

A different view of independent sets in bipartite graphs

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A different view of independent sets in ~~bipartite~~ graphs

counting/sampling independent sets in general graphs:



polynomial time sampler for $\Delta \leq 5$ (Dyer, Greenhill '00, Luby, Vigoda '99, Weitz '06).



no polynomial time sampler (unless NP=RP) for $\Delta \geq 25$ (Dyer, Frieze, Jerrum '02).



Glauber dynamics does not mix in polynomial time for **6-regular bipartite** graphs (example: union of 6 random matchings) (Dyer, Frieze, Jerrum '02).

Δ = maximum degree of G

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~~no polynomial time sampler (unless NP=RP) for $\Delta \geq 25$ (Dyer, Frieze, Jerrum '02).
(max independent set in bipartite graph \Leftrightarrow max matching)~~



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Δ = maximum degree of G

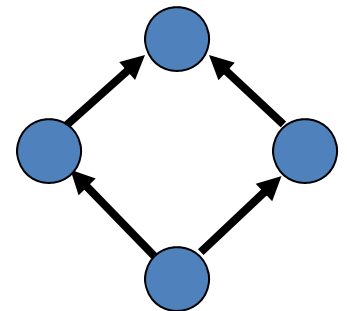
How hard is counting/sampling independent sets in bipartite graphs?

Why do we care?

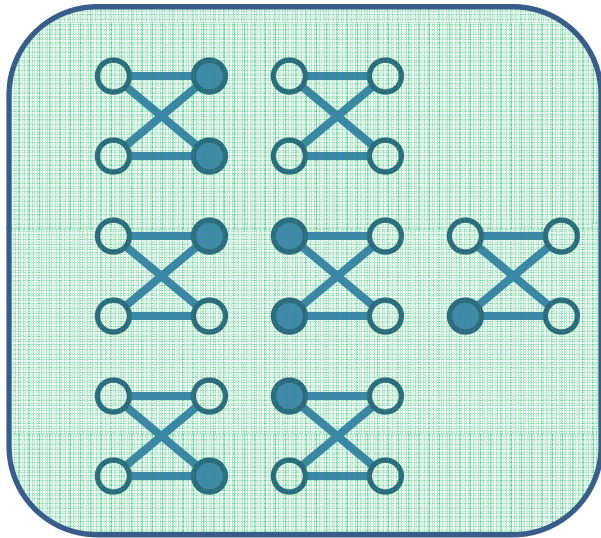
- * **bipartite independent sets**

equivalent to

- * enumerating solutions of a linear Datalog program
- * downsets in a poset
(Dyer, Goldberg, Greenhill, Jerrum'03)
- * ferromagnetic Ising with mixed external field
(Goldberg, Jerrum'07)
- * stable matchings
(Chebolu, Goldberg, Martin'10)

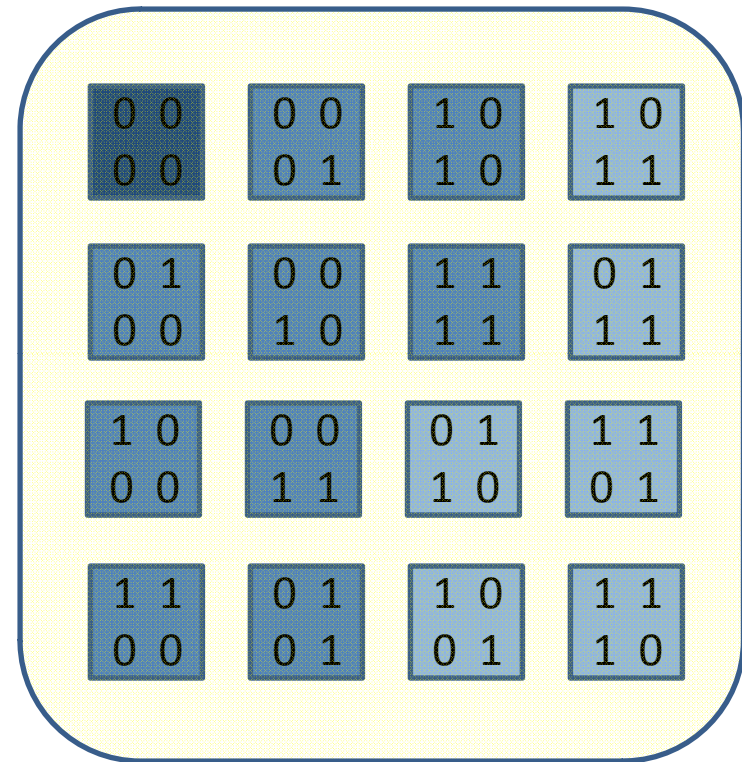


A different view of independent sets in bipartite graphs



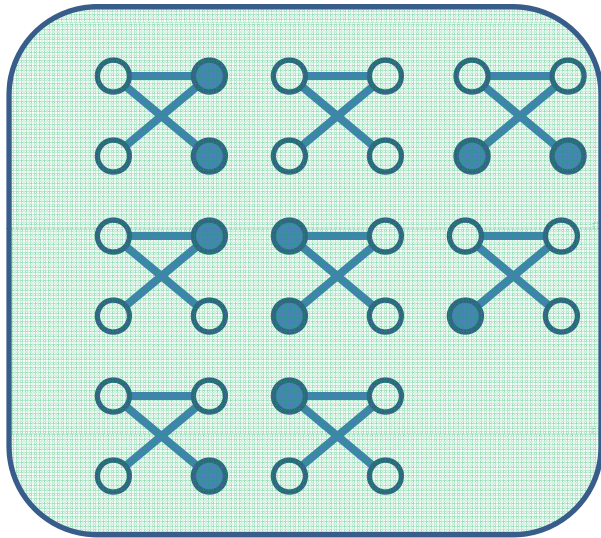
Independent sets in a bipartite graph.

Ge, Štefankovič '09

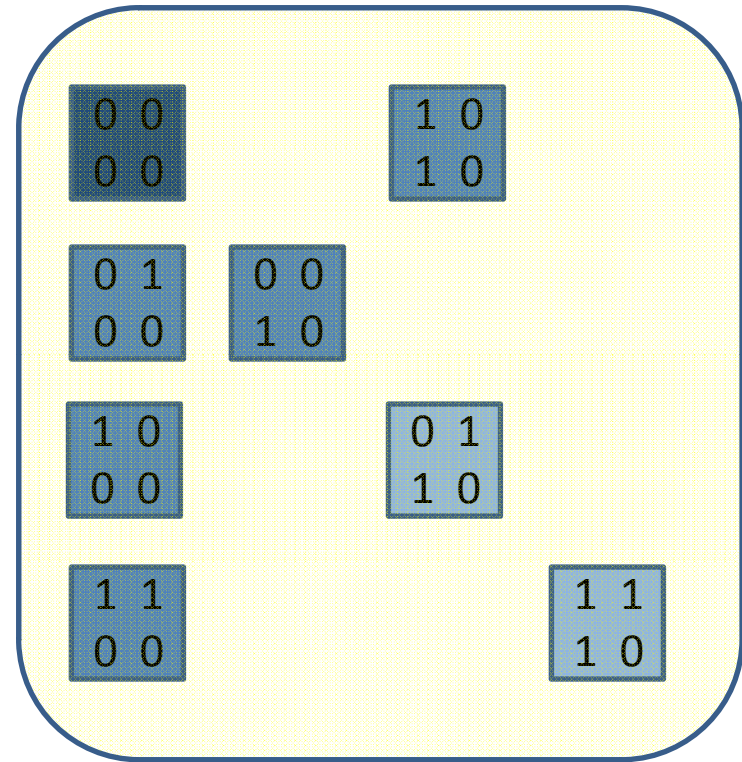


0-1 matrices weighted by $(1/2)^{\text{rank}}$ (1 allowed at A_{uv} if uv is an edge)

A different view of independent sets in bipartite graphs



Ge, Štefankovič '09



Independent sets in a bipartite graph.

$$\#IS = 2^{|V \cup U| - |E|} \sum_{A \leq B} 2^{-\text{rk}(A)}$$

0-1 matrices weighted by $(1/2)^{\text{rank}}$ (1 allowed at A_{uv} if uv is an edge)

A different view of independent sets in bipartite graphs

Question:

Is there a polynomial-time sampler that produces matrices $A \leq B$ with

$$P(A) \propto 2^{-\text{rank}(A)}$$

$$B_{ij}=0 \Rightarrow A_{ij}=0$$

(everything over the F_2)

Natural MC

**flip random entry +
Metropolis filter.**

**$A = X_t$ with random (valid)
entry flipped**

**if $\text{rank}(A) \leq \text{rank}(X_t)$
then $X_{t+1} = A$**

if $\text{rank}(A) > \text{rank}(X_t)$ then

$X_{t+1} = A$ w.p. $\frac{1}{2}$

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BAD NEWS:

**Goldberg, Jerrum'10: the chain is exponentially
slow for some graphs.**

A different view of independent sets in
bipartite graphs

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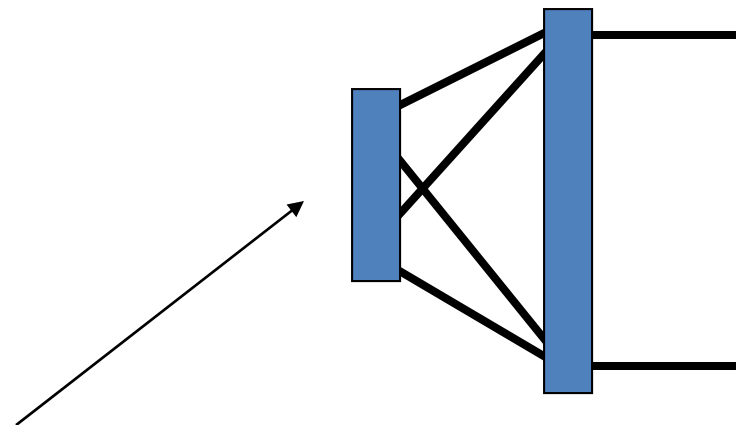
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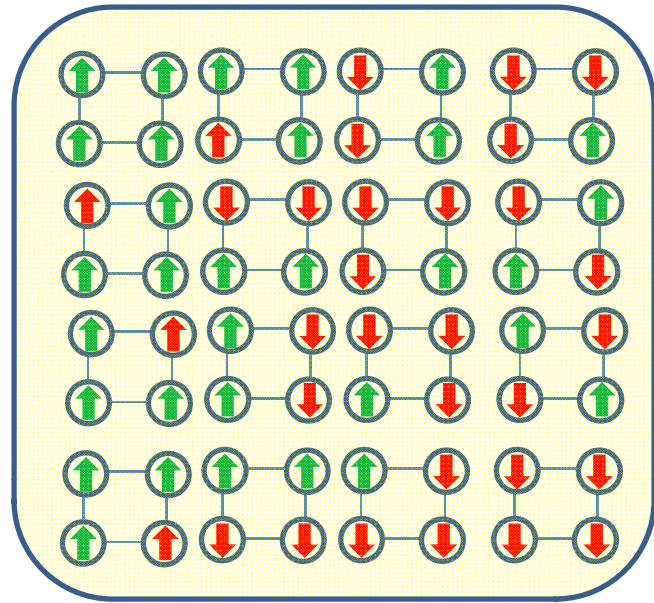
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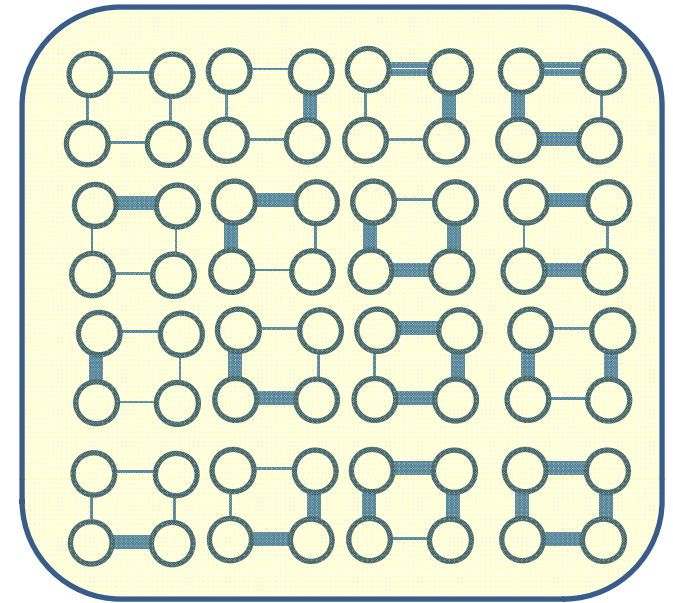
we conjectured it is mixing



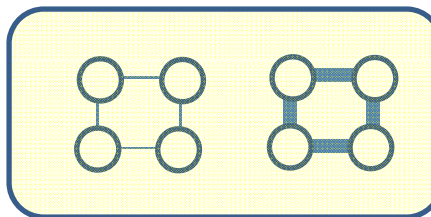
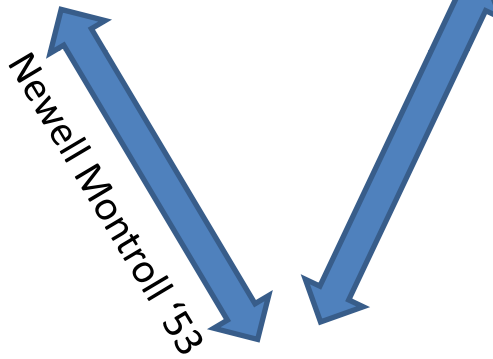
Our inspiration (Ising model):



Ising model: assignment of spins to sites weighted by the number of neighbors that agree



Random cluster model: subgraphs weighted by the number of components and the number of edges



High temperature expansion: even subgraphs weighted by the number of edges

Random cluster model

$$Z(\mathbf{G}, \mathbf{q}, \mu) = \sum_{S \subseteq E} \mathbf{q}^{\kappa(S)} \mu^{|S|}$$

number of connected
components of (G, S)

(Tutte polynomial)

Ising model

Potts model

chromatic polynomial

Flow polynomial

Random cluster model

$$Z(\mathbf{G}, \mathbf{q}, \mu) = \sum_{S \subseteq E} \mathbf{q}^{\kappa(S)} \mu^{|S|}$$

↑
number of connected components of (\mathbf{G}, S)

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Ising model

Potts model

chromatic polynomial

Flow polynomial

R_2 model

$$R_2(\mathbf{G}, \mathbf{q}, \mu) = \sum_{S \subseteq E} \mathbf{q}^{\text{rk}_2(S)} \mu^{|S|}$$

↑
rank (over F_2) of the adjacency matrix of (\mathbf{G}, S)

Matchings

Perfect matchings

Independent sets

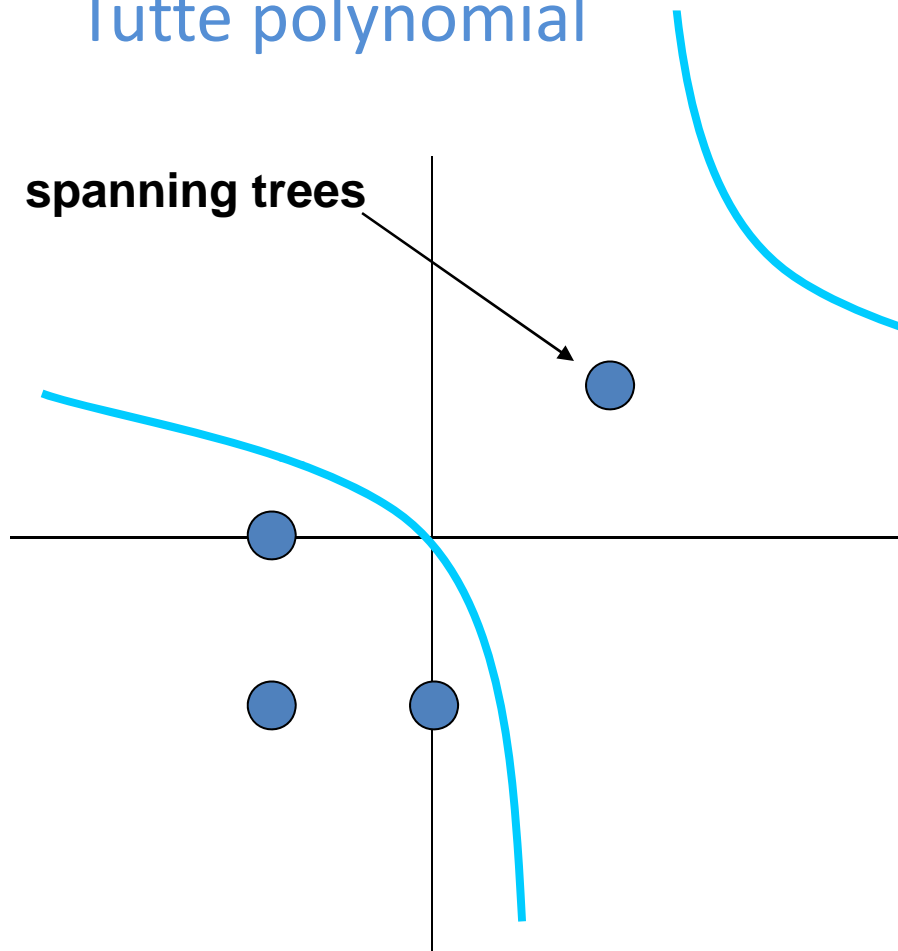
(for bipartite only!)

More ?

Complexity of exact evaluation

$$R_2'(G, q, \mu) = \sum_{S \subseteq E} q^{rk_2(S)} \mu^{|S|}$$

Tutte polynomial

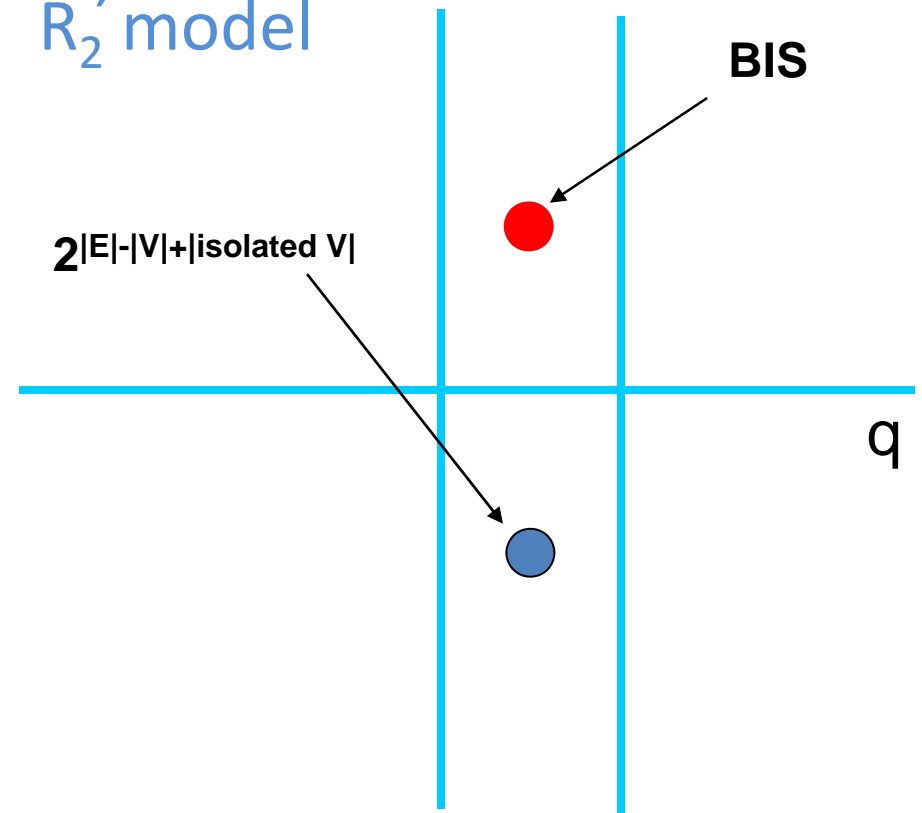


Jaeger, Vertigan, Welsh '90

easy if $(x-1)(y-1)=1$, or
 $(1,1), (-1,-1), (0,-1), (-1,0)$

#P-hard elsewhere

R_2' model



Ge, Štefankovič '09

easy if $q \in \{0, 1\}$
 or $\mu = 0$, or $(1/2, -1)$

#P-hard elsewhere (GRH)

“high-temperature expansion”

$$2^{|E|} \#BIS = \sum_{U \rightarrow \{0,1\}} \sum_{V \rightarrow \{0,1\}} \prod_{\{u,v\} \in E} (1 - \chi(\sigma(u), \sigma(v)))$$

where

$$\chi(1,1) = 1$$

$$\chi(0,1) = \chi(1,0) = \chi(0,0) = -1$$

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$$= \begin{cases} 0 & \text{if some } v \in V \text{ has an odd number of neighbors} \\ & \text{in } (U \cup V, S) \text{ labeled by 1} \\ (-2)^{|V|} & \text{otherwise} \end{cases}$$

“high-temperature expansion”

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$$= \sum_{S \subseteq E} (-1)^{|S|} \sum_{U \rightarrow \{0,1\}} \sum_{V \rightarrow \{0,1\}} \prod_{\{u,v\} \in S} \chi(\sigma(u), \sigma(v))$$

$$= 2^{|V|} \sum_{S \subseteq E} \text{number of } u \text{ such that } u^T A = 0 \pmod{2}$$

↑
bipartite adjacency matrix of $(U \cup V, S)$

$$= 2^{|V|+|U|} \sum_{S \subseteq E} 2^{-\text{rank}_2(A)}$$

“high-temperature expansion” – curious

$$f(A, \lambda) = \sum \lambda^{|\nu|_1} \left(\frac{1-\lambda}{1+\lambda} \right)^{|A\nu|_1}$$

$$f(A, 1) = 2^{\text{rank}_2(A)}$$



$$f(A, 1) = f(A^T, 1)$$

But in fact:

$$f(A, \lambda) = f(A^T, \lambda)$$

Questions:

Is there a polynomial-time sampler that produces matrices $A \leq B$ with $P(A) \propto 2^{-\text{rank}(A)}$?

$$R_2(\mathbf{G}, \mathbf{q}, \mu) = \sum_{S \subseteq E} q^{\text{rk}_2(S)} \mu^{|S|}$$

What other quantities does the R_2 polynomial encode ?