string graph = intersection graph of a set of curves in the plane
Examples:

\( K_n \) is a string graph for any \( n \)

Any planar graph is a string graph
Examples:

\( K_n \) is a string graph for any \( n \)

Any planar graph is a string graph
Given a graph G, decide if it is a string graph?
Given a graph $G$, decide if it is a string graph?

Not a string graph!
History of the Question:

**S. Benzer:** On the topology of the genetic fine structure '1959

**F.W. Sinden:** Topology of thin RC circuits '1966

conductors of with some capacitance between them do intersect, other conductors do not intersect
Related older problem:

Diagrammatic reasoning

Is it possible that some A is B, some B is C, but no A is C?
Is it possible that some A is B, some B is C, but no A is C?
...has old history

J. Vives
J. Sturm
G. Leibniz
J. Lambert
L. Euler
Can any true statement of type*

“Is it possible that some A is B, some B is C, but no A is C?”

be represented using diagrams?

(in the plane, each class being represented by a region homeomorphic to a disc)

*) i.e. for any two concepts we specify whether they must/can/can not intersect.
Can any true statement of type

“Is it possible that some A is B, some B is C, but no A is C?”

be represented using diagrams?

NO
Can we for any true statement of type

“Is it possible that some A is B, some B is C, but no A is C?”

decline whether it can be represented by diagrams?
Can we for any true statement of type

"Is it possible that some A is B, some B is C, but no A is C?"

declare whether it can be represented by diagrams?

Given a graph G, decide if it is a string graph?
Some known results:

Ehrlich, Even, Tarjan ’76:
computing the chromatic number of a string graph is NP-complete

Kratochvíl ’91:
recognizing string graphs is NP-hard

induced minor closed, infinitely many non-isomorphic forbidden induced minors
An interesting question:

Kratochvíl, Matoušek ’91:
Can we give an upper bound on the number of intersections of the smallest realization?
Weak realizability:

given a graph $G$ and a set of pairs of edges $R$ – is there a drawing of $G$ in which only edges in $R$ may intersect?

e.g. for $R=0$ \[\rightarrow\] planarity
string graph $\rightarrow$ weak realizability

any edge from $\bullet$ and $\bullet$ from $\bullet$ and $\bullet$

KM '91
Can we give an upper bound on the number of intersections of the smallest weak realization?

SURPRISE! [KM’91]

There are graphs whose smallest weak representation has exponentially many intersections!

Conjecture[KM’91]: at most exponentially many many intersections
Theorem: A graph with $m$ edges has weak realization with at most $m^{2^m}$ intersections.

Deciding string graphs is in NEXP.
Given a graph $G$ and pairs of edges which are allowed to intersect (some set $R$).

(e.g. $K_5$ with 2 edges allowed to intersect)

If $(G,R)$ can be realized in the plane, can we give an upper bound on the number of intersections in the smallest realization?
Idea: if there are too many intersections on an edge we will be able to redraw the realization to reduce the number of intersections.

color the edges
suppose there are $>2^m$ intersections on $e$

Then there is a non-trivial segment of $e$ where each color occurs even number of times (possibly 0).

(nontrivial = with $>0$ intersections)
suppose there are \( >2^m \) intersections on \( e \)

Then there is a non-trivial segment of \( e \) where each color occurs even number of times (possibly 0).

\[ \uparrow \text{vector of parities of the colors to the left} \]

\((2^m \text{ pigeonholes})\)

Then there is a non-trivial segment of \( e \) where each color occurs even number of times (possibly 0).
look at the segment:

(a circle)

axis (a mirror)
number the intersections with circle:

2-3, 6-7, ..., 4k-2 – 4k-1- connected outside

4-5, ..., 4k – 4k+1- also connected outside
look at the connections 2-3, 6-7, ...:

(for all colors, respecting allowed intersections)

2-3, 6-7, ..., 4k-2-4k-1 - connected outside
clear the inside and bring them inside

(a circle)

(for all colors, respecting allowed intersections)

2-3, 6-7, ..., 4k-2-4k-1- connected outside
clear the inside and bring them inside

(a circle)

(4-5,...,4k – 4k+1- connected outside)
use mirror – now everything is connected.

What about e?
use upper or lower half of the circle as $e$

Decreased the # of intersections!

(Thus in a realization with minimal number of intersections $<m2^m$ of them)
consequences to topological inference:
can decide realizability for more complex formulas:

\begin{itemize}
  \item disjoint
  \item meet
  \item covered
  \item inside
  \item overlap
\end{itemize}
in NEXP
Conclusion:

STRING GRAPHS

NEXP
EXP
PSPACE
PH
NP