CONTROL THEORY

- Huge Field
- Practical ... Abstract
- Many idiosyncratic engineering analysis tricks?

- Linear Systems - common approx.
- Discrete & Continuous
  (Digital & Analog)

- Common Applications
  - Set point control (e.g. thermostat)
  - Fly by wire
  - Autopilot
  - Anti-shake, stabilization ...

(1)
\(n^{th}\) order linear differential eqn.

Constant or time-varying coefficients

\[ a_n(t) \frac{d^n y}{dt^n} + a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1(t) \frac{dy}{dt} + a_0(t)y = g(t). \]

For complete solution, need initial condition(s).

E.g. \( \frac{dy}{dt} = 1 \) \( y(t) = t + b \)

I.C. e.g. what is \( y \) when \( t = 0 \)?

\( \text{linear Systems (Dudek Ch 4.)} \)
2\textsuperscript{nd} - Order System

(constant coef.)

\[ m, k, b \text{ parameters} \]

\[ m, k, b \text{ determine behavior.} \]

With no outside applied forces:

\[ m\ddot{x} + b\dot{x} + kx = 0 \quad \text{Homogeneous} \]

\[ F = ma \quad \text{friction spring eqn.} \]

Solution depends on ICs. of position and velocity. There are 2 ICs:

2 \text{nd order system.}
Characteristic Equation Method

$$a_n \frac{d^n z}{dt^n} + a_{n-1} \frac{d^{n-1} z}{dt^{n-1}} + \ldots + a_0 z = 0$$

Suppose $z = e^{st}$ for some constants $s$ is a solution, then...

$$a_n s^n e^{st} + a_{n-1} s^{n-1} e^{st} + \ldots + a_0 e^{st} = 0$$

$e^{st} > 0$ so divide it out

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0 = 0 \quad (\star)$$

$e^{st}$ is solution $\implies s$ satisfies $$(\star)$$

Also $$(\star) \implies s$ satisfies $$(\star)$$

$s$ satisfies $$(\star) \implies e^{st}$ is solution.

Note: Can add $\pm A \text{ const.}$ add $\pm \text{ const.}$
E.g.

\[ \frac{dy}{dt} = ay \]
\[ s - a = 0 \]
\[ s = a \] is a solution:

\[ y(t) = Ke^{at} \]
\[ K \text{ is a constant determined by I.C.} \]

As in:

\[ R \frac{dq}{dt} = -\frac{1}{C} q \]
\[ \frac{dq}{dt} = -\frac{1}{RC} q \]

E.g.

\[ \frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0 \]

\[ s^2 - 5s + 6 = 0 \quad (s-2)(s-3) = 0 \]
\[ s = 2, \ s = 3 \quad \text{are roots} \]

\[ y(t) = c_1 e^{2t} + c_2 e^{3t} \]
2nd Order Systems Again

\[ m, b, k > 0 \implies \text{physical} \implies \text{stable} \]

\[ m \ddot{x} + b \dot{x} + kx = 0 \]

\[ m \xi^2 + bs + k = 0 \]

\[ s_1 = \frac{-b}{2m} + \frac{\sqrt{b^2 - 4km}}{2m} \]

\[ s_2 = \frac{-b}{2m} - \frac{\sqrt{b^2 - 4km}}{2m} \]

\( s_1, s_2 \) are so-called poles of system.

(Control-theory speak).

Describe its behavior. Dominant poles \( \implies \) can approximate nth-order syst. as 2nd-order.
$s_1, s_2$ can be

- real and unequal
  $b^2 > 4nk$: friction dominates
  overdamped
- complex
  $b^2 < 4nk$: stiffness (spring) dominates
  oscillatory
- real and equal
  $b^2 = 4nk$

Fastest won't oscillating, response critically damped.

**Real Roots:**

- $x$
  \[ \frac{t}{\text{time}} \]

What do oscillations have to do with $e^{st}$?
\[ e^x = 1 + \frac{1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \]

\[ e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots \]

\[ = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \]

\[ + \ i x - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \cdots \]

\[ = \cos x + i \sin x \]

(Euler's Formula)
Real & Unequal Roots

\[ x(t) = e^{s_1 t} + e^{s_2 t} \]

\( s_1, s_2 \) from the quadratic formula solution to char. eq., recall?

Sluggish motion - overdamped.

If one pole is much greater in magnitude than other, can ignore it. "bigger" \( e^{-s_1 t} \to \text{zero faster.} \)
E.G.

\[ n = 1 \]
\[ b = 5 \]
\[ k > 0 \]

release \( q + \) rest block from \( x = -1 \)

\[ s^2 + 5s + 6 = 0 \]
\[ s_1 = -2, \quad s_2 = -3 \]

\[ x(t) = c_1 e^{-2t} + c_2 e^{-3t} \]

\[ x(0) = -1, \quad x'(0) = 0, \quad \text{so} \]

\[ c_1 + c_2 = -1 \]
\[ -2c_1 - 3c_2 = 0, \quad \text{so} \]

\[ c_1 = -3, \quad c_2 = 2, \quad \text{so} \]

\[ x(t) = -3e^{-2t} + 2e^{-3t} \]
complex roots

\[ s_1 = \lambda + i\mu \]
\[ s_2 = \lambda - i\mu \]

Euler's formula:

\[ e^{ix} = \cos x + i\sin x \]

and

\[ x(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t) \]

(good exercise)

\[ c_1, c_2 \text{ depend on } \lambda, \mu. \]

Further, say \( c_1 = r\cos\delta \)
\[ c_2 = r\sin\delta \]

\[ x(t) = re^{\lambda t} \cos(\mu t - \delta) \]
\[ r = \sqrt{c_1^2 + c_2^2} \]
\[ \delta = \arctan2(c_2, c_1) \]
E.G.

\[ m = 1 \quad b = 1 \quad k = 1 \quad x = -1 \]

\[ s^2 + s + 1 = 0 \]

\[ s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \]

\[ x(t) = e^{-\frac{t}{2}} \left[ c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t \right] \]

\[ x(0) = -1 \quad , \quad x'(0) = 0 \]

so \[ c_1 = -1 \quad , \quad -\frac{1}{2} c_1 - \frac{\sqrt{3}}{2} c_2 = 0 \]

or \[ c_1 = -1 \quad , \quad c_2 = \frac{\sqrt{3}}{2} \]

\[ x(t) = e^{-\frac{t}{2}} \left[ \cos \frac{\sqrt{3}}{2} t - \frac{\sqrt{3}}{2} \sin \frac{\sqrt{3}}{2} t \right] \]

\[ x(t) = \frac{2\sqrt{3}}{3} e^{-\frac{t}{2}} \cos \left( \frac{\sqrt{3}}{2} t + 120^\circ \right) \]
**REAL & EQUAL ROOTS**

\[ x(t) = c_1 e^{s_1 t} + c_2 \int e^{s_2 t} f(t) \, dt \]

\[ s_1 = s_2 = -\frac{b}{2m}, \quad s_0 \]

\[ x(t) = (c_1 + c_2 t) e^{-\frac{b}{2m} t} \]

\[ m = 1, \quad b = 4, \quad k = 4, \quad x = -1 \]

\[ s^2 + 4s + 4 \]

\[ s_1 = s_2 = -2 \]

\[ x(t) = (c_1 + c_2 t) e^{-2t} \]

\[ \int x(t) \, dt = (-1 - 2t + e^{-2t}) \]

\[ -2c_1 + c_2 = 0; \quad c_2 = 1, \quad c_2 = -2 \]
CONTROL OF 2nd-ORDER SYSTEMS

Suppose our 2nd-order system isn't what we want. It's too lively or too sluggish... maybe there's no spring.

- Add sensors, actuator for applying force $f$ on block.

Propose this: A control law:

$$f = -K_p x - K_v x$$
**Position Regulation**

\[ f = -k_p x - k_v \dot{x} \]

Clearly that's what this is doing!

(there are clearly other things controllers can do, like following a trajectory).

So...

\[ m \ddot{x} + b \dot{x} + k x = -k_p x - k_v \dot{x} \]

\[ m \ddot{x} + (b + k_v) \dot{x} + (k + k_p) x = 0 \]

\[ m \ddot{x} + b' \dot{x} + k' x = 0 \]

So by picking CONDITIONS \( k_v, k_p \),

can make any system we like!
E.g. $b' = 2 \sqrt{mk}$ get critical damping

if $m=b=k=1$, how make the system critically damped with closed-loop stiffness (equivalent spring) 16?

if $k' = 16.0$, $b' = 2 \sqrt{mk'} = 8$

$k = 1$, $b = 1$

$k_p = 15$

$k_v = 7$
MODEL-BASED CONTROL
("computed torque", e.g.).
(state of the art)

A.K.A. control-law partitioning

- model-based partition
- servo partition
- put system parameters in the
  model partition, so servo is
  independent.
- generalizes to non-linear systems.
\[ mx^2 + bx + kx = f \]

If we know \( m, b, k \); why not use the model-based controller to turn system into a unit, frictionless, springless mass? THAT would be easy to control.

Model-based control law:

\[ f = \alpha f' + \beta \]

\( \alpha, \beta \) are functions or constants, chosen so that if \( f' \) is the input, system behaves like unit mass.
\[ m \ddot{x} + b \dot{x} + k x = \alpha f' + \beta \]

If \( f' \) should make system act like unit mass, \( \ddot{x} = f' \).

So ... \( \alpha = m \)
\[ \beta = b \dot{x} + k x \]

Now we can go as before treating this as open-loop dynamics of a system:

\[ f' = -k_v \dot{x} - k_p x \quad \text{control law} \]

\[ \ddot{x} + k_v \dot{x} + k_p x = 0 \]

\[ k_v = 2 \sqrt{k_p} \quad \text{for critical damping} \]

\[ m, b, k, k_p \]
CLOSED-LOOP CONTROL w/ PARTITIONED CONTROL