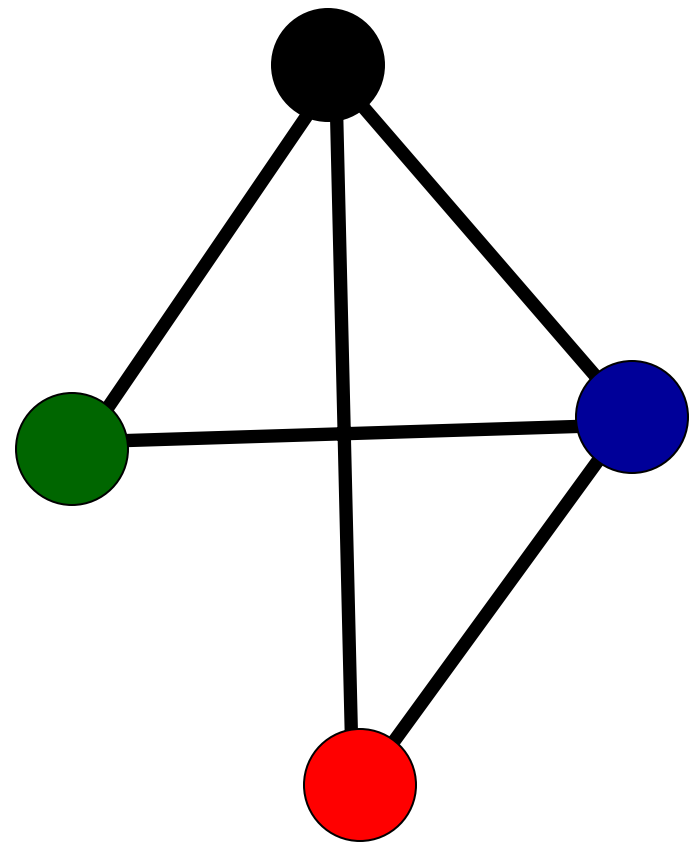
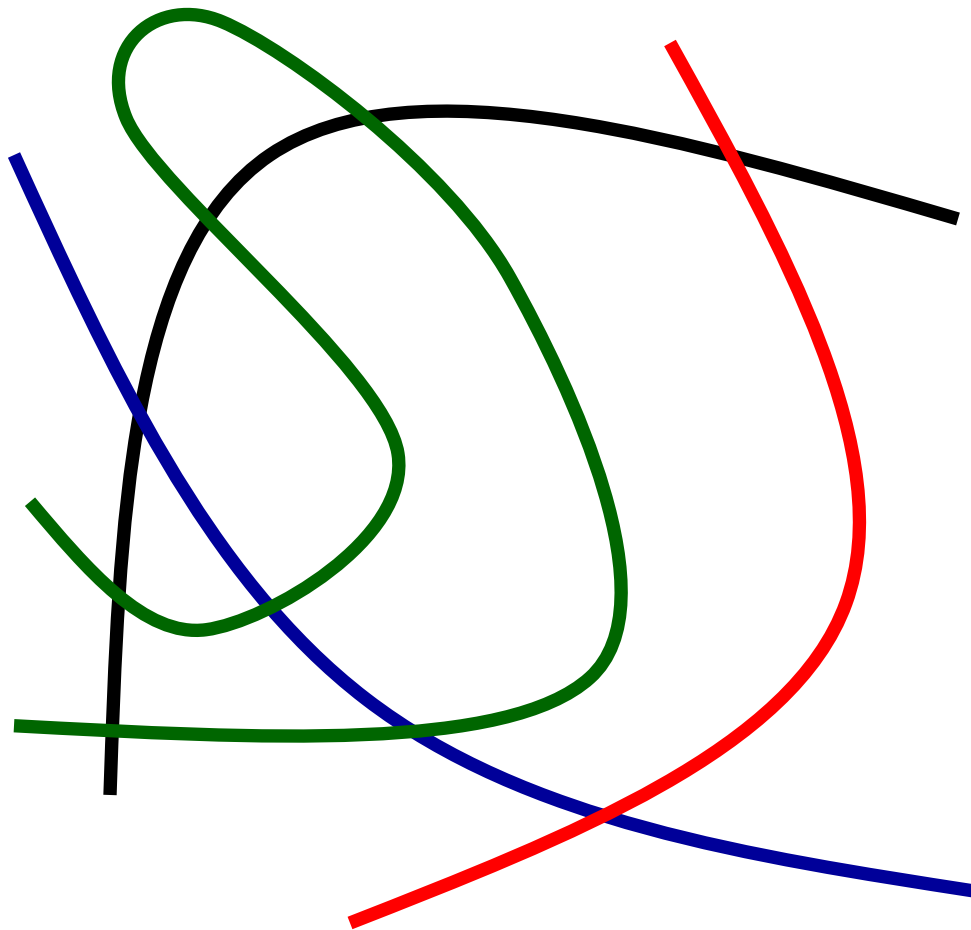


Decidability of String Graphs

Marcus Schaefer

Daniel Štefankovič

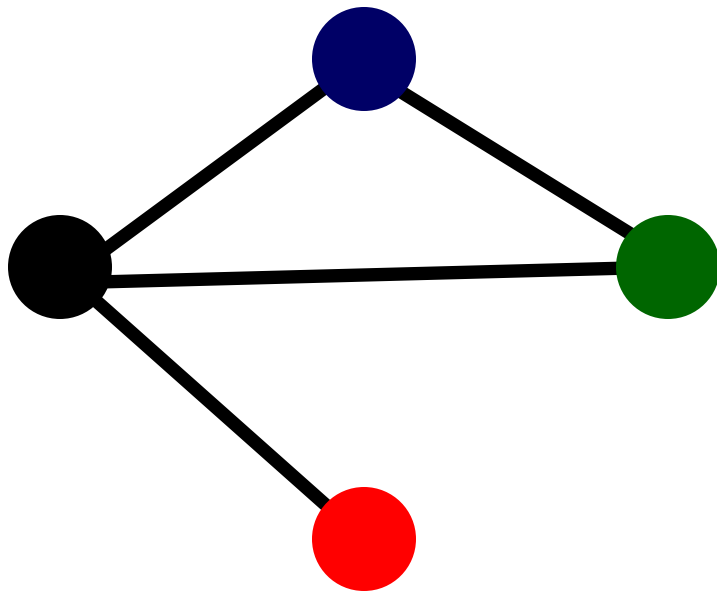
string graph = intersection
graph of a set of curves
in the plane



Examples:

K_n is a string graph for any n

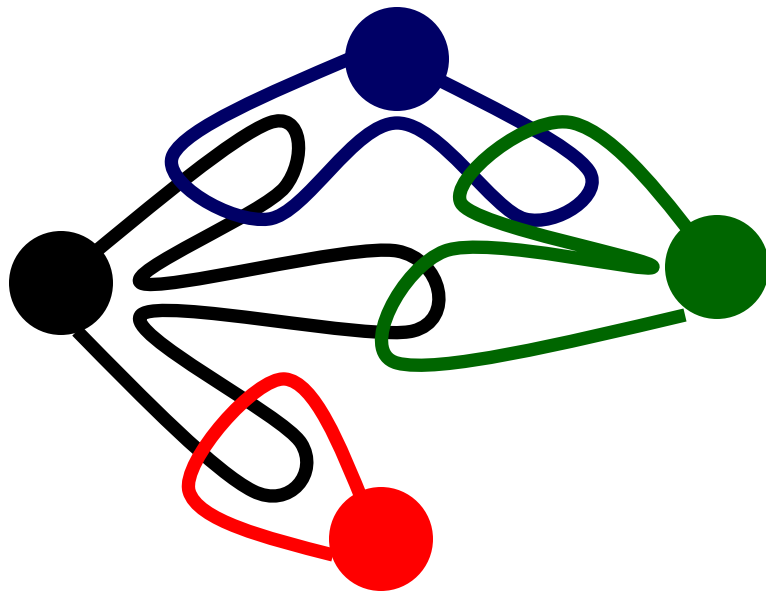
Any planar graph is a string graph



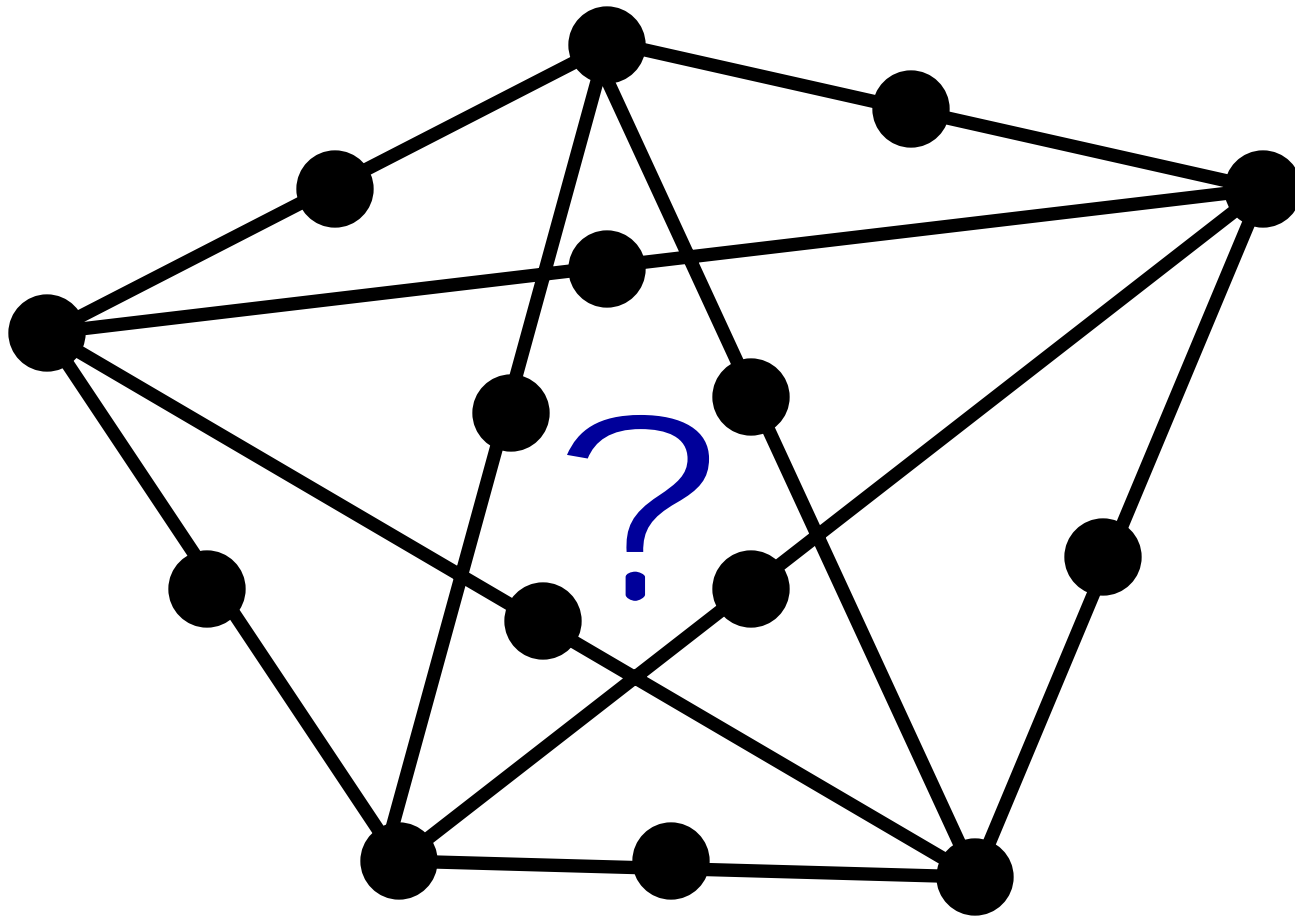
Examples:

K_n is a string graph for any n

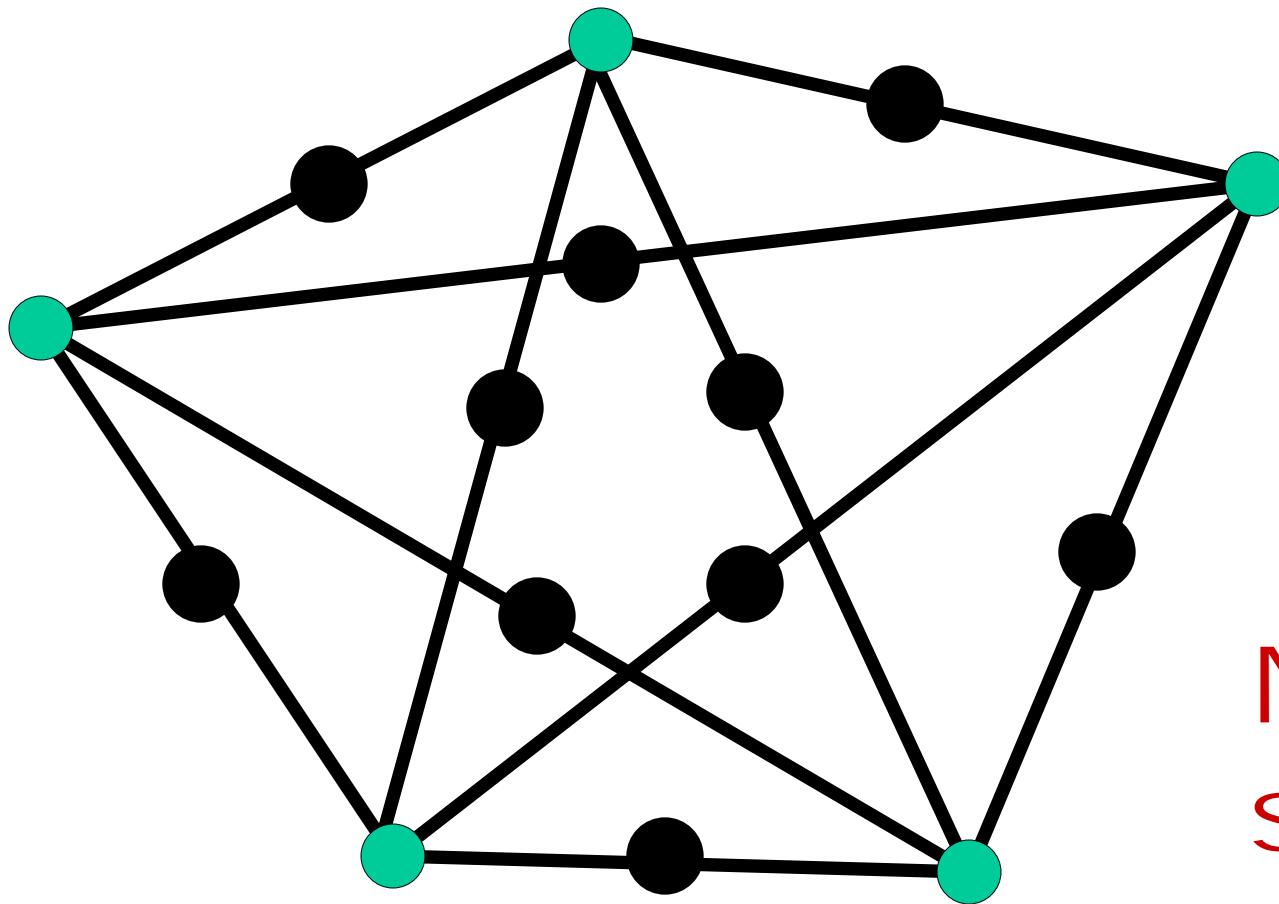
Any planar graph is a string graph



Given a graph G , decide if it is a string graph?



Given a graph G , decide if it is a string graph?



Not a string graph!

History of the Question:

S.Benzer: On the topology of the genetic fine structure '1959

F.W.Sinden: Topology of thin RC circuits '1966

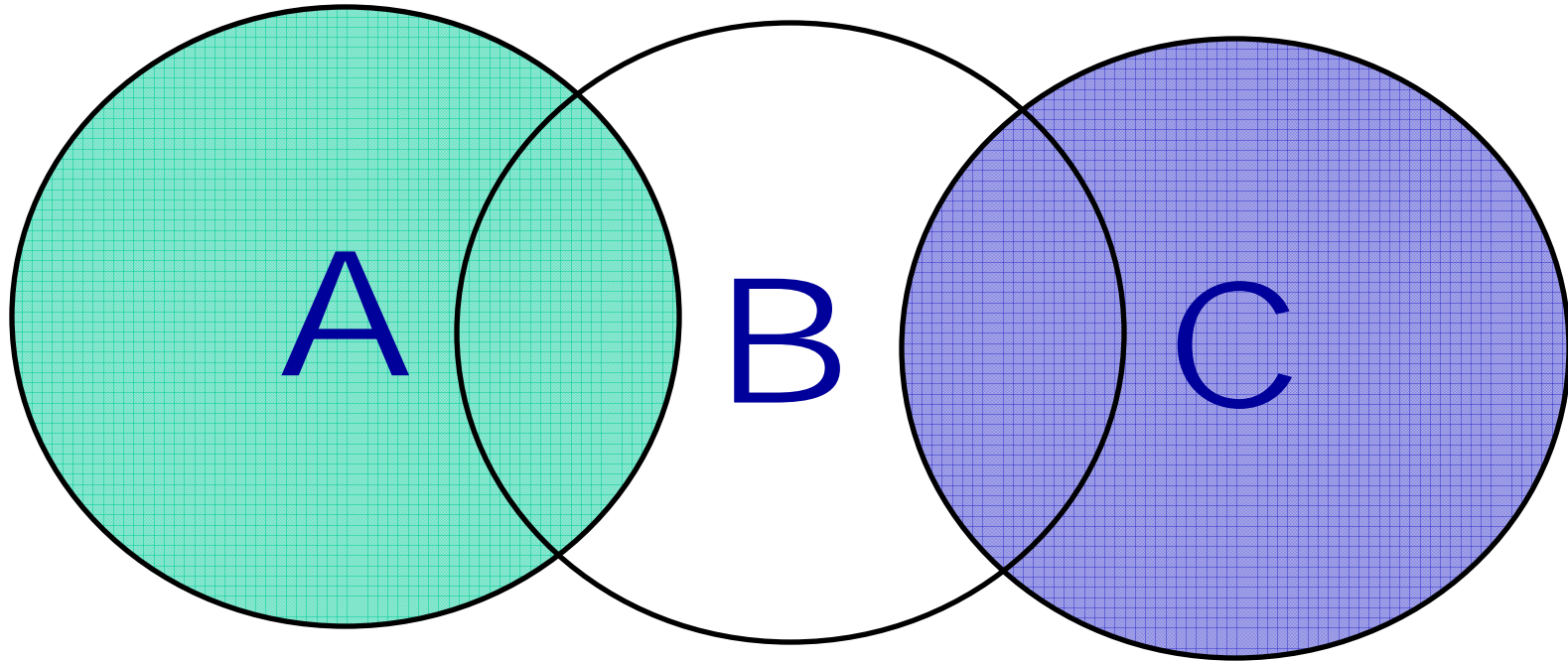
conductors of with some capacitance between them do intersect, other conductors do not intersect

Related older problem:

Diagrammatic reasoning

Is it possible that some A is B,
some B is C, but no A is C?

Is it possible that some A is B,
some B is C, but no A is C?



...has old history

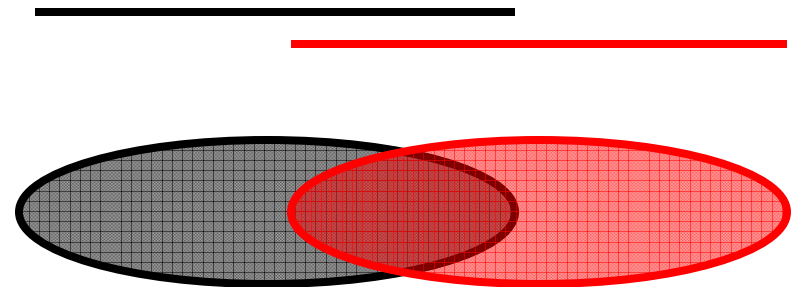
J. Vives

J. Sturm

G. Leibniz

J. Lambert

L. Euler



Can any true statement of type*

“Is it possible that some A is B,
some B is C, but no A is C?”

be represented using diagrams?

(in the plane, each class being represented
by a region homeomorphic to a disc)

*) i.e. for any two concepts we specify
whether they **must/can/can** not intersect.

Can any true statement of type

“Is it possible that some A is B,
some B is C, but no A is C?”

be represented using diagrams?

NO

Can we for any true statement
of type

“Is it possible that some A is B,
some B is C, but no A is C?”

decide whether it can be
represented by diagrams?

?

Can we for any true statement
of type

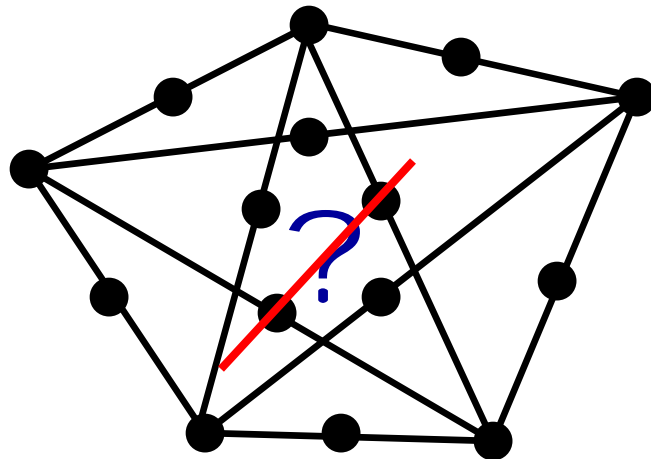
“Is it possible that some A is B,
some B is C, but no A is C?”

decide whether it can be
represented by diagrams?



NEXP

Given a graph G, decide if it is
a string graph?



Some known results:

Ehrlich, Even, Tarjan '76:

computing the chromatic number of a string graph is NP-complete

Kratochvíl '91:

recognizing string graphs is NP-hard

induced minor closed, infinitely many non-isomorphic forbidden induced minors

An interesting question:

Kratochvíl, Matoušek '91:

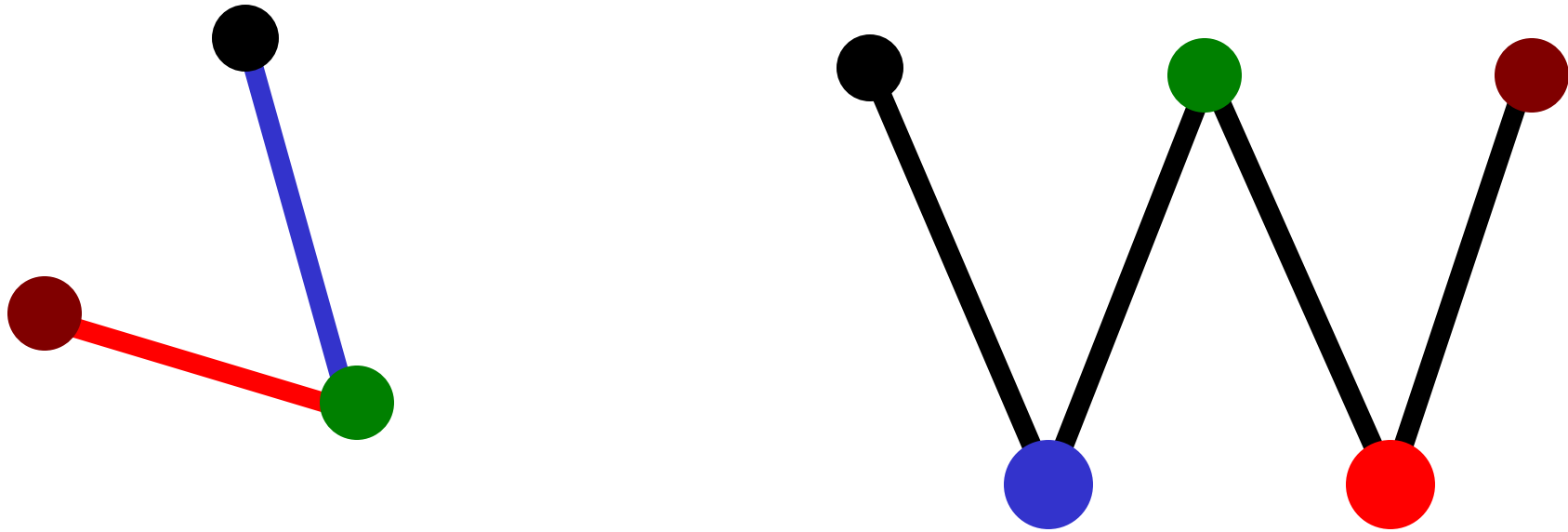
Can we give an upper bound on the number of intersections of the smallest realization?

Weak realizability:

given a graph G and a set of pairs of edges R – is there a drawing of G in which only edges in R may intersect?

e.g. for $R=0 \longrightarrow$ planarity

string graph \rightarrow weak realizability



any edge from ● and ●
from ● and ●

KM '91

Kratochvíl, Matoušek'91:

Can we give an upper bound on the number of intersections of the smallest weak realization?

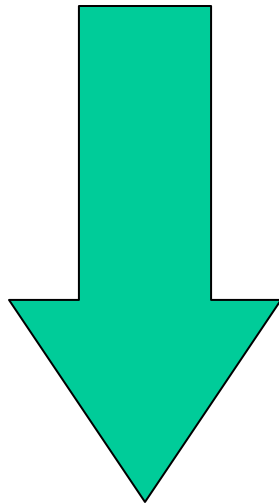
SURPRISE!

[KM'91]

There are graphs whose smallest weak representation has exponentially many intersections!

Conjecture[KM'91]: at most exponentially many intersections

Theorem: A graph with m edges has weak realization with at most $m2^m$ intersections.



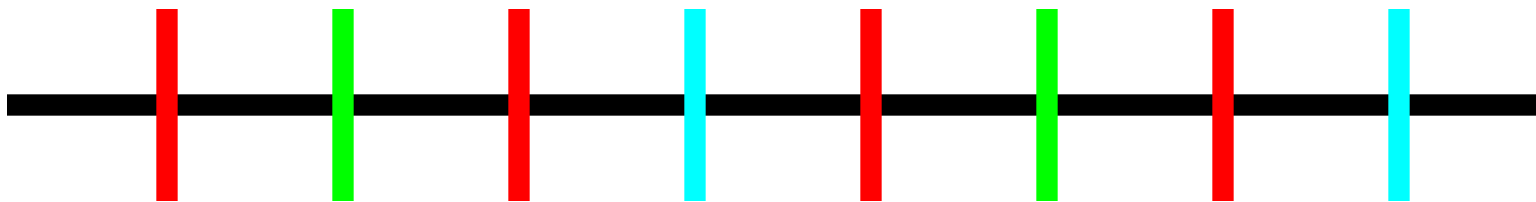
Deciding string graphs is in NEXP.

Given a graph G and pairs of edges which are allowed to intersect (some set R).

(e.g. K_5 with 2 edges allowed to intersect)

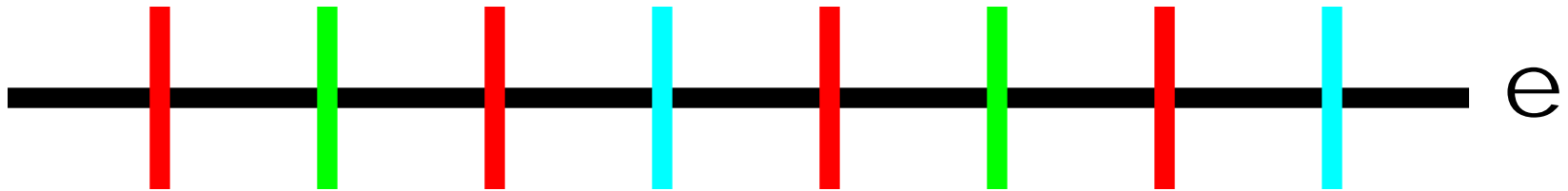
If (G, R) can be realized in the plane, can we give an upper bound on the number of intersections in the smallest realization?

Idea: if there are too many intersections on an edge we will be able to redraw the realization to reduce the number of intersections.



color the edges

suppose there are $>2^m$ intersections on e

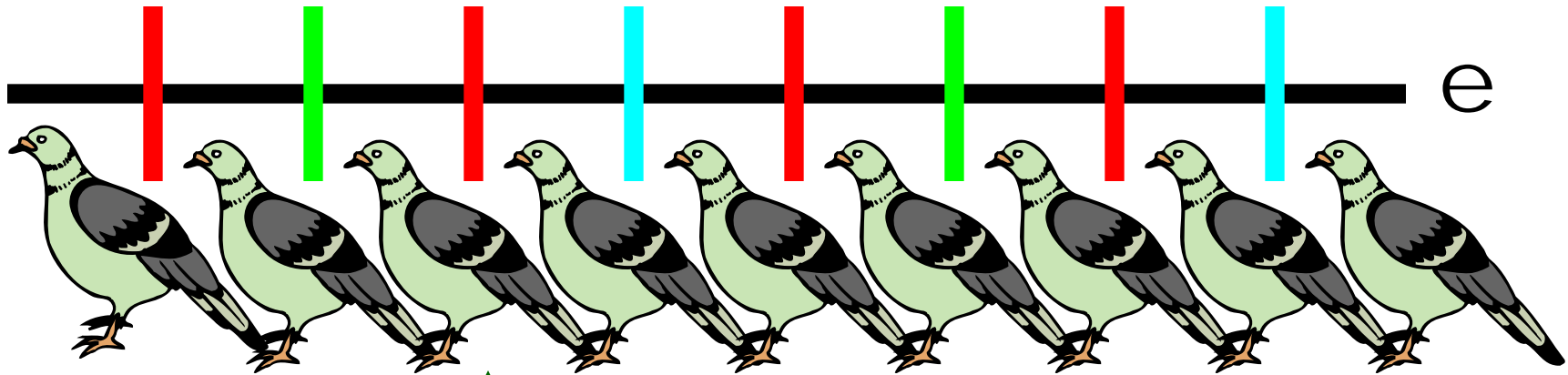


(nontrivial = with >0 intersections)



Then there is a non-trivial segment of e where each color occurs even number of times (possibly 0).

suppose there are $>2^m$ intersections on e

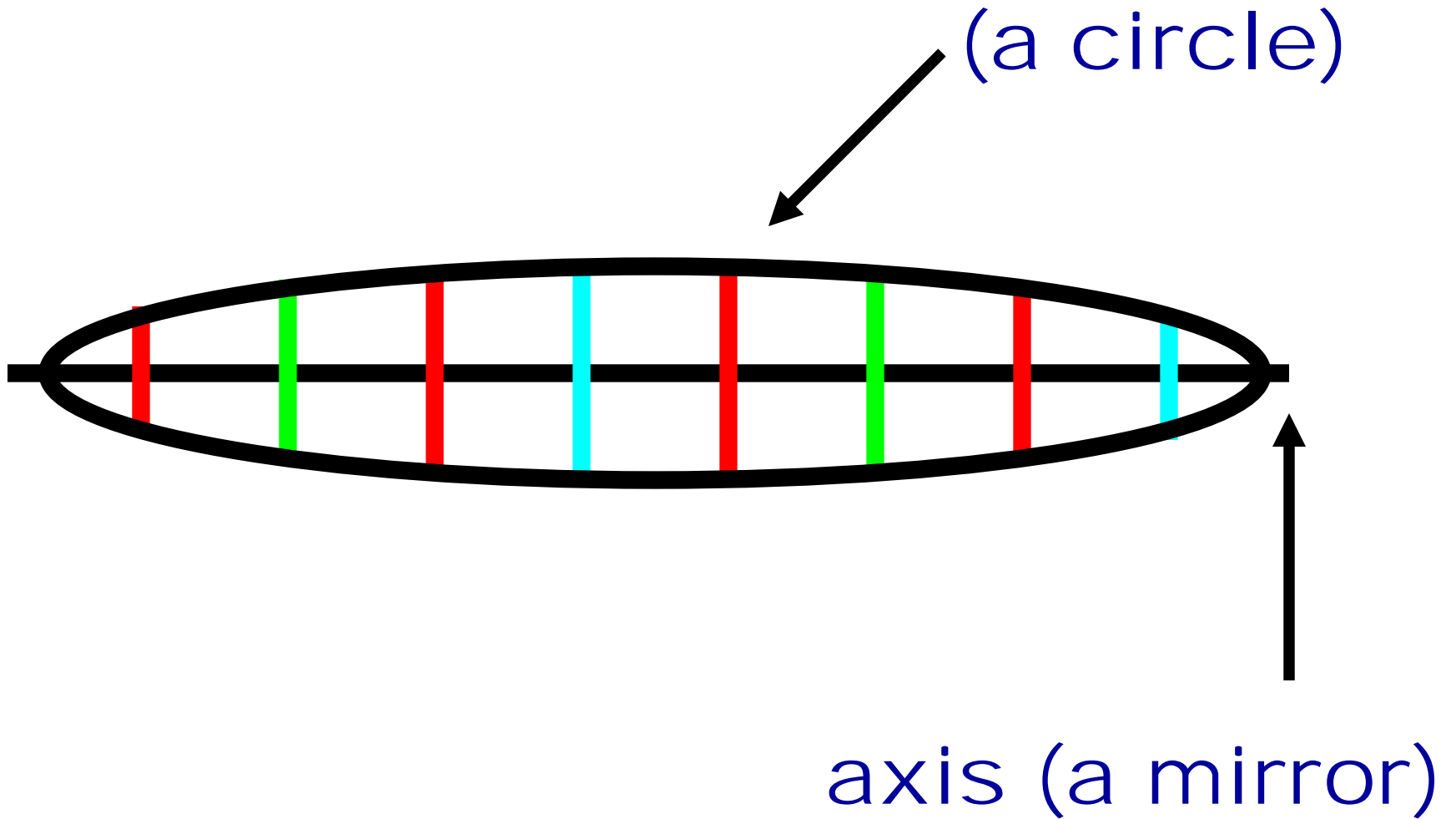


↑ vector of parities of
the colors to the left

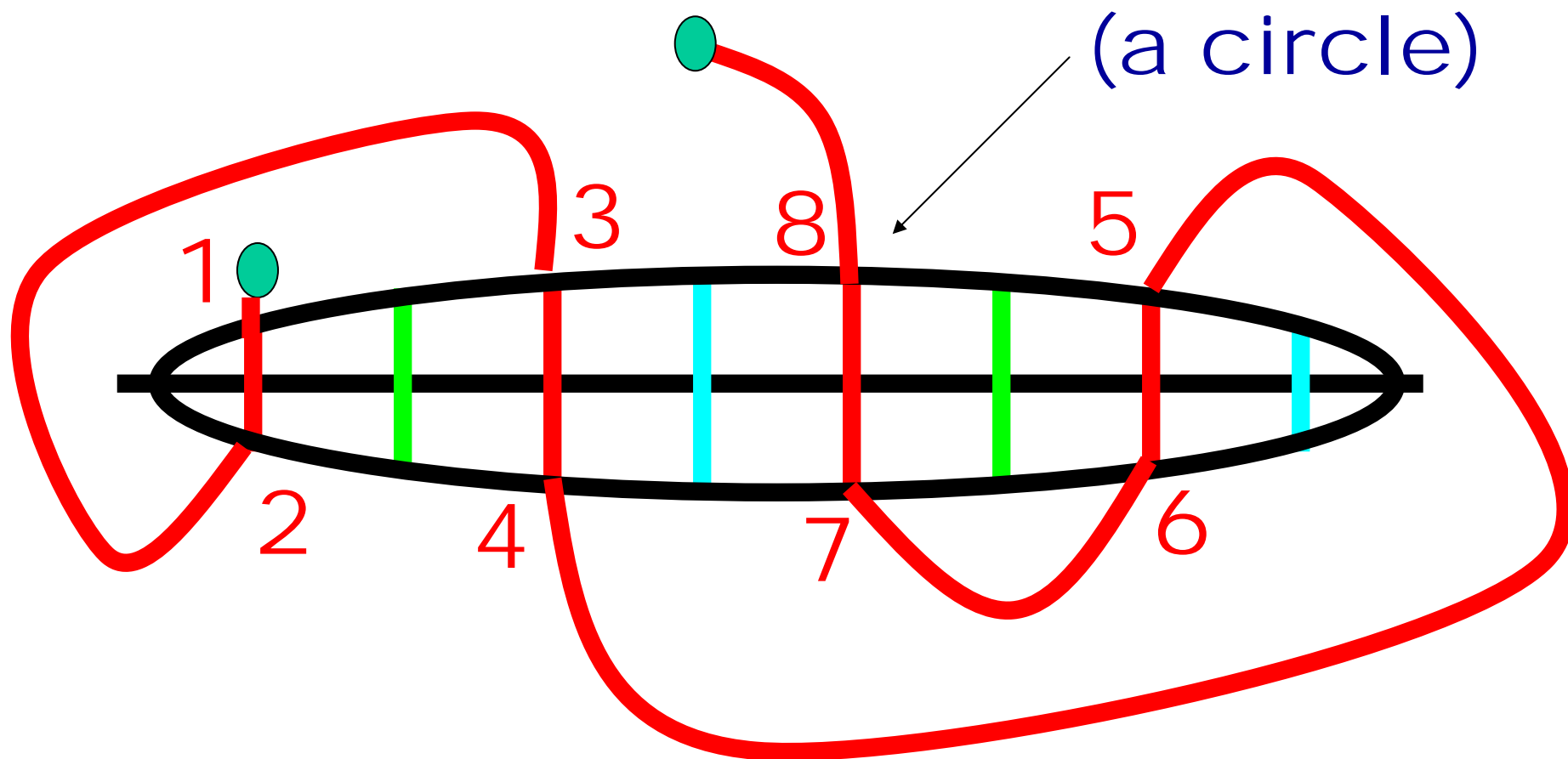
(2^m pigeonholes)

Then there is a non-trivial segment of e where each color occurs even number of times (possibly 0).

look at the segment:



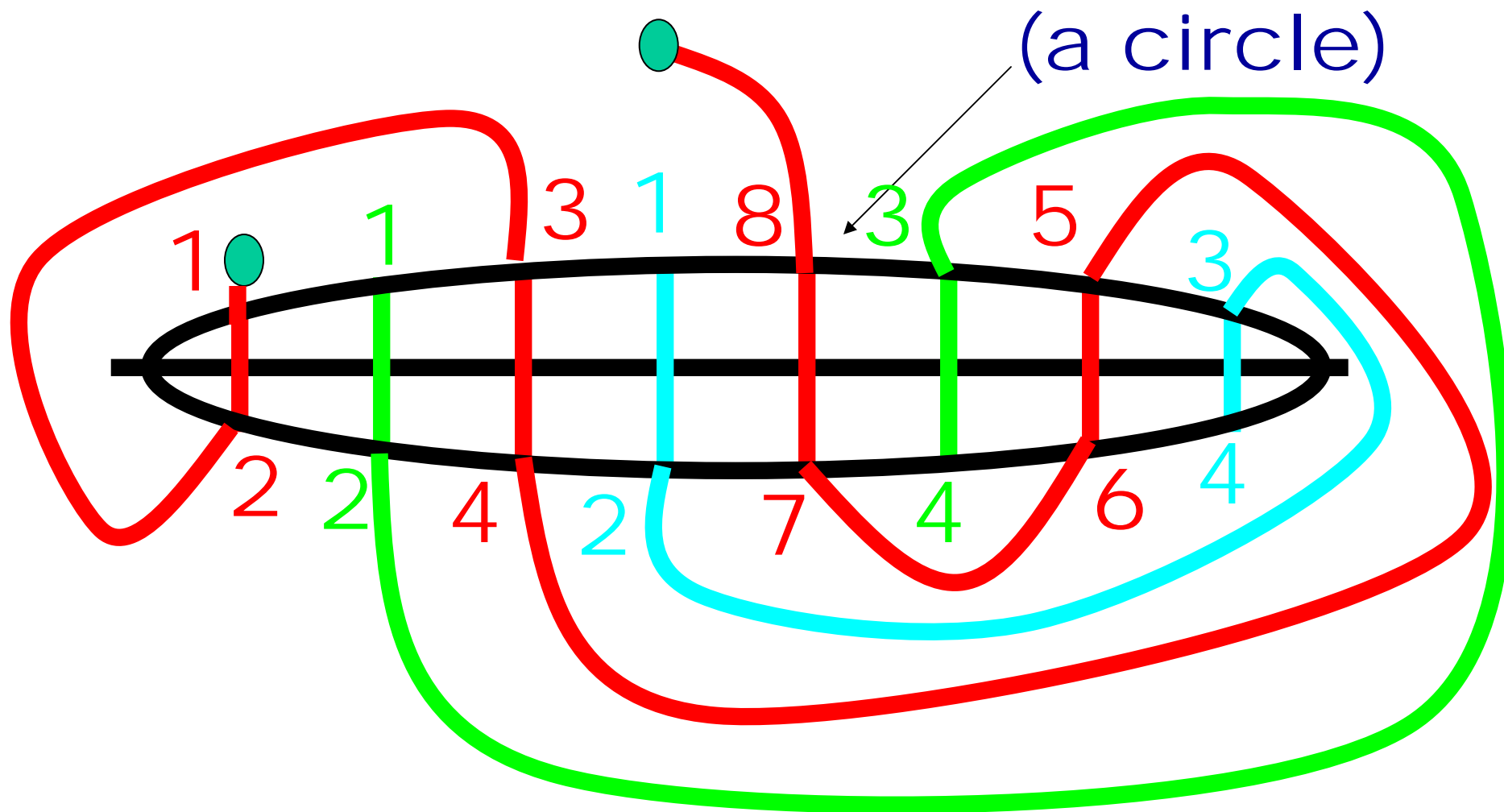
number the intersections with circle:



2-3, 6-7, ..., $4k-2 - 4k-1$ - connected outside

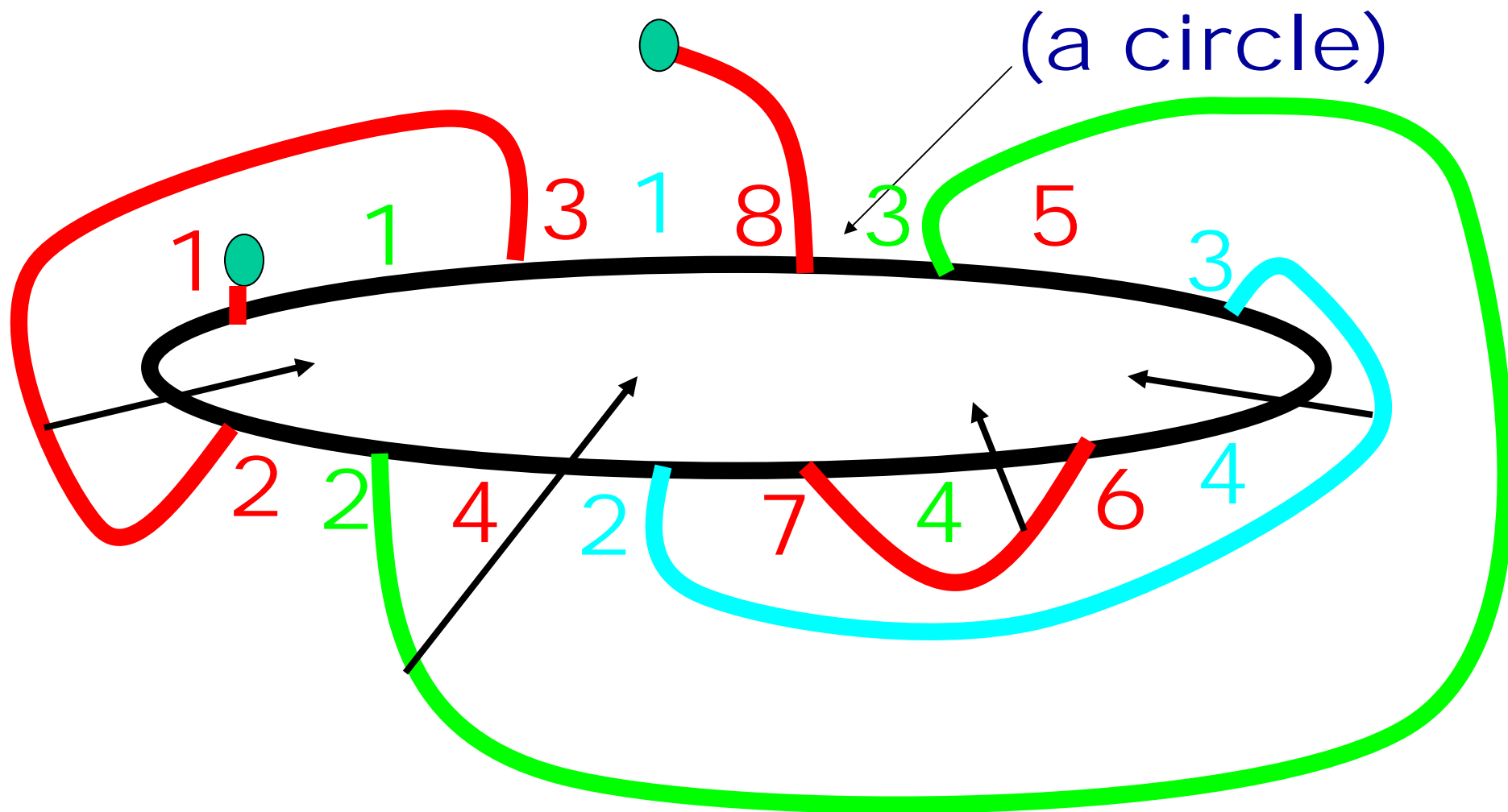
4-5, ..., $4k - 4k+1$ - also connected outside

look at the connections 2-3,6-7,...:



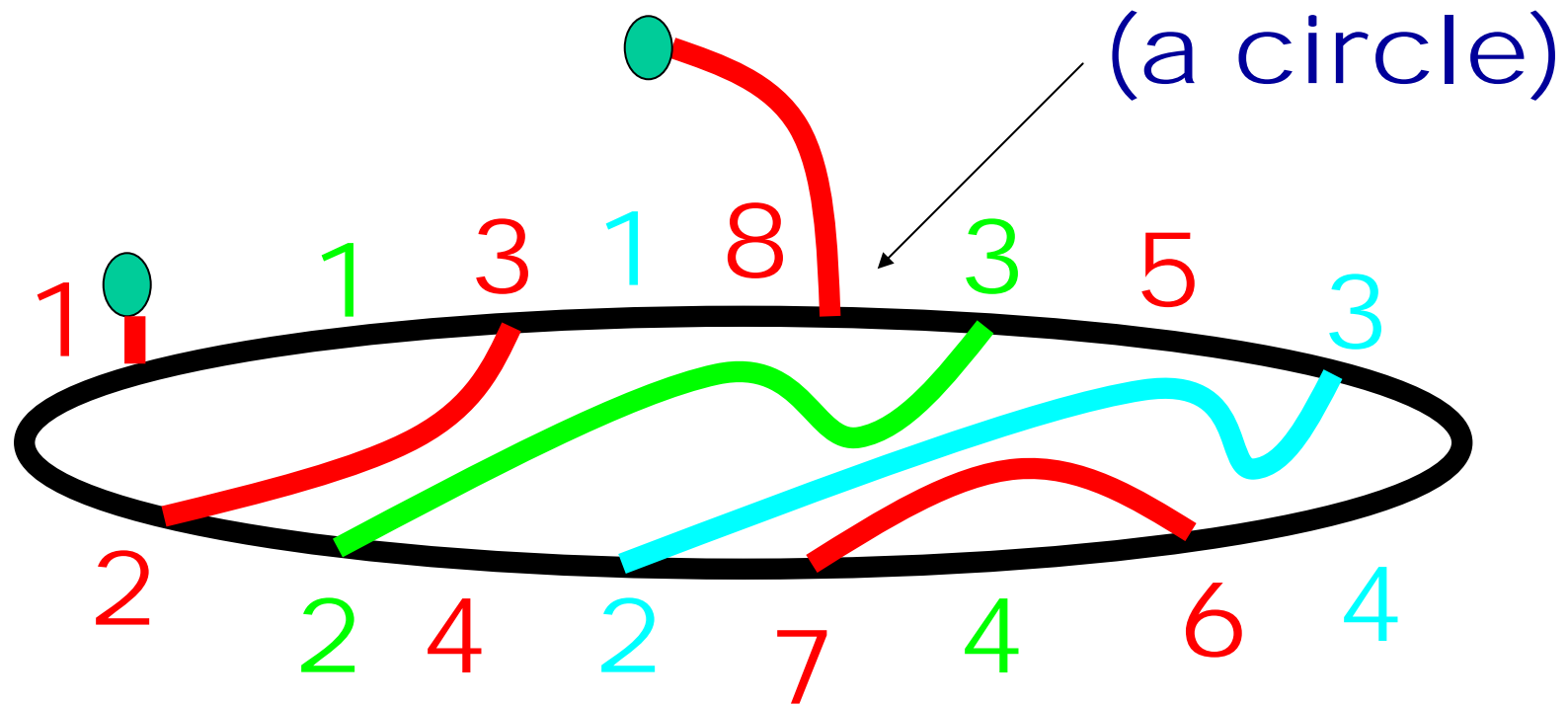
(for all colors, respecting allowed intersections)
2-3,6-7,...,4k-2-4k-1- connected outside

clear the inside and bring them inside

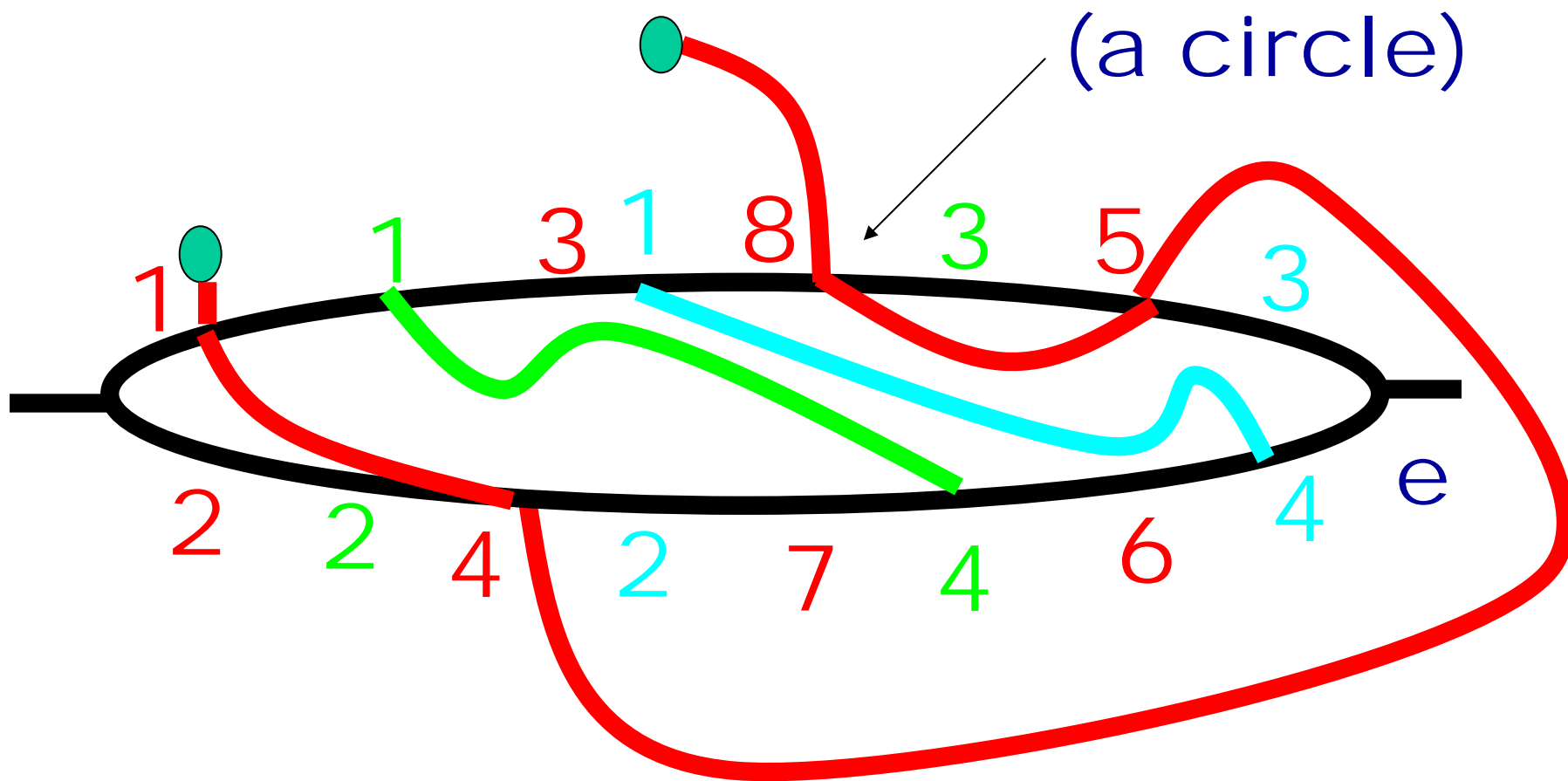


(for all colors, respecting allowed intersections)
2-3, 6-7, ..., $4k-2-4k-1$ - connected outside

clear the inside and bring them inside



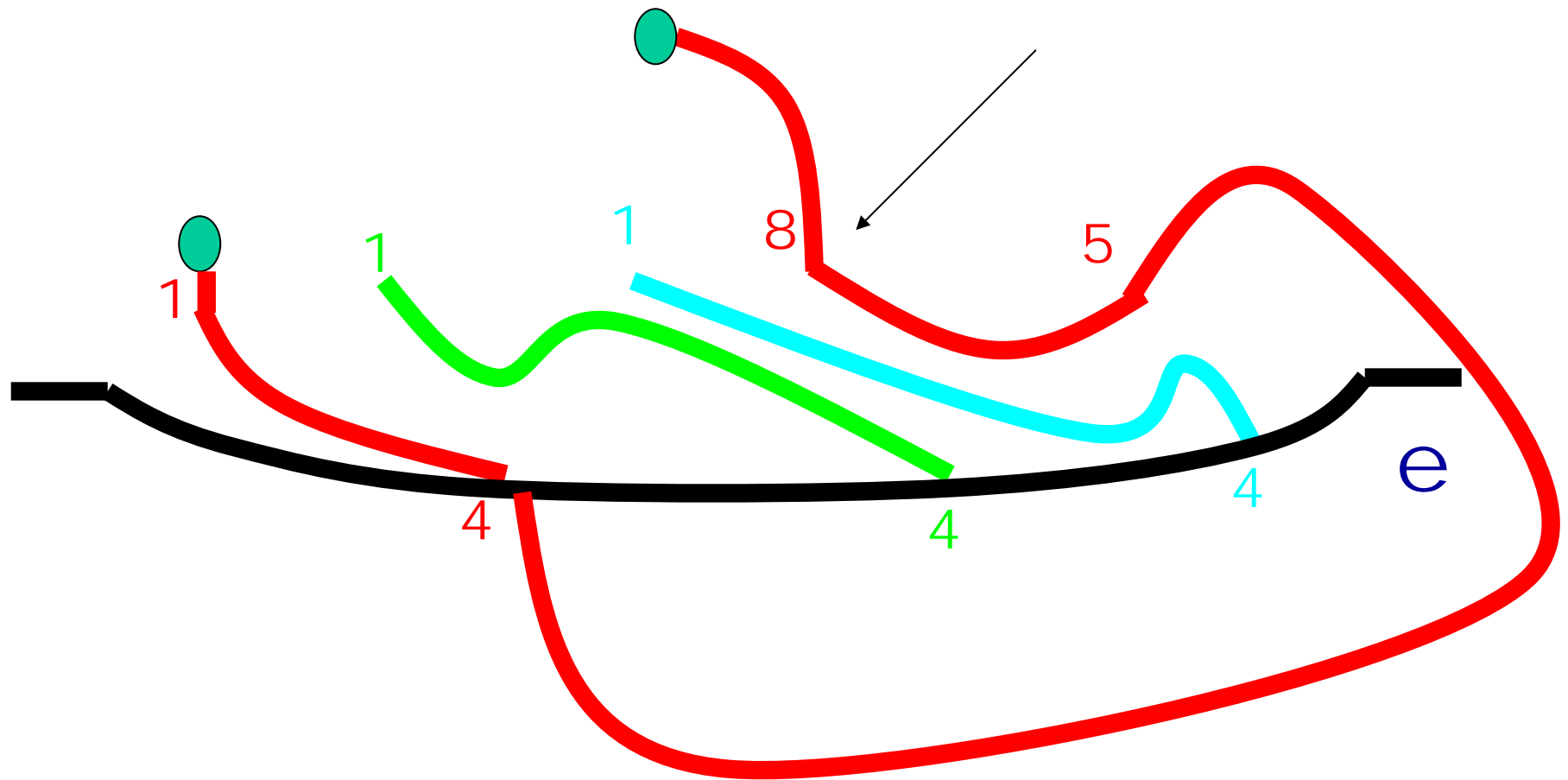
(4-5, ..., 4k - 4k+1- connected outside)



use mirror – now everything is connected.

What about e?

use upper or lower half of the circle as e

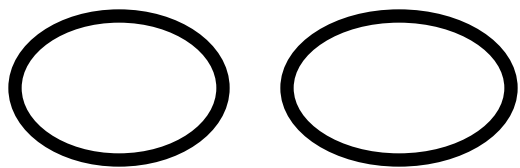


Decreased the # of intersections!

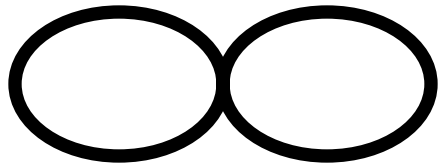
(thus in a realization with minimal number of intersections $< m2^m$ of them)

consequences to topological inference:

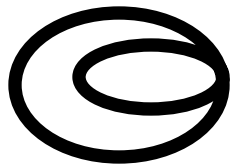
can decide realizability for more complex formulas:



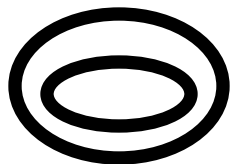
disjoint



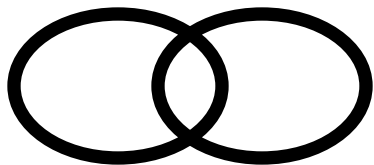
meet



covered



inside



overlap

in NEXP

Conclusion:

NEXP

EXP

PSPACE

STRING GRAPHS

?

PH

⋮

NP

