Linear programming (part 3 of CSC 282),

http://www.cs.rochester.edu/~stefanko/Teaching/15CS282

IMPORTANT: **Write each problem on a separate sheet of paper. Write your name on each sheet.** (Why? To speed up the grading we will use parallelization—each problem will be graded by a different TA. At the beginning of the class (on the date when the homework is due) there will be a separate pile for each problem.)

Homework 4 due: Nov. 5 (Thursday); collected before class in B&L 109.

Problem sessions before Homework 4 is due:
Nov. 2 (Monday) 6:30pm - 7:30pm in Goergen 108
Nov. 3 (Tuesday) 6:00pm - 7:00pm in Goergen 108
Nov. 4 (Wednesday) 6:15pm - 7:15pm in Goergen 108
Nov. 4 (Wednesday) 7:30pm - 8:30pm in Goergen 108

Exam 3 is on: Nov. 12 (Thursday).

Problem sessions before Exam 3:
Nov. 9 (Monday) 6:30pm - 7:30pm in Goergen 108
Nov. 10 (Tuesday) 6:00pm - 7:00pm in Goergen 108
Nov. 11 (Wednesday) 6:15pm - 7:15pm in Goergen 108
Nov. 11 (Wednesday) 7:30pm - 8:30pm in Goergen 108

1 Homework - solve and turn in

4.1 (**due Nov 5**) Krusty the Clown purchased a new super-efficient SUV with two engines. The first engine runs on a 2:1 mixture of gasoline and ethanol and gets 12 mpg. The second engine runs on a 1:2 mixture of gasoline and ethanol and gets 22 mpg. Krusty has 12 gallons of gasoline and 14 gallons of ethanol. How far can he get?

(a) Write a linear program for the problem. You are allowed to have only two variables in your linear program.

(b) Solve the linear program (you do not need to use the simplex method).

(c) Write down the dual of your linear program from part (a).

(d) Solve the dual program (you do not need to use the simplex method).

4.2 (**due Nov 5**) Compute the value of the zero-sum two person game given by the following payoff matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

That is, solve the following linear program:

$$\min z$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} z \\ z \end{pmatrix}$$

$$x_1 + x_2 = 1$$

$$x_1 \geq 0, x_2 \geq 0,$$

(we are using the convention that the entries in the matrix are the payoff for the row player).

4.3 (**due Nov 5**) Zofka is trying to measure the length of a race track using a car. The (digital) odometer in the car shows the number of kilometers traveled since the car was made. She drives around the track and each time she crosses the finish line she writes down the number on the odometer. She wants to infer the length of the race track from these numbers in the following sense: 1) she wants to know the minimal possible track length that is consistent with the measurements and 2) she wants to know the maximal possible track length that is consistent with the measurements.

Example input: suppose Zofka took one lap around the track. When she crossed the finish line at the beginning of the first lap the odometer showed 713. When she crossed the finish line at the end of the first lap the odometer showed 715. The minimal track length is 1 kilometer (if the car went from 713.999... km to 715.00km). The maximal track length is 3 kilometers (if the car went from 713.000km to 715.999... km).[1]

The input of the problem is n—the number of laps Zofka did around the track and $n+1$ numbers $a_0, a_1, a_2, \ldots, a_{n+1}$ (the values the odometer showed when she crossed the finish line). Give a linear problem that computes the minimal possible track length that is consistent with the measurements (the linear problem that computes the maximal possible track length that is consistent with the measurements is the same with min replaced by max). **The linear program should have exactly two variables.** Clearly explain the meaning of variables in your programs.

Show how your general solution works for the example input.

4.4 (**due Nov 5**) Let a be a constant and x be a variable. What is the value of the following linear program? Give a simple mathematical expression.

$$\begin{aligned} \min\, & x \\ & x \geq \quad a \\ & x \geq -a \end{aligned}$$

Let a, b be constants and x, y be variables. What is the value of the following linear program? Give a simple mathematical expression.

$$\begin{aligned} \min\, & x + y \\ & x \geq \quad a \\ & x \geq -a \\ & y \geq \quad b \\ & y \geq -b \end{aligned}$$

(If you don't know where to start—plug in concrete values for a and b (make sure to use negative numbers too), solve the resulting linear program (either by hand or using a solver) and formulate a conjecture based on the experiments.)

Let a, b be constants and x, y be variables. What is the value of the following linear program? Give a simple mathematical expression.

$$\begin{aligned} \min\, & x - y \\ & x \geq \quad a \\ & x \geq -a \\ & y \geq \quad b \\ & y \geq -b \end{aligned}$$

[1] Technically we should be talking about suprema instead of maxima and infima instead of minima. Ignore this unimportant technical detail (if it worries you—then you are allowed strict inequalities in your linear program for this problem).

2 Bonus Homework - solve and turn in

4.5 (**due Nov 5**)
PART A:
We are given n pairs of real numbers $(x_1, y_1), \ldots, (x_n, y_n)$. We want to find real numbers a, b which minimize

$$\max_{i \in \{1,\ldots,n\}} |ax_i + y_i - b|.$$

Write a linear program for this problem.
PART B:
$(x_1, y_1), \ldots, (x_n, y_n)$. We want to find real numbers a, b which minimize

$$\sum_{i \in \{1,\ldots,n\}} |ax_i + y_i - b|.$$

Write a linear program for this problem.
PART C:
We are given n pairs of real numbers $(x_1, y_1), \ldots, (x_n, y_n)$. We want to find real numbers a, b which minimize

$$\sum_{i \in \{1,\ldots,n\}} (ax_i + y_i - b)^2.$$

Find a formula for a and b (you do NOT need linear programming for this one).

4.6 (**due Nov 5**) Give a mathematical expression (sums are allowed; linear programs are NOT allowed) for the value of the two person zero-sum game with the following $n \times n$ payoff matrix

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n \end{pmatrix}.$$

(More precisely, the matrix P is defined by $P_{ii} = i$ and $P_{ij} = 0$ for $i \neq j$).

3 Additional problems from the book (do not turn in)

Try to solve the following problems. A few of them may be on the quiz. We will go over the ones that you choose in the problem sessions.

- 7.1, 7.2, 7.3, 7.4, 7.6, 7.7, 7.8, 7.9, 7.11, 7.12, 7.13, 7.14, 7.15, 7.19, 7.27, 7.28.