1 Schedule

Problem sessions:

The homework is due Nov 18, 2008.
The QUIZ will be on Tuesday, Nov. 25.

2 List of algorithms covered in the class

(B-basic, I-intermediate, A-advanced):

I: Gaussian elimination (p. 219, DSV).
I: Computing the dual of a linear program (p. 206, DSV).
A: The simplex algorithm (p. 213, DSV).
B: Zero-sum games using linear programming (p. 208, DSV).
A: Max-flow using linear programming (p. 198, DSV).

3 Basic material

Important concepts, problems, theorems, and algorithms:

- system of linear equations, rank of a matrix,
- linear program, dual linear program,
- basic matrix notation.

Testing method:

- Solve a small linear program (2-3 variables).
- Solve a system of equations using Gaussian elimination (up to 4 variables).
- Given a system of equations, write it in a matrix form.
- Compute rank of a matrix.
- Solve a zero-sum game.

Example problems:

5.1 (due Nov 18, 2008) Solve the following linear program:

\[
\begin{align*}
\text{max } & x_1 + x_2 \\
& x_1 + 2x_2 \leq 3 \\
& 3x_1 + x_2 \leq 4 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{align*}
\]
5.2 (due Nov 18, 2008) Solve the following system of equations using Gaussian elimination:

\[ \begin{align*}
y_1 + y_2 + y_4 &= 4, \\
y_1 + y_3 &= 10, \\
y_3 + 4y_4 &= 3, \\
y_2 + y_4 &= 1. \\
\end{align*} \]

5.3 (due Nov 18, 2008) Write the following system of equations in the matrix form \( Ax = b \):

\[ \begin{align*}
y_1 + y_2 + y_4 &= 4, \\
y_1 + y_3 &= 10, \\
y_3 + 4y_4 &= 3, \\
y_2 + y_4 &= 1. \\
\end{align*} \]

5.4 (due Nov 18, 2008) Compute the rank of the following two matrices

\[
\begin{pmatrix} 2 & 4 & 9 \\ 1 & -3 & 5 \\ 0 & 10 & -1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 4 & 9 & 3 \\ 1 & -3 & 5 & -1 \\ 0 & 10 & -1 & 5 \end{pmatrix}
\]

Does the system \( 2x + 4y + 9z = 3, x - 3y + 5z = -1, 10y - z = 5 \) have a solution?

5.5 (removed)

4 Additional homework

5.6 (due Nov 18, 2008) Solve the following linear program:

\[
\max \sum_{i=1}^{9} x_i, \quad x_1 + x_2 \leq 1, \\
x_2 + x_3 \leq 1, \\
x_3 + x_4 \leq 1, \\
x_4 + x_5 \leq 1, \\
x_5 + x_6 \leq 1, \\
x_6 + x_7 \leq 1, \\
x_7 + x_8 \leq 1, \\
x_8 + x_9 \leq 1, \\
x_9 + x_1 \leq 1, \\
x_i \geq 0, \ i = 1, \ldots, 9.
\]

Use a linear programming solver to obtain the solution (for example you can use freeware lpsolve or function Maximize in Mathematica (installed in most labs)).

5.7 (due Nov 18, 2008) There are \( n \) bottles which contain different mixtures of three chemicals called \( A, B, C \). The \( i \)-th bottle contains the chemicals in ratio \( a_i : b_i : c_i \) (thus, \( a_i / (a_i + b_i + c_i) \) fraction of the \( i \)-th bottle is chemical \( A \), \( b_i / (a_i + b_i + c_i) \) fraction of the \( i \)-th bottle is chemical \( B \), and \( c_i / (a_i + b_i + c_i) \) fraction of the \( i \)-th bottle is chemical \( C \)). We want to know whether it is possible to obtain a mixture containing the chemicals \( A, B, C \) in ratio \( a : b : c \) by mixing various amounts from the bottles. Give an efficient algorithm for this problem.

For example, if the input is \( n = 2 \), the ratios in the bottles are \( 1 : 1 : 2 \) and \( 3 : 3 : 1 \), and we want to obtain mixture with ratio \( 1 : 1 : 1 \) then the answer is YES (we can take 2 parts from the first bottle and 1 part from the second bottle).
5.8 (due Nov 18, 2008) Construct the linear program dual to the following linear program:

\[
\begin{align*}
\text{max } & \quad x_1 + 2x_2 + 3x_3 + 4x_4 \\
\text{subject to } & \quad x_2 + x_3 + x_4 \leq 1 \\
& \quad x_1 + x_3 + x_4 \leq 2 \\
& \quad x_1 + x_2 + x_4 \leq 3 \\
& \quad x_1 + x_2 + x_3 \leq 4 \\
& \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.
\end{align*}
\]

Find the optimal solution of the primal and the dual problem. Use a linear programming solver to obtain the solutions.

5 Additional problems from the book (do not turn in)

Try to solve the following problems. A few of them will be on the quiz. We will go over the ones that you choose in the problem sessions.

- 7.1, 7.2, 7.3, 7.4, 7.7, 7.8, 7.11, 7.12, 7.13, 7.15, 7.19, 7.27.