CSC 172– Data Structures and Algorithms

Lecture #6 Spring 2018

- visualizing what happens when a recurrence is iterated

RECURSION TREES

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Binary Search

• Let's go back to the textbook

 <u>http://lti.cs.vt.edu/LTI_ruby/Books/CS172/html/A</u> <u>nalProgram.html#binary-search-practice-exercise</u>

Analyzing Binary Search



But we want the runtime in big oh notation!



$T(n) = O(\log n)$



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How?

• We will learn today.

• Recursion/Recurrence Tree

• Let's review our math before we start

Mathematical Background

Logarithms

• Arithmetic series

• Geometric series

Logarithms

A more interesting observation we will repeatedly use:

$$n^{\log_b(m)} = m^{\log_b(n)}$$

a consequence of $n = b^{\log_b n}$

$$n^{\log_b (m)} = (b^{\log_b(n)})^{\log_b(m)}$$
$$= b^{\log_b(n)} \log_b(m)$$
$$= (b^{\log_b(m)})^{\log_b(n)}$$
$$= m^{\log_b(n)}$$

Arithmetic series

Each term in an arithmetic series is increased by a constant value (usually 1) :

$$0+1+2+3+ + n = \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

Geometric series

The next series we will look at is the geometric series with common ratio *r*:

$$\sum_{k=0}^{n} r^{k} = \frac{1 - r^{n+1}}{1 - r}$$

and if |r| < 1 then it is also true that

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

EXAMPLES

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Example #1

T(n) = 5T(n/3) + 1

Recursion Tree



Asymptotic Analysis

1.
$$T(n) = 5T(n/3) + 1$$

$$T(n) = 5T(n/3) + 1 = 5^{k}T(n/3^{k}) + \sum_{i=0}^{k-1} 5^{i} = 5^{k}T(n/3^{k}) + \frac{5^{k}-1}{4}$$

Hence, by setting $k = \log_3 n$, we have

$$T(n) = 5^k T(1) + \Theta(5^k) = \Theta(5^k) = \Theta(5^{\log_3 n}) = \Theta(n^{\log_3 5}) \approx \Theta(n^{1.465})$$

Example #2

$$T(n) = 7T(n/3) + n$$

Recursion Tree



Asymptotic Analysis

2.
$$T(n) = 7T(n/3) + n$$

$$T(n) = 7T(n/3) + n = 7^{k}T(n/3^{k}) + \sum_{i=0}^{k-1} 7^{i}(n/3^{i}) = 7^{k}T(n/3^{k}) + n\sum_{i=0}^{k-1} (7/3)^{i})$$
$$= 7^{k}T(n/3^{k}) + n\frac{(7/3)^{k} - 1}{(4/3)}$$

Hence, by setting $k = \log_3 n$, we have

$$T(n) = 7^k T(1) + \Theta(n \cdot (7/3)^k) = \Theta(7^k) + \Theta(7^k) = \Theta(7^{\log_3 n}) = \Theta(n^{\log_3 7}) \approx \Theta(n^{1.771})$$

Example #3

$$T(n) = 4T(n/4) + n$$

Recursion Tree



Asymptotic Analysis

$$T(n) = 4T(n/4) + n$$

$$T(n) = 4T(n/4) + n = 4^{k}T(n/4^{k}) + \sum_{i=0}^{4-1} 4^{i}(n/4^{i}) = 4^{k}T(n/4^{k}) + n\sum_{i=0}^{k-1} (4/4)^{i}$$
$$= 4^{k}T(n/4^{k}) + n \cdot k$$

Hence, by setting $k = \log_4 n$, we have

$$T(n) = 4^k T(1) + \Theta(n \cdot k) = nT(1) + n \cdot \log_4 n = \Theta(n \log n)$$

Example #4

$$T(n) = 6T(n/2) + n^3$$

Recursion Tree



Asymptotic Analysis

 $T(n) = 6T(n/2) + n^3$

$$T(n) = 6T(n/2) + n^3 = 6^k T(n/2^k) + \sum_{i=0}^{k-1} 6^i (n/2^i)^3 = 6^k T(n/2^k) + n^3 \sum_{i=0}^{k-1} (6/8)^i)$$
$$= 6^k T(n/2^k) + n^3 \frac{1 - (6/8)^k}{(2/8)}$$

Hence, by setting $k = \log_2 n$, we have

$$T(n) = 6^k T(1) + \Theta(n^3 \cdot (1 - (6/8)^k)) = \Theta(6^k) + \Theta(n^3) = \Theta(n^{\log_2 6}) + \Theta(n^3) = \Theta(n^3)$$

Example #5

$$T(n) = 9T(n/3) + n^2$$

Recursion Tree



Asymptotic Analysis

$$\begin{split} T(n) &= 9T(n/3) + n^2 \\ T(n) &= 9T(n/3) + n^2 = 9^k T(n/3^k) + \sum_{i=0}^{k-1} 9^i (n/3^i)^2 = 9^k T(n/3^k) + n^2 \sum_{i=0}^{k-1} (1)^i) \\ &= 9^k T(n/3^k) + n^2 \cdot k \end{split}$$

Hence, by setting $k = \log_3 n$, we have

$$\begin{split} T(n) &= 9^k T(1) + \Theta(n^2 \cdot k) = \Theta(9^k) + n^2 \Theta(k) \\ &= \Theta(n^{\log_3 9}) + \Theta(n^2 k) = \Theta(n^2) + \Theta(n^2 \log n) = \Theta(n^2 \log n) \end{split}$$

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