

IMPORTANT: Write each problem on a separate sheet of paper. Write your name on each sheet. (Why? To speed up the grading we will use parallelization—each problem will be graded by a different TA. At the beginning of the class (on the date when the homework is due) there will be a separate pile for each problem.)

Homework 3 due: Oct. 15 (Thursday); collected before class in B&L 109.

Problem sessions before Homework 3 is due:

Oct. 12 (Monday) 6:30pm - 7:30pm in Goergen 108

Oct. 13 (Tuesday) 6:00pm - 7:00pm in Goergen 108

Oct. 14 (Wednesday) 6:15pm - 7:15pm in Goergen 108

Oct. 14 (Wednesday) 7:30pm - 8:30pm in Goergen 108

Exam 2 is on: Oct. 22 (Thursday).

Problem sessions before Exam 2:

Oct. 19 (Monday) 6:30pm - 7:30pm in Goergen 108

Oct. 20 (Tuesday) 6:00pm - 7:00pm in Goergen 108

Oct. 21 (Wednesday) 6:15pm - 7:15pm in Goergen 108

Oct. 21 (Wednesday) 7:30pm - 8:30pm in Goergen 108

1 Homework - solve and turn in

3.1 (due Thursday, Oct 16) Suppose we want to run Dijkstra’s algorithm on a graph whose edge weights are in the range $\{0, \dots, W\}$. Show how Dijkstra’s algorithm can be made to run in time $O(W|V| + |E|)$.

3.2 (due Thursday, Oct 16) We are given a weighted directed graph $G = (V, E, w)$. (The weights represent widths of roads in a road network.) The width of a path P is the minimum of the weights of edges in P . (Intuitively, the width of a path is the maximum width of an object that can be transported on that path without the object ever “sticking out”.) Given two vertices s, t we want to compute the widest path from s to t . Give an algorithm for the problem; your algorithm should be as fast as possible; state the running time of your algorithm.

3.3 (due Thursday, Oct 16) Consider the following modified Dijkstra’s algorithm (The modification is on the last line—in the original Dijkstra’s algorithm we never reinsert a finalized vertex into the queue.)

```

⟨ ... initialization ... ⟩
H := makequeue(V)
while H is not empty
    v := deletemin(H)
    for each e = (v, z) ∈ E, z not marked
        if dist(v) + w(v,z) < dist(z) then
            dist(z) = dist(v) + w(v,z)
            if z is in the queue H then decreasekey(H,z) else insert(H,z)

```

PART A: Prove that the algorithm correctly computes shortest paths in graphs with no negative cycles.

PART B: Consider the following weighted directed graph $G = (V, E, w)$: $V = \{a_0, \dots, a_n, b_0, \dots, b_{n-1}\}$, $E = \bigcup_{i \in \{0, \dots, n-1\}} \{(a_i, a_{i+1}), (a_i, b_i), (b_i, a_{i+1})\}$, $w(a_i, a_{i+1}) = 10^{n-i}$, $w(a_i, b_i) = 10^{n+1-i}$, $w(b_i, a_{i+1}) = -9 \cdot 10^{n-i} - 1$. What is the running time of the modified Dijkstra’s algorithm on this graph (computing distances from a_0) ?

2 Bonus Homework - solve and turn in

Definitions needed for problems 3.5 - 3.7:

Let $G = (V, E)$ be an undirected graph. A *matching* of G is a set of edges $E' \subseteq E$ such that every vertex of G is in *at most one* edge of E' . A *perfect matching* of G is a set of edges $E' \subseteq E$ such that every vertex of G is in *exactly one* edge of E' .

Let $G = (V, E, w)$ be an undirected graph where w are edge-weights. Let $E' \subseteq E$ be a matching of G . The weight of E' is

$$\sum_{e \in E'} w(e).$$

We will consider the following two problems:

MAX-WEIGHT MATCHING:

INPUT: weighted graph $G = (V, E, w)$

OUTPUT: a matching $E' \subseteq E$ of G with the maximum weight.

MAX-WEIGHT PERFECT MATCHING:

INPUT: weighted graph $G = (V, E, w)$

OUTPUT: a perfect matching $E' \subseteq E$ of G with the maximum weight.

You need the following information for problems 3.5 and 3.7:

FACT: there is a polynomial-time algorithm to find a maximum-weight matching.

3.4 (warm-up problem—do not turn in) We are given n rectangles of sizes $a_1 \times b_1, \dots, a_n \times b_n$. We want to build the highest tower out of the rectangles. In a tower, if a rectangle of width w is on top of a rectangle of width w' then we require $w \leq w'$. We are allowed to rotate the rectangles (i. e., an $a \times b$ rectangle can be changed into a $b \times a$ rectangle). Give an $O(n)$ algorithm which finds the height of the highest possible tower.

(For example if the input is $11 \times 11, 8 \times 2, 1 \times 10$ then the solution is a tower of height $29 = 11 + 8 + 10$.)

3.5 (due Thursday, Oct 16) Suppose that in Problem 3.4 we change the requirement $w \leq w'$ to $w < w'$, i. e., a rectangle on top of another rectangle has to be strictly thinner. Note that now it can happen that not all rectangles get used (e. g., if we have two squares of the same size). Give a polynomial-time algorithm for the modified problem. (Hint: try to cast the problem as MAX-WEIGHT MATCHING, see definitions below.)

(For example if the input is $2 \times 11, 2 \times 10, 10 \times 10$ then the solution is a tower of height $22 = 2 + 10 + 10$.)

3.6 (due Thursday, Oct 16)

PART A: Suppose that you have an algorithm (let's call the algorithm A) which solves the MAX-WEIGHT MATCHING problem on any weighed graph $G = (V, E, w)$ in time $O(m + n)$ (where $n := |V|$, $m := |E|$). Show how you can use algorithm A to solve the MAX-WEIGHT PERFECT MATCHING problem in time $O(m + n)$. (HINT: consider the weighted graph $G' = (V, E, w')$ where $w'(e) = w(e) + C$ where C is some large number.)

PART B: Suppose that you have an algorithm (let's call the algorithm B) which solves the MAX-WEIGHT PERFECT MATCHING problem on any weighed graph $G = (V, E, w)$ in time $O(m + n)$ (where $n := |V|$, $m := |E|$). Show how you can use the algorithm B to solve the MAX-WEIGHT MATCHING problem in time $O(m + n)$.

3.7 (due Thursday, Oct 16) We have n jobs and m machines. We are given $n \times m$ table T where T_{ij} is the time to complete job i on machine j . Our task is to schedule the jobs on the machines to minimize the **average completion time** (that is, for each job you take the time when it finished and then average over all jobs). Give a polynomial-time algorithm for the problem. (Hint: try to cast the problem as minimum-weight matching of given cardinality.)

(For example if we have 3 jobs and 2 machines and

$$T = \begin{pmatrix} 1 & 3 \\ 3 & 2 \\ 4 & 1 \end{pmatrix},$$

then the following schedule is optimal:

- machine 1: 1;
- machine 2: 3, 2.

The completion times of jobs are: 1, 3, 1, the average completion time is $5/3$.)

3 Additional problems from the book (same as previous handout)

Try to solve the following problems. We will go over the ones that you choose in the problem sessions.

- 3.1, 3.2, 3.3, 3.4, 3.6, 3.7, 3.8, 3.9, 3.11, 3.12, 3.13, 3.15, 3.18, 3.19, 3.21, 3.22, 3.23, 3.24, 3.25, 3.26, 3.27,
- 4.1, 4.2, 4.3, 4.4, 4.5, 4.7, 4.8, 4.10, 4.11, 4.13, 4.15, 4.17, 4.18, 4.19, 4.20, 4.21,
- 5.1, 5.2, 5.3, 5.7.