \leq d$$

$$a + c \geq d$$

$$cr \geq ac + bd$$

0	ac	b(d-a)	*
-ac	0	ab	a(c-b)
b(a-d)	-ab	0	bd
*	a(b-c)	-bd	0

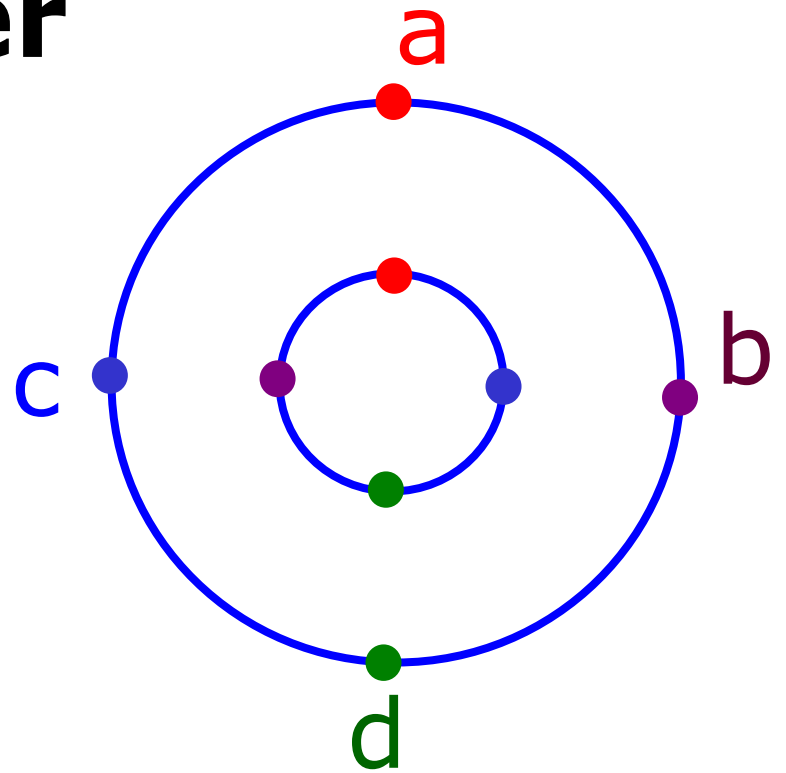
# Putting it together

$$ocr \leq ad + bc$$

$$cr \geq ac + bd$$

$$a \leq b \leq c \leq d$$

$$a + c \geq d$$

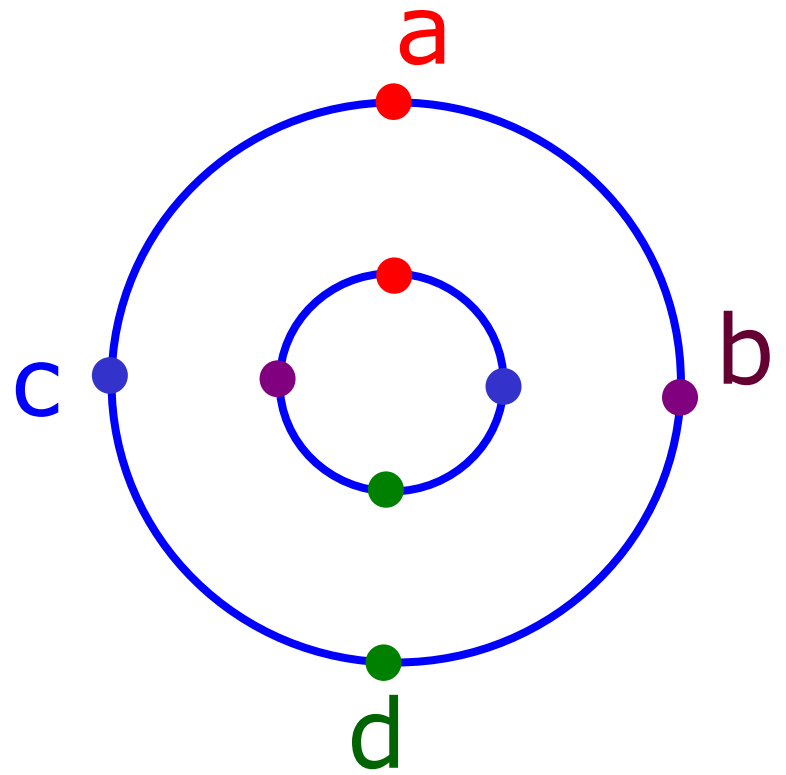


$$b = c = 1, a = (\sqrt{3} - 1)/2 \sim 0.37, d = a + c$$

$$ocr/cr = \sqrt{3}/2 \sim 0.87$$

# Crossing number

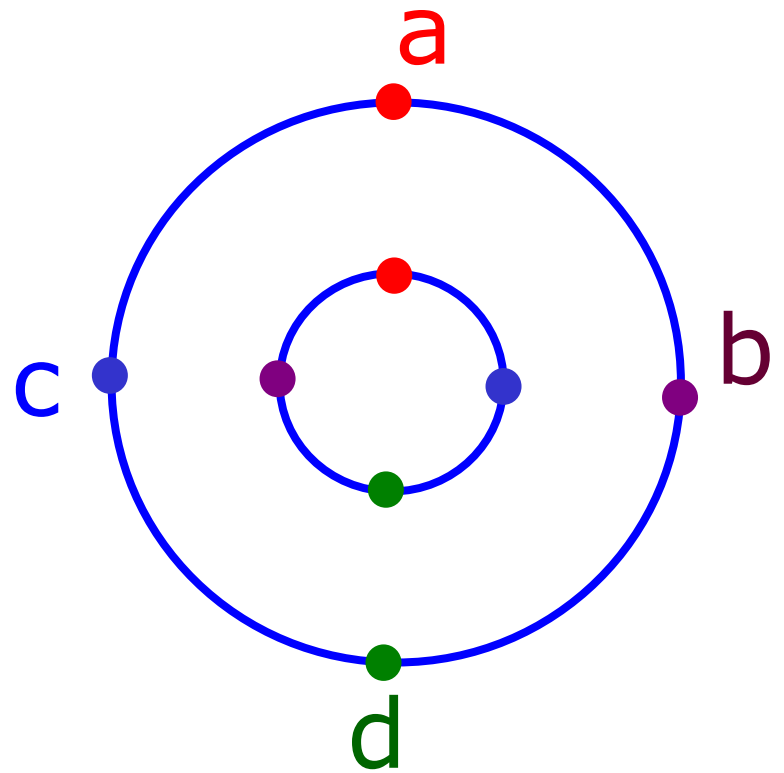
$$\text{ocr}/\text{cr} = \sqrt{3}/2 \sim 0.87$$



# Crossing number

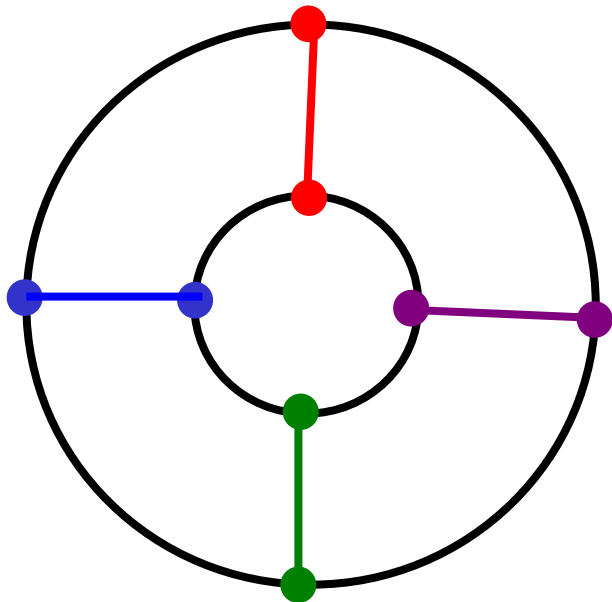
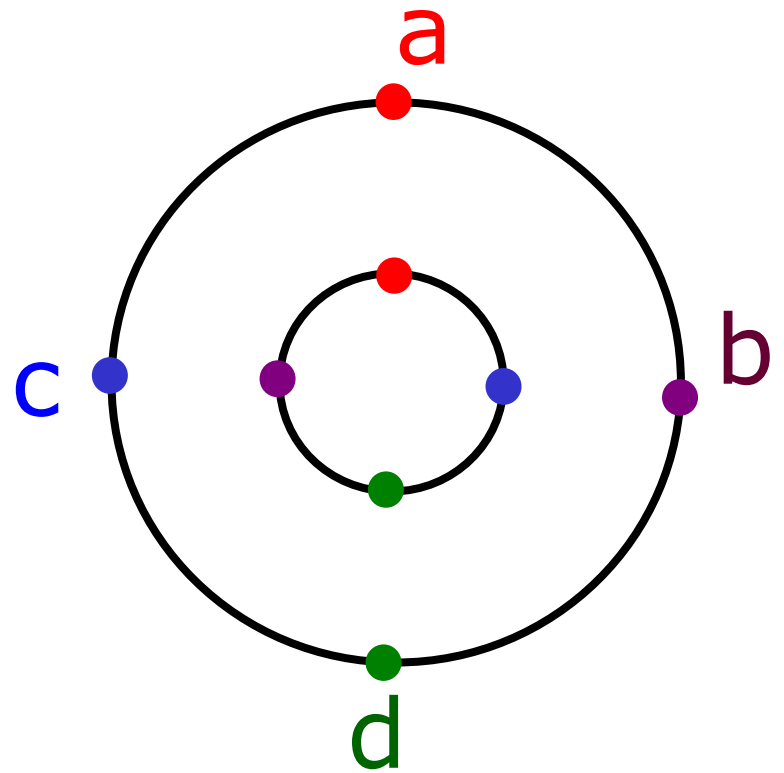
$$\text{ocr}/\text{cr} = \sqrt{3}/2 \sim 0.87$$

for graphs?



# Crossing number

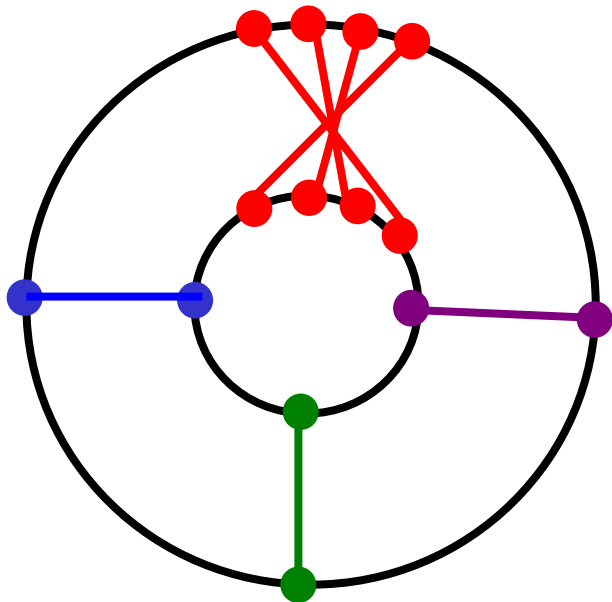
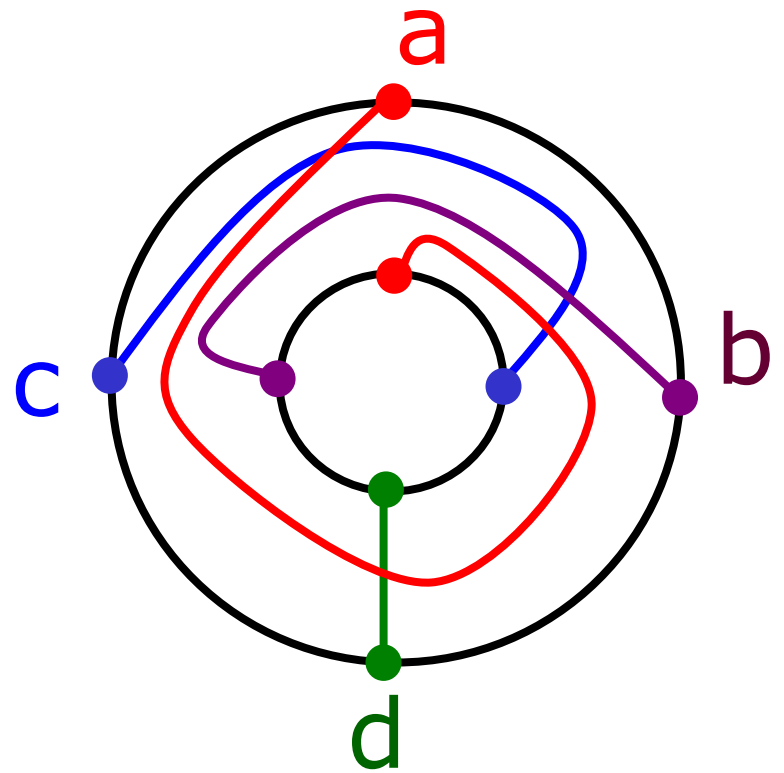
$$\text{ocr}/\text{cr} = \sqrt{3}/2 \sim 0.87$$



cr=?

# Crossing number

$$\text{ocr}/\text{cr} = \sqrt{3}/2 \sim 0.86$$



$\text{cr} = ?$

# Crossing number for graphs

Theorem:

$(\forall \varepsilon > 0) (\exists \text{ graph})$  with  
$$\text{ocr}/\text{cr} \leq \sqrt{3}/2 + \varepsilon.$$



**Is  $cr = O(ocr)$ ?**

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Is  $cr = O(ocr)$  on annulus?

**Is  $cr = O(ocr)$ ?**

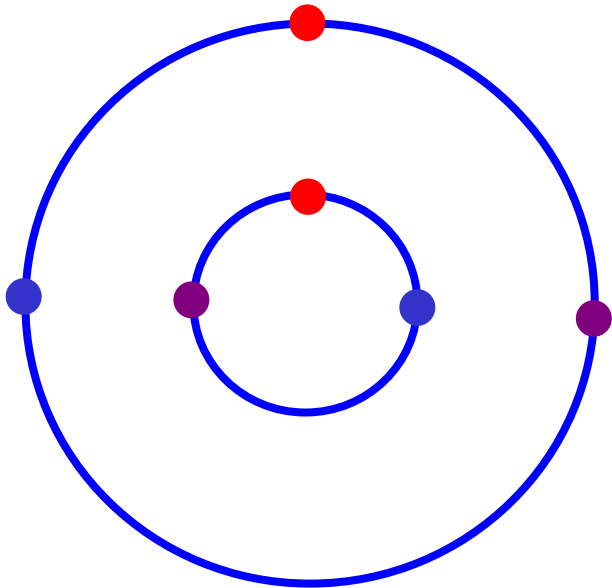
Is  $cr = O(ocr)$  on annulus?

Theorem:

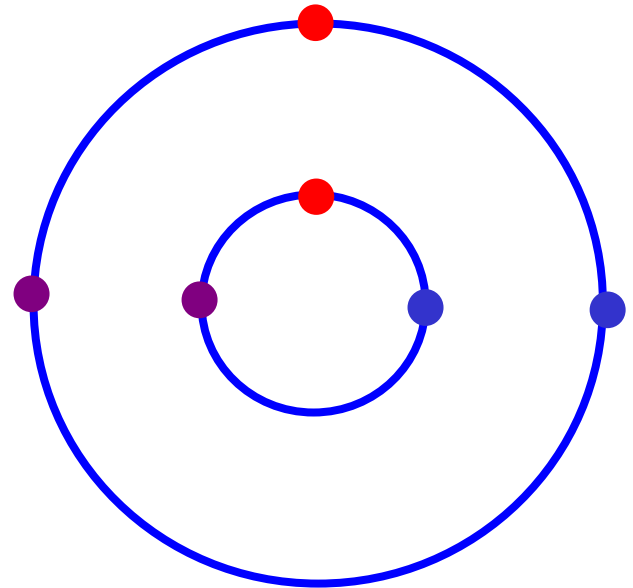
On annulus  $cr \leq 3ocr$

Theorem:

On annulus  $cr \leq 3ocr$

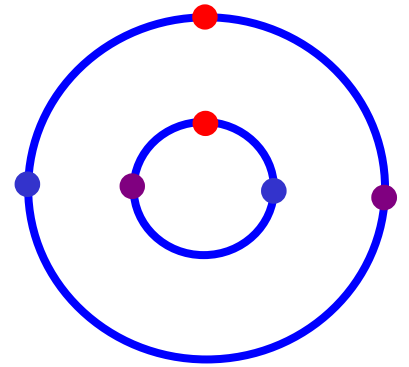


BAD triple

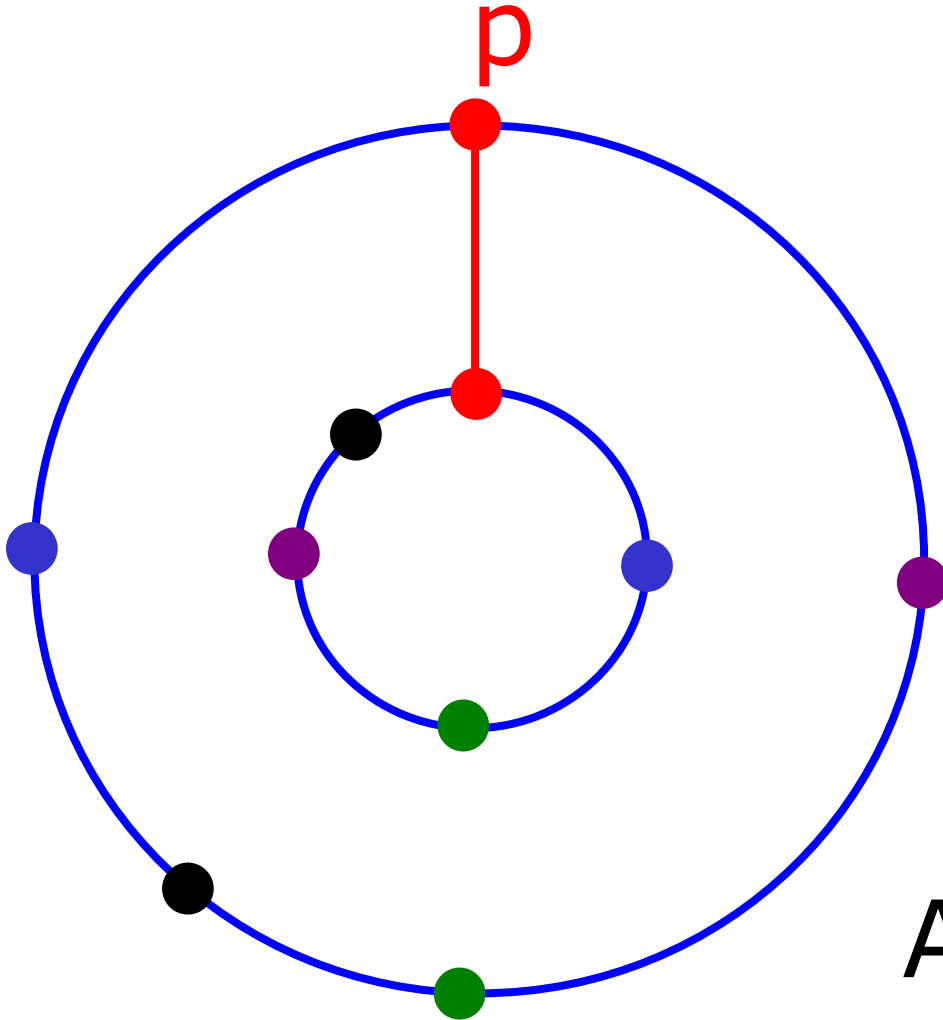


GOOD triple

$$n.CR \leq 3 \#BAD$$



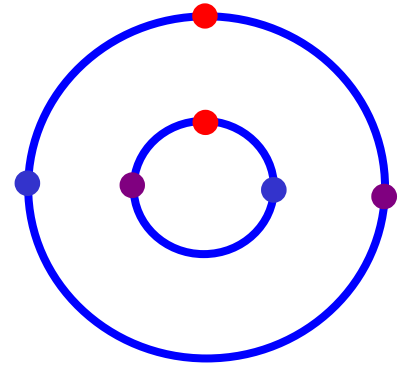
BAD triple



Pay: # of bad  
triples  $\{p, i, j\}$

Average over  $p$ .

$$\# \text{BAD} \leq n \cdot \text{OCR}$$



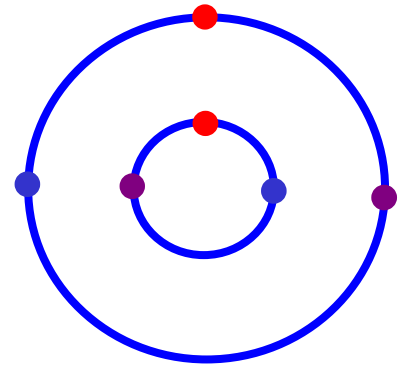
BAD triple

random  $i, j, k$   
 $X = \# \text{ odd pairs}$

$$\frac{\# \text{BAD}}{\binom{n}{3}} \leq E[X] \leq \frac{3\text{OCR}}{\binom{n}{2}}$$

$$\# \text{BAD} \leq n \cdot \text{OCR}$$

$$n \cdot \text{CR} \leq 3 \# \text{BAD}$$



BAD triple

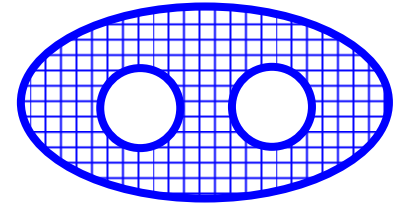
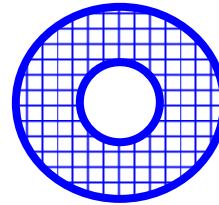
$$\text{CR} \leq 3 \text{OCR}$$

(on annulus)

# Crossing number for graphs

There exists graph with  
 $\text{ocr}/\text{cr} \leq \sqrt{3}/2 + \varepsilon.$

On annulus  
 $\text{ocr}/\text{cr} \geq 1/3$



Experimental evidence:

$\text{ocr}/\text{cr} \geq \sqrt{3}/2$  on annulus and pair of pants

**Bold (wrong) conjecture:**

**For any graph**

$$\text{ocr}/\text{cr} \geq \sqrt{3}/2$$



# Questions

crossing number of maps  
with  $d$  vertices in poly-time?  
(true for  $d \leq 2$ )

**Bold (wrong) conjecture:**

**For any graph**

$$\text{ocr}/\text{cr} \geq \sqrt{3}/2$$

(map = graph + rotation system)

# Open questions - classic

Guy's conjecture:

$$\text{cr}(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

Zarankiewicz's conjecture:

$$\text{cr}(K_{m,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$$

Better approx algorithm for cr.

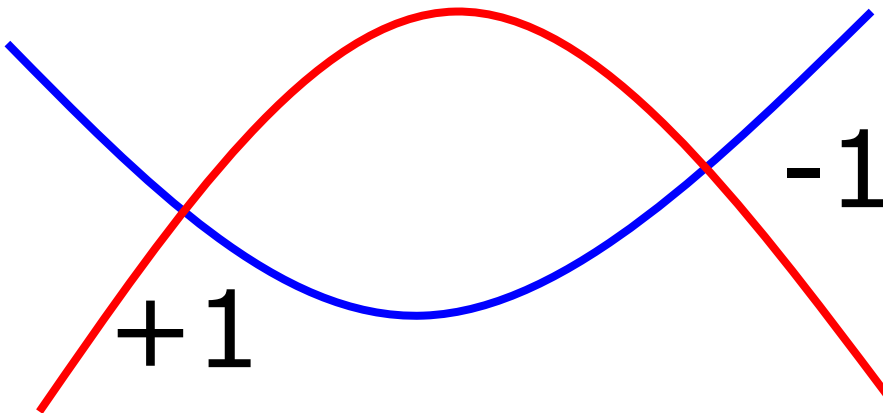
# Crossing number for graphs

pair crossing number (pcr)

# number of pairs of crossing edges

algebraic crossing number (acr)

$\Sigma$  algebraic crossing number of edges



# Crossing numbers

