CSC 252/452: Computer Organization
Fall 2025: Lecture 3

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Department of Computer Science
University of Rochester

Announcement

- Programming Assignment 1 is out
  - Details: https://www.cs.rochester.edu/courses/252/fall2025/labs/assignment1.html
  - Due on Sep 15th, 11:59 PM
  - You have 3 slip days
  - Seek help from TAs.
    - TAs are best positioned to answer your questions about programming assignments!!!
- Only 53 students enrolled in Piazza
- Academic Honesty Policy
  - No copy/paste from GPT/Gemini/Google
- Pay attention to Blackboard announcements and website updates
  - There are changes about the office hours

Last Lecture

- Why Binary (bits)?
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary

Encoding Negative Numbers

- Two’s Complement

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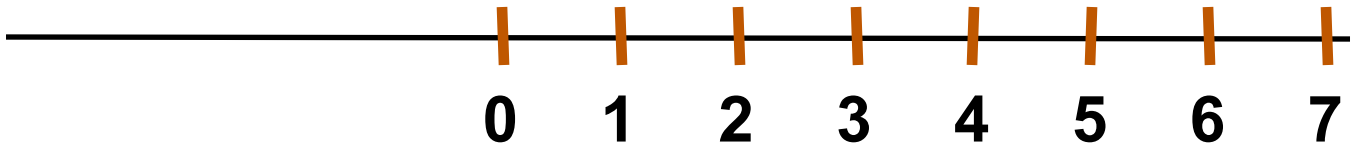

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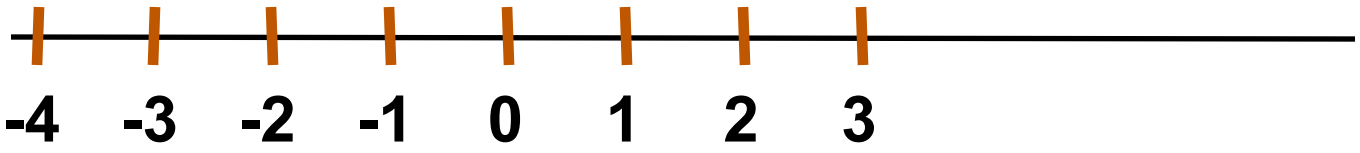

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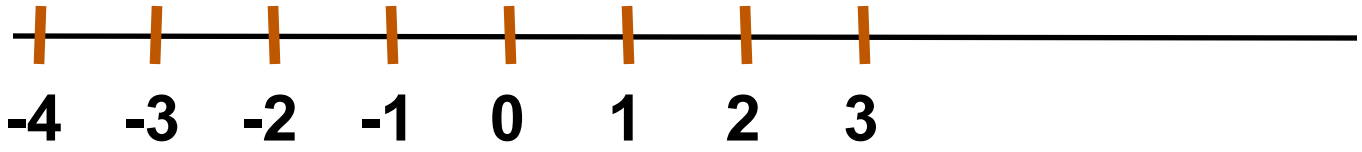

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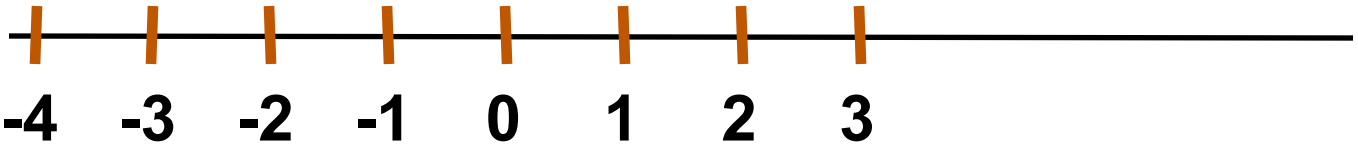
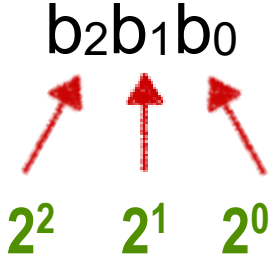


Weights in Unsigned



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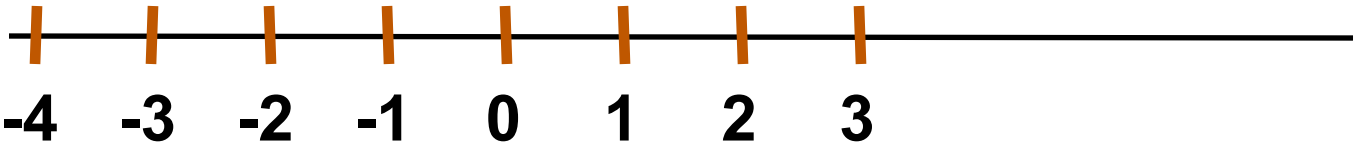

$b_2 b_1 b_0$

Weights in Unsigned: 2^2 2^1 2^0

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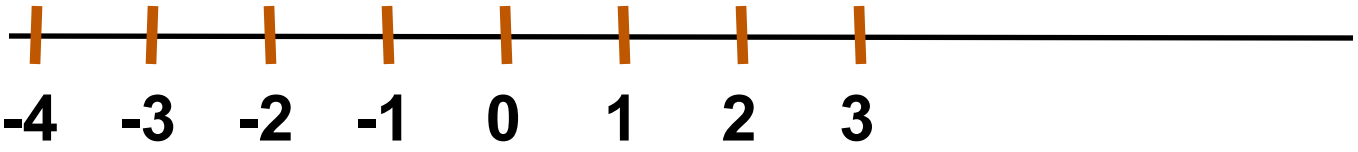

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$101_2 = 1*2^0 + 0*2^1 + (-1*2^2) = -3_{10}$

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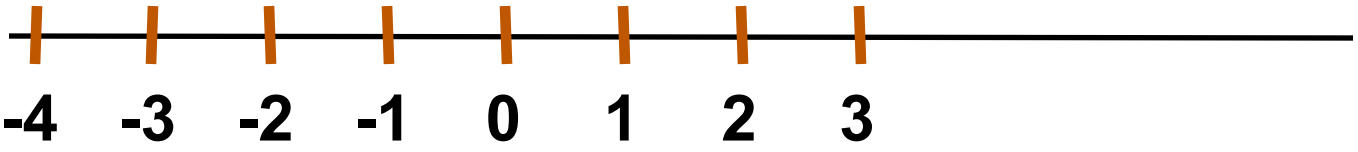

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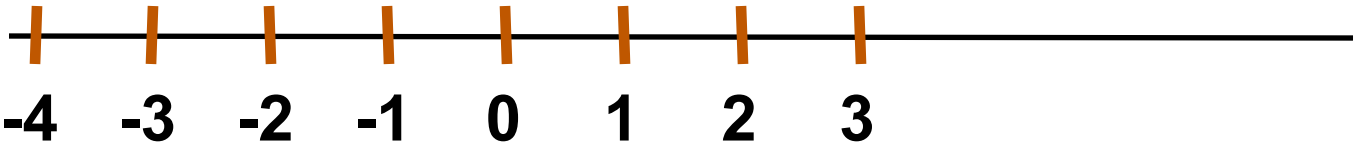

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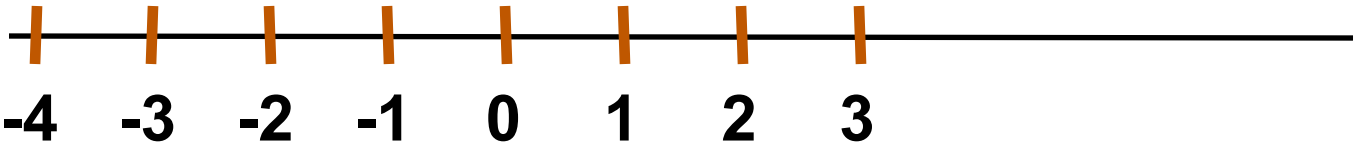

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Two-Complement Implications

- Only 1 zero
- There is (still) a bit that represents sign!
- Unsigned arithmetic still works

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   010
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- 3 + 1 becomes -4 (called overflow. More on it later.)

Data Types (in C)

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  - Integer
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- Suppose you want to define a variable that represents a person's age. What assumptions can you make about this variable's numerical value?
  - Integer
  - Non-negative
  - Between 0 and 255 (8 bits)
- Define a data type that captures all these attributes: `unsigned char` in C
  - Internally, an **`unsigned char`** variable is represented as a 8-bit, non-negative, binary number

Data Types (in C)

- What if you want to define a variable that could take negative values?

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Data Types (in C)

- What if you want to define a variable that could take negative values?
  - That's what signed data types (e.g., **`int`**, **`short`**, etc.) are for
- How are `int` values internally represented?
  - Theoretically could be either signed-magnitude or two's complement
  - The C language designers chose two's complement

Data Types (in C)

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

C Data Type	32-bit	64-bit
`(unsigned) char`	1	1
`(unsigned) short`	2	2
`(unsigned) int`	4	4
`(unsigned) long`	4	8

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- C Language
  - `#include <limits.h>`
  - Declares constants, e.g.,
    - `ULONG_MAX`
    - `LONG_MAX`
    - `LONG_MIN`
  - Values platform specific

Mapping Between Signed & Unsigned

- Mappings between unsigned and two's complement numbers: Keep bit representations and reinterpret

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0	0	000
1	1	001
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Mapping Signed ↔ Unsigned

Bits	Signed		Unsigned
0000	0		0
0001	1		1
0010	2		2
0011	3	=	3
0100	4		4
0101	5	T2U →	5
0110	6		6
0111	7		7
1000	-8	← U2T	8
1001	-7		9
1010	-6	+/- 16	10
1011	-5		11
1100	-4		12
1101	-3		13
1110	-2		14
1111	-1		15

Today: Representing Information in Binary

- Why Binary (bits)?
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
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The Problem

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
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`char`	1
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- Should we preserve the value?
- Can we preserve the value?
- How?

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- Converting from smaller to larger integer data type
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- Can we preserve the value?
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	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

Signed Extension

- Task:
  - Given *w*-bit signed integer *x*
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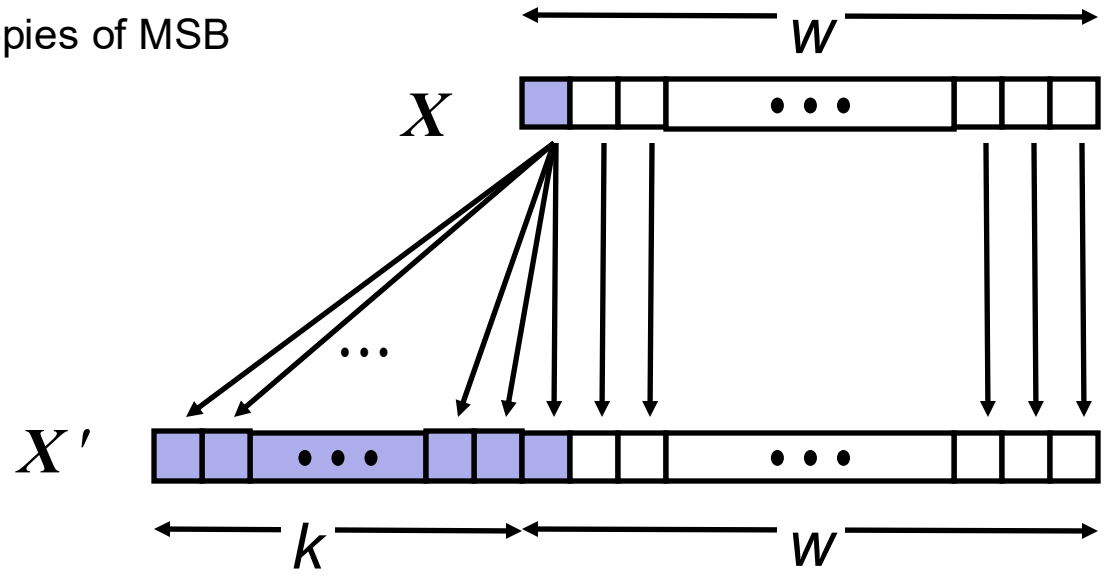


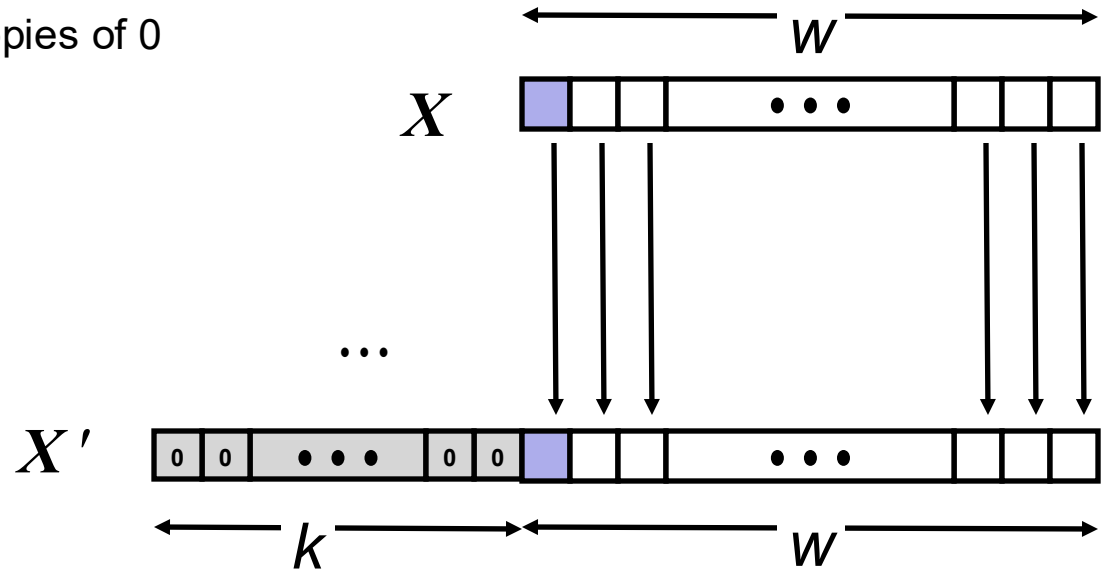
Another Problem

```
unsigned short x = 47981;
unsigned int  ux = x;
```

	Decimal	Hex	Binary
x	47981	BB 6D	10111011 01101101
ux	47981	00 00 BB 6D	00000000 00000000 10111011 01101101

Unsigned (Zero) Extension

- Task:
  - Given *w*-bit unsigned integer *x*
  - Convert it to (*w*+*k*)-bit integer with same value
- Rule:
  - Simply pad zeros:
  - $X' = \underbrace{0, \ldots, 0}_{k \text{ copies of } 0}, x_{w-1}, x_{w-2}, \ldots, x_0$



Yet Another Problem

```
int    x =  53191;
short sx = (short) x;
```

	Decimal	Hex	Binary
x	53191	00 00 CF C7	00000000 00000000 11001111 11000111
sx	-12345	CF C7	11001111 11000111

Yet Another Problem

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int    x =  53191;
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	Decimal	Hex	Binary
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- Truncating (e.g., int to short OR unsigned int to unsigned short)
  - C's implementation: leading bits are truncated, results reinterpreted
  - So can't always preserve the numerical value

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  - Addition, negation, multiplication, shifting
  - Summary
- Representations in memory, pointers, strings

Unsigned Addition

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Normal Case

```
   010          2
+) 101       +) 5
-----        ----
   111          7
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Overflow Case

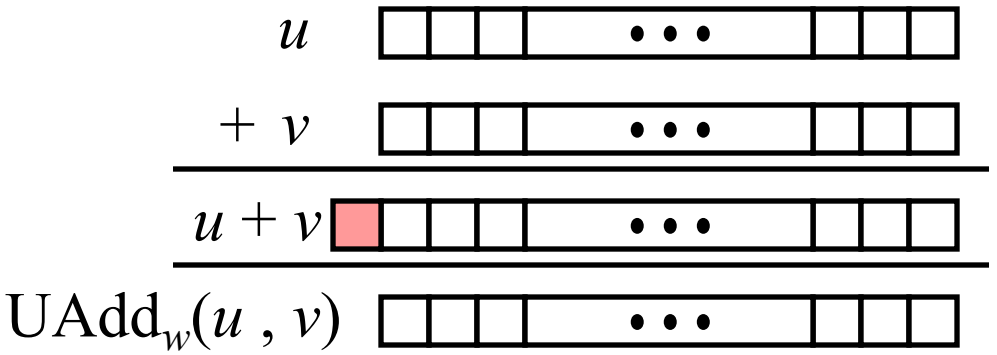
```
  110        6
+) 101     +) 5
-----      ----
 1011       11   <-- True Sum
  011        3   <-- Sum with same bits
```

Unsigned Addition in C

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



Two’s Complement Addition

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- Has identical bit-level behavior as unsigned addition (a big advantage over sign-magnitude)

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- Overflow can also occur

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Overflow Case

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   110        -2
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  1011        -5
```

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Min →

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```

Overflow Case

```
   110        -2          011          3
+) 101     +) -3      +) 001      +)   1
-----      -----      -----      -----
  1011        -5         0100         4
   011         3
```

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```

Overflow Case

```
  110        -2          011          3
+) 101     +) -3      +) 001      +)  1
-----      -----      -----      -----
 1011       -5         0100         4
  011        3          100        -4
```

Negative Overflow

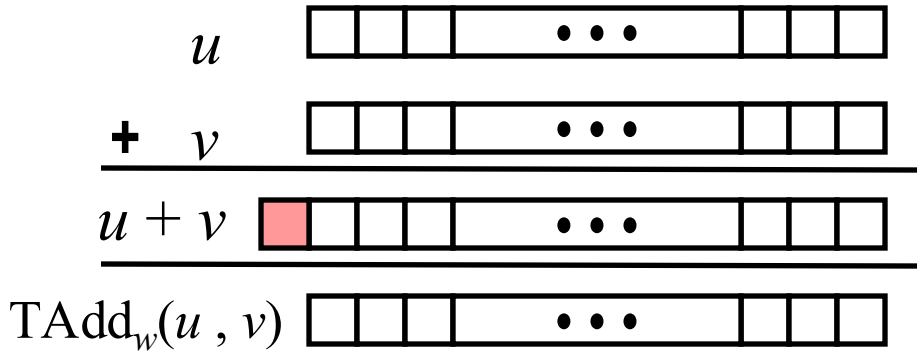
Positive Overflow

Two's Complement Addition in C

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



Is This Signed Addition an Overflow?

```
   111
+) 110
-------
  1101
```

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Is This Signed Addition an Overflow?

```
   111
+) 110
-------
  1101
```

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

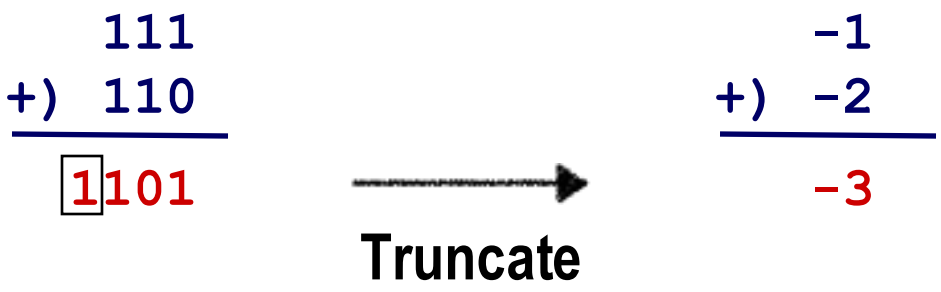
Is This Signed Addition an Overflow?

```
   111
+) 110
-----
  1101
```

Truncate

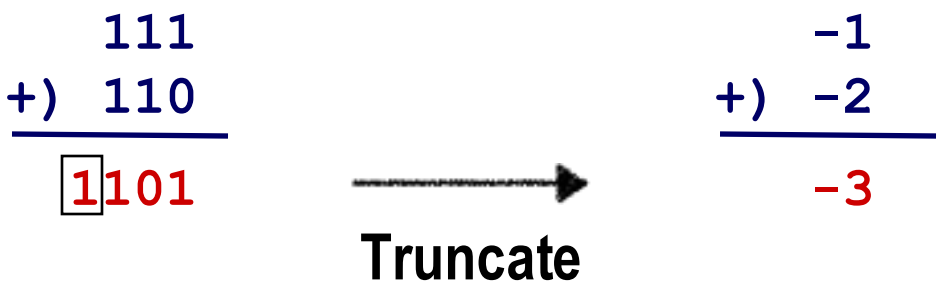
Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Is This Signed Addition an Overflow?



Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

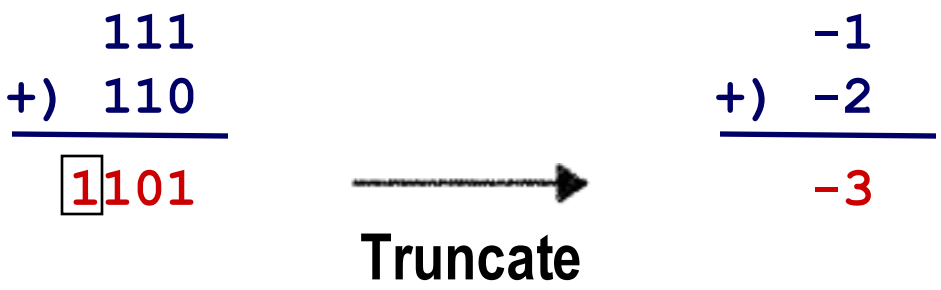
Is This Signed Addition an Overflow?



Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

- This is not an overflow by definition

Is This Signed Addition an Overflow?



Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

- This is not an overflow by definition
- Because the actual result can be represented using the bit width of the datatype (3 bits here)

Multiplication

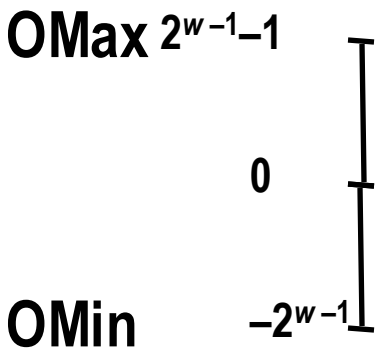
Multiplication

- Goal: Computing Product of w-bit numbers x, y

Multiplication

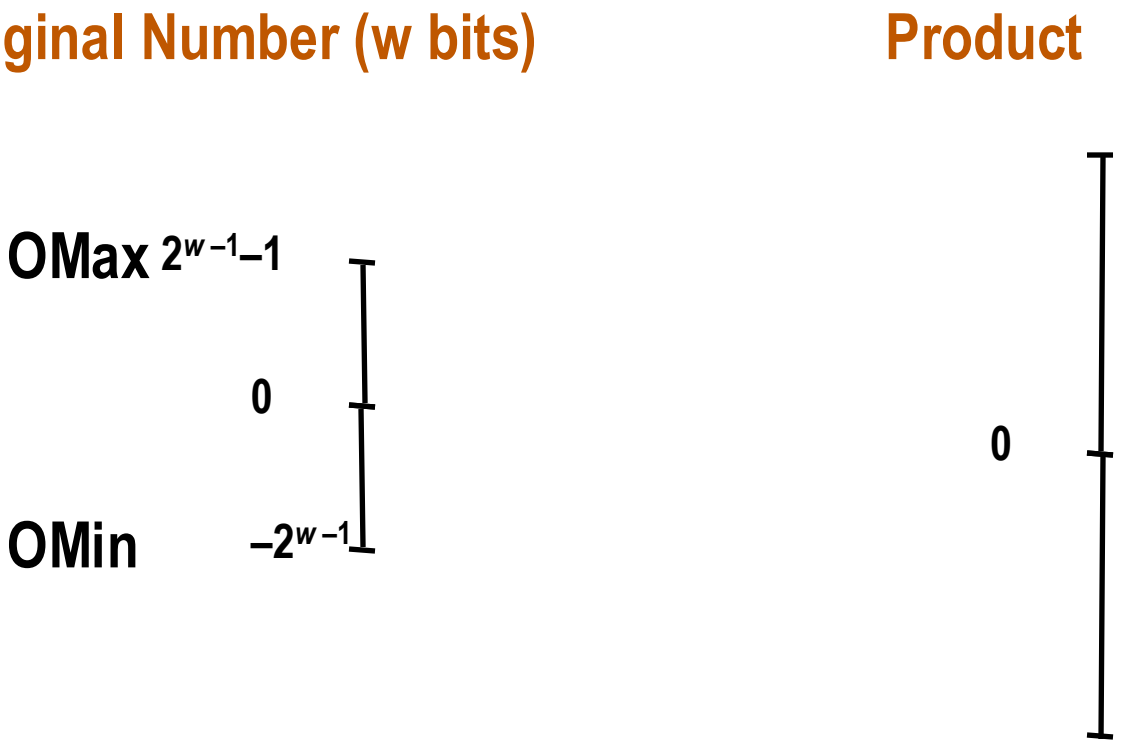
- Goal: Computing Product of *w*-bit numbers *x*, *y*

Original Number (w bits)



Multiplication

- Goal: Computing Product of *w*-bit numbers *x*, *y*



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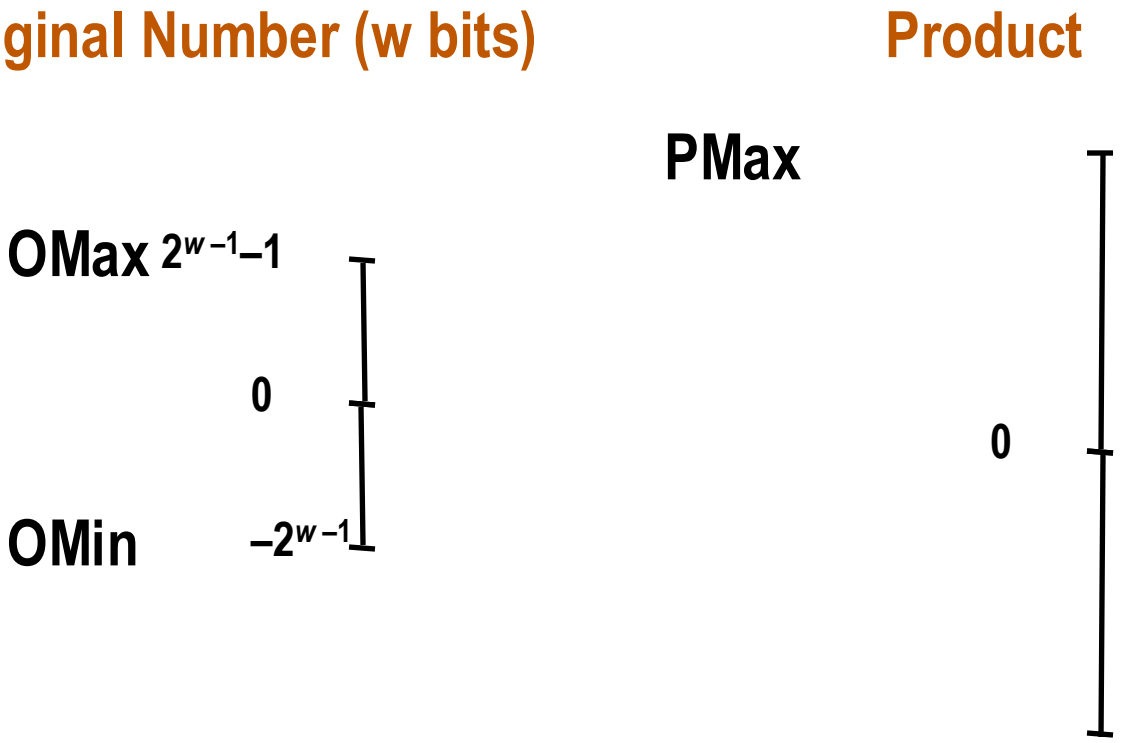

Multiplication

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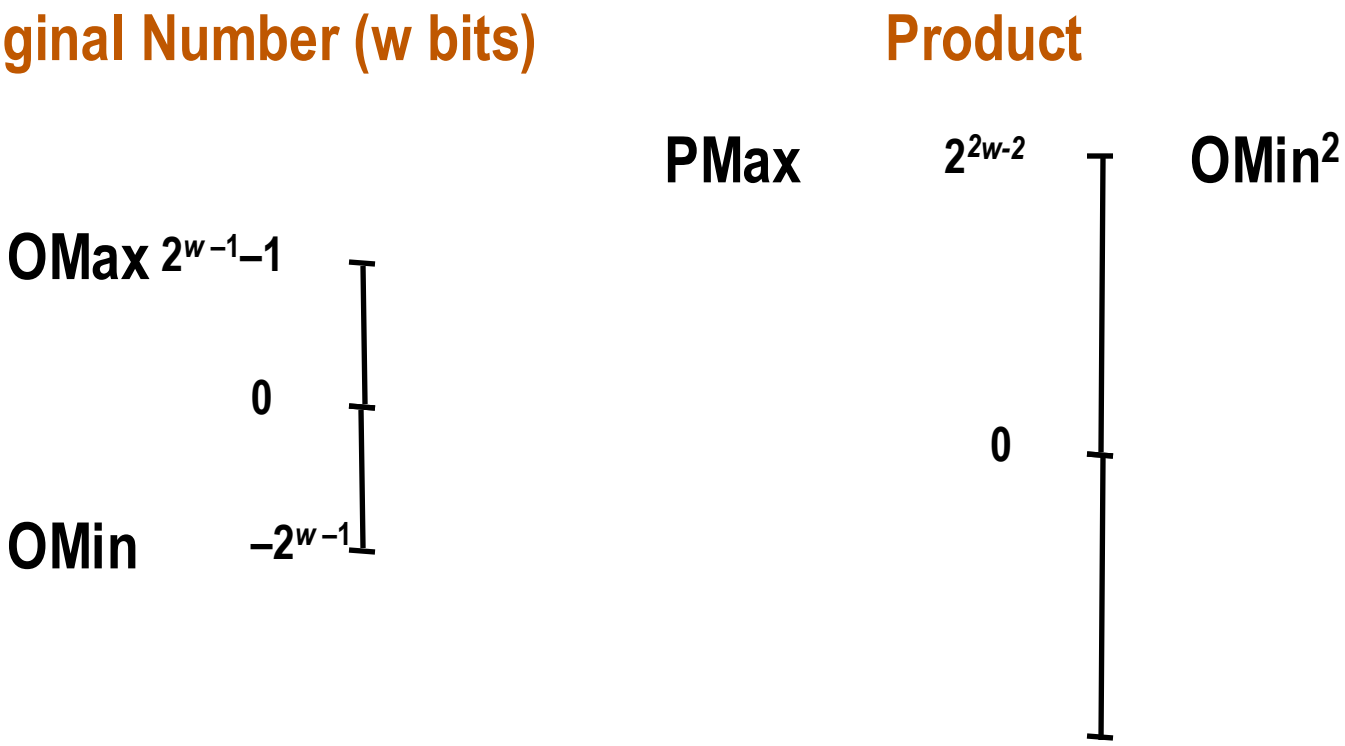

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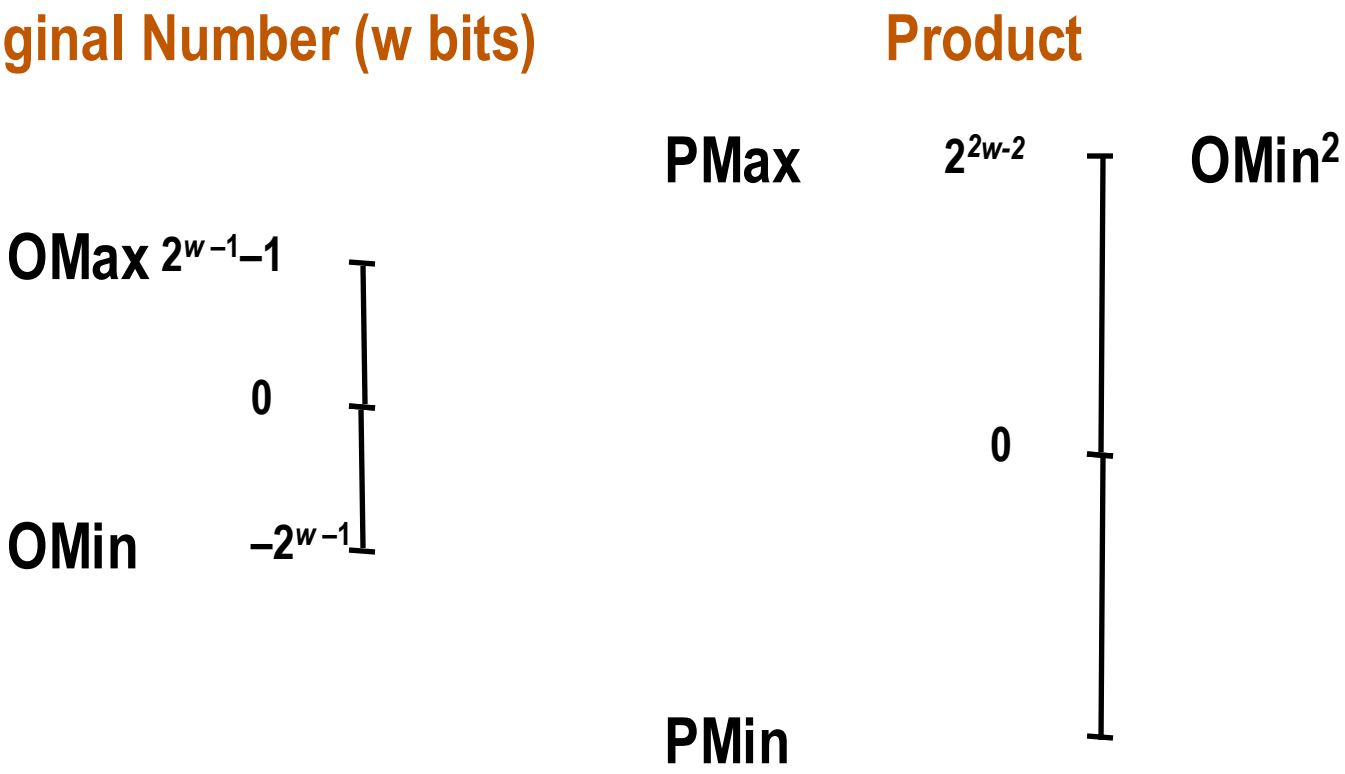

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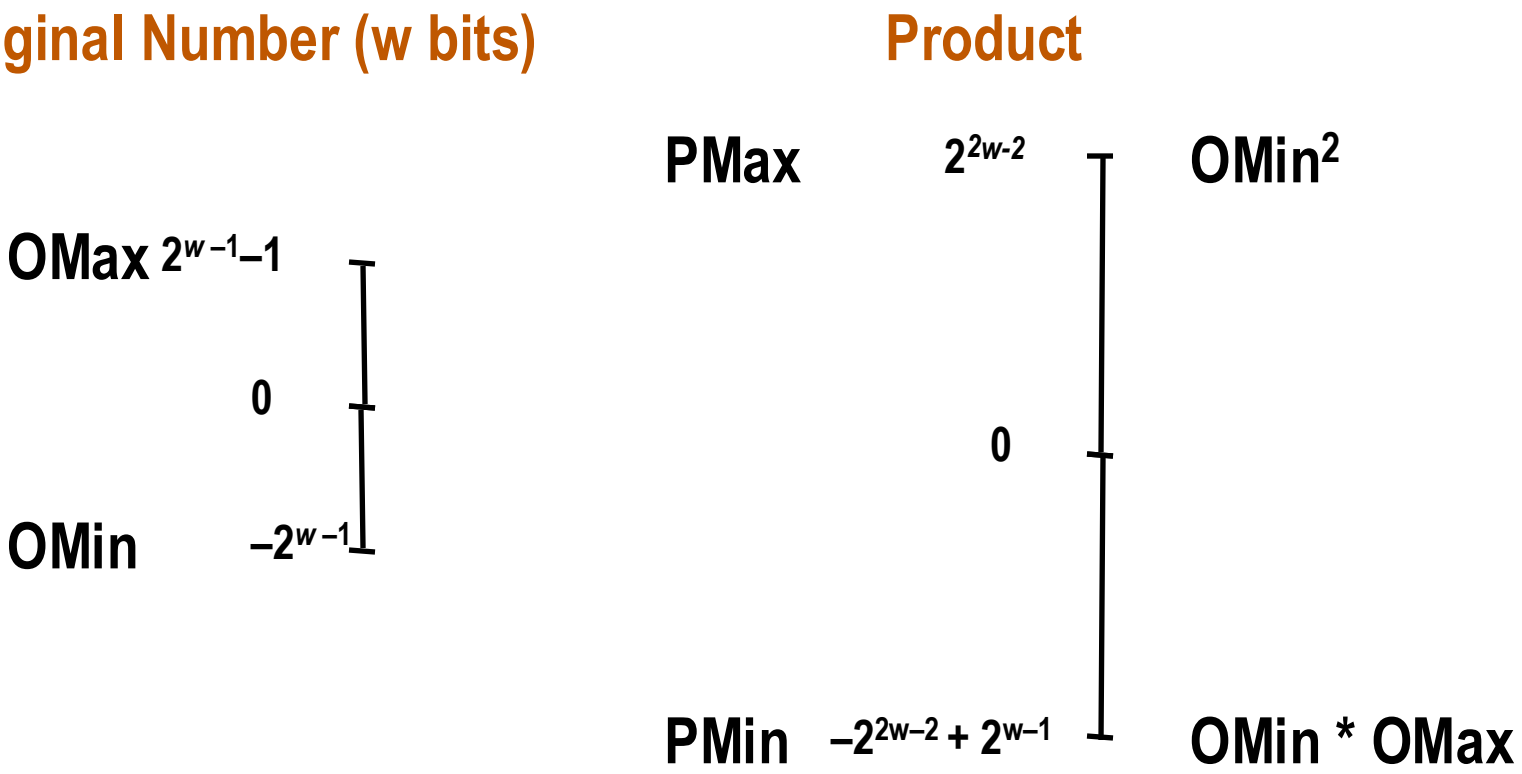

Multiplication

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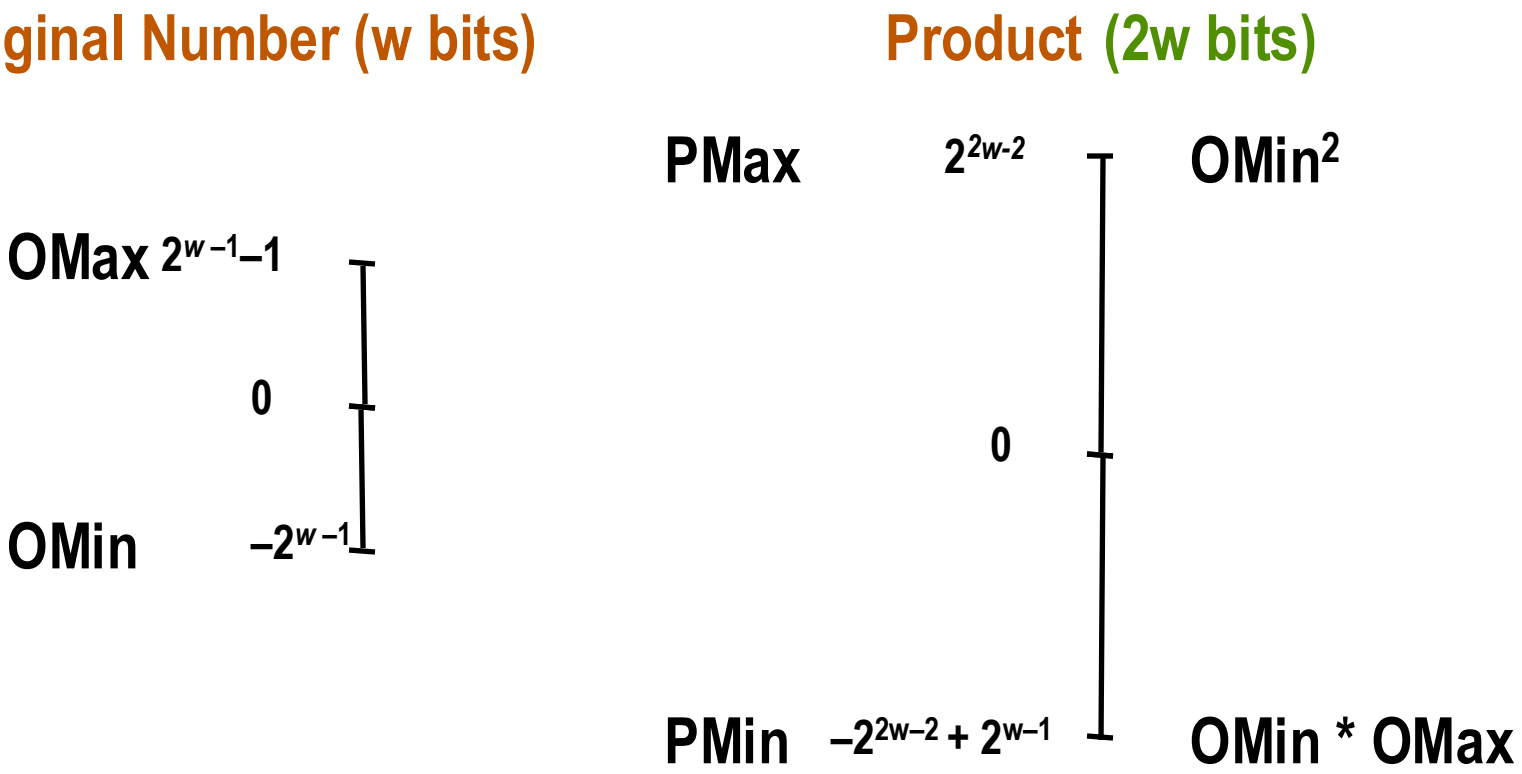
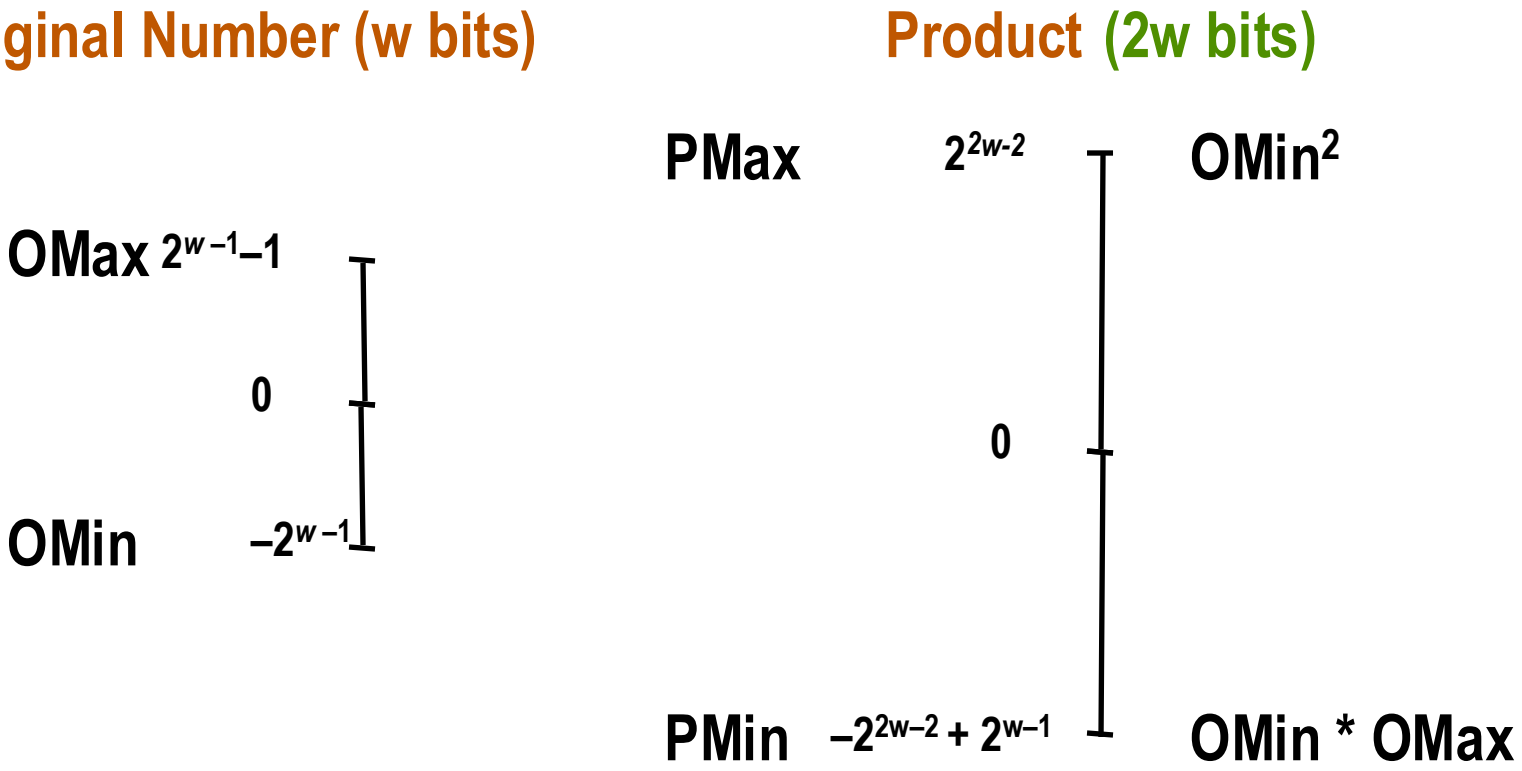
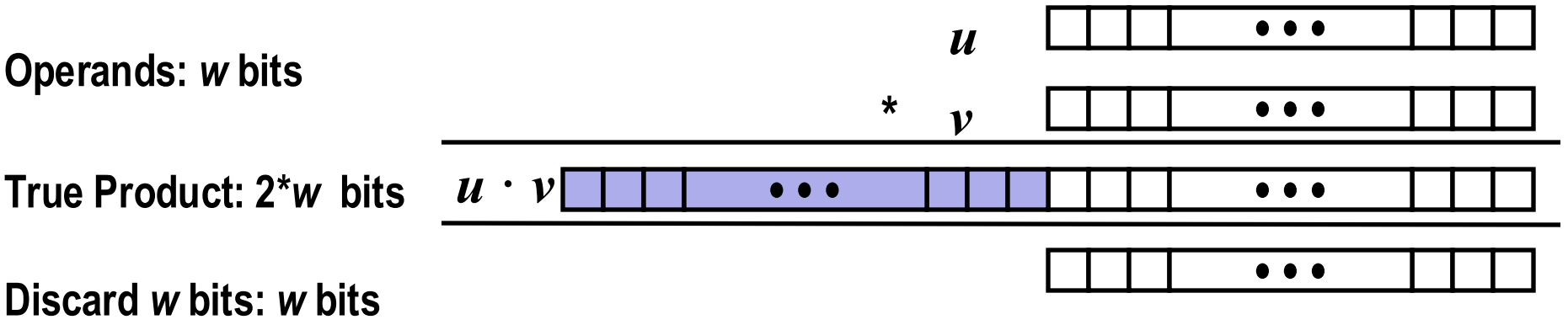


Multiplication

- Goal: Computing Product of *w*-bit numbers *x*, *y*
- Exact results can be bigger than *w* bits
  - Up to *2w* bits (both signed and unsigned)



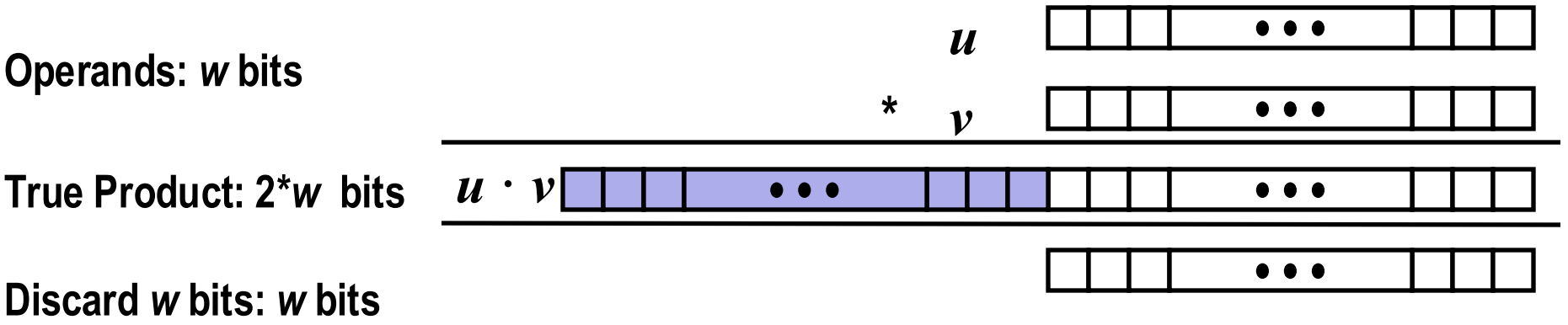
Unsigned Multiplication in C



- Standard Multiplication Function
  - Ignores high order w bits
- Effectively Implements the following:

  $\text{UMult}_w(u, v) = u \cdot v \mod 2^w$

Unsigned Multiplication in C



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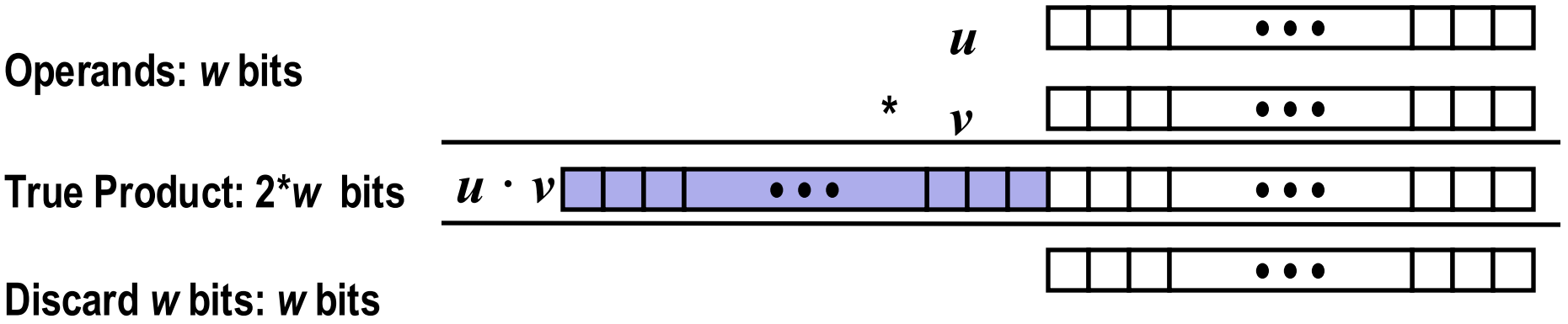
$\text{UMult}_w(u, v) = u \cdot v \bmod 2^w$

```
                1110 1001      E9      223
*               1101 0101   *  D5   *  213
-----------------------------------------
 **** ****      1101 1101      C1DD    47499
-----------------------------------------
                1101 1101      DD      221
```

Signed Multiplication in C



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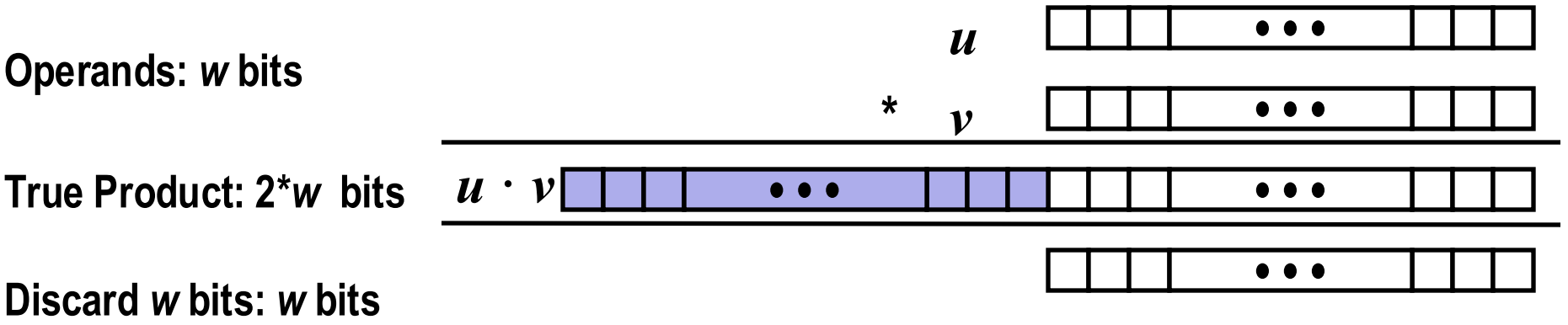
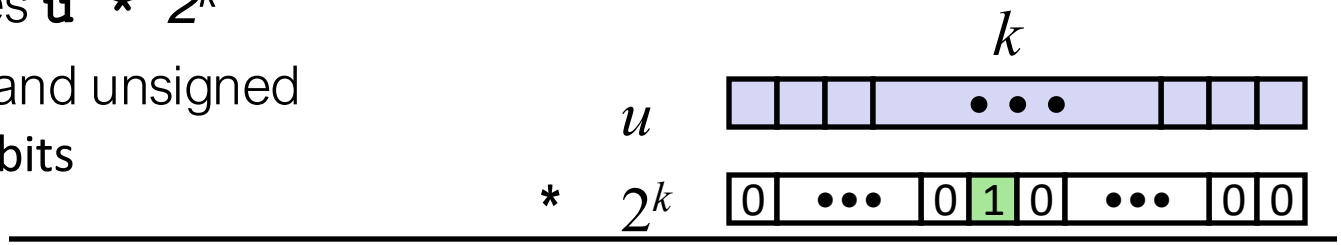


- Standard Multiplication Function
  - Ignores high order w bits
  - Some of which are different for signed vs. unsigned multiplication
  - Lower bits are the same

```
                  1110 1001      E9      -23
*                 1101 0101    * D5    * -43
-------------------------------------------
 **** ****        1101 1101     03DD     989
-------------------------------------------
                  1101 1101      DD      -35
```

Power-of-2 Multiply with Shift

- Operation
  - **`u << k`** gives **`u * `** 2^k
  - Both signed and unsigned
    Operands: *w* bits

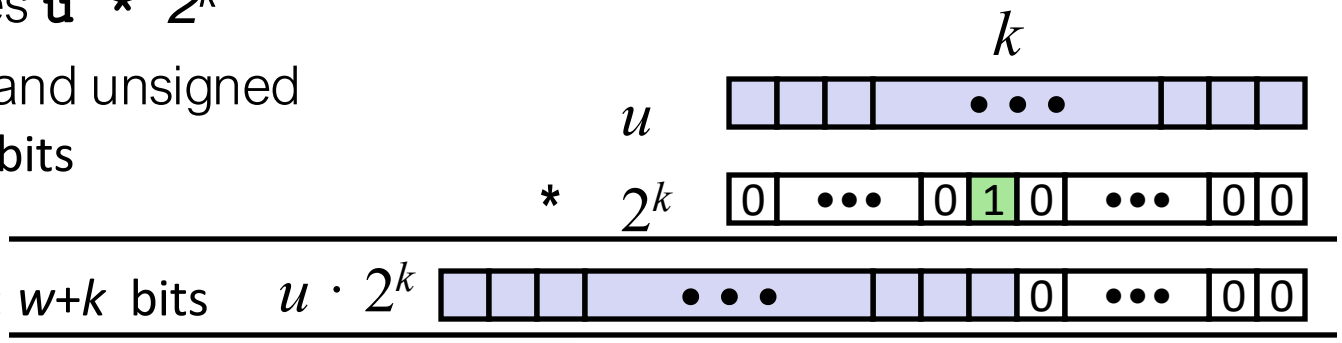


Power-of-2 Multiply with Shift

- Operation
  - **`u << k`** gives **`u * `** 2^k
  - Both signed and unsigned

    Operands: *w* bits

    True Product: *w*+*k* bits



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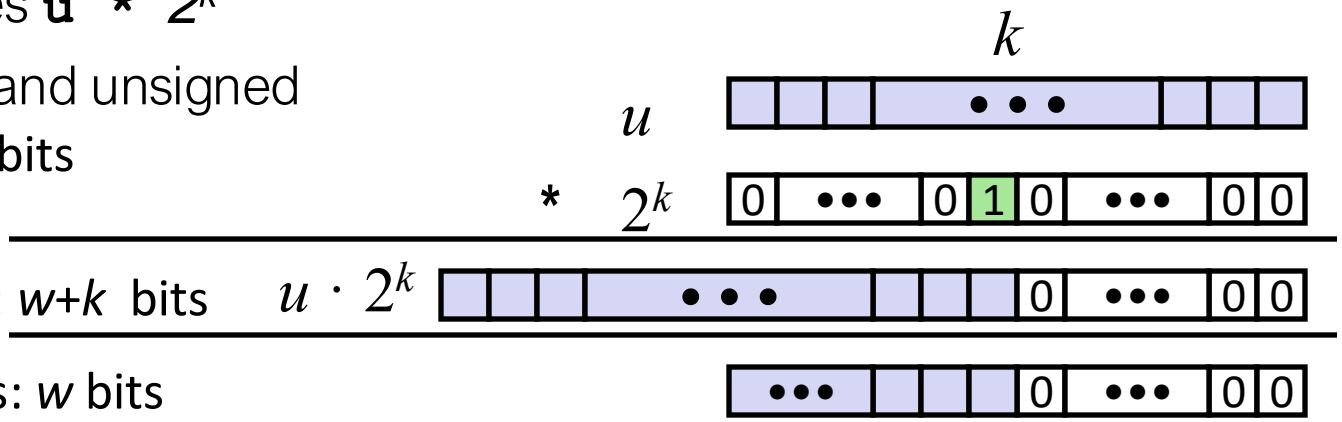

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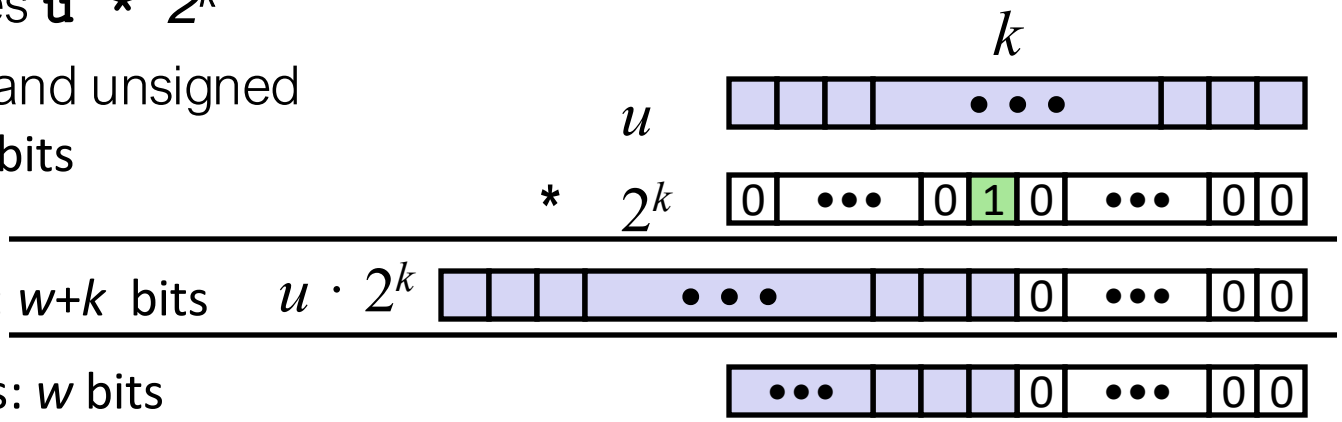

- Examples
  - **`u << 3 == u * 8`**
  - **`(u << 5) - (u << 3) == u * 24`**
  - Most machines shift and add faster than multiply
  - Compiler generates this code automatically

Today: Representing Information in Binary

- Why Binary (bits)?
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary

Arithmetic: Basic Rules

- Addition:
  - Unsigned/signed: Normal addition followed by truncate, same operation on bit level
- Multiplication:
  - Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
  - Shift: Power-of-2 Multiply

Why Should I Use Unsigned?

- *Don't* use without understanding implications
  - Easy to make mistakes

```
unsigned int i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - Can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

Why Should I Use Unsigned? – Bit Set

- *Use bits to represent my availability of the week*

b_6	b_5	b_4	b_3	b_2	b_1	b_0
Sun	**Mon**	**Tue**	**Wed**	**Thu**	**Fri**	**Sat**
1	0	1	1	0	0	1

- Use 1 bit per day, 7 bits in total.
- If bit x is set to 1 then I'm available on day mapped to bit x.

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- Use 1 bit per day, 7 bits in total.
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- In C: `unsigned int aval;`

$$aval = 1*2^0 + 0*2^1 + 0*2^2 + 1*2^3 + 1*2^4 + 0*2^5 + 1*2^6 = 89_{10}$$

Today: Floating Point

- Background: Fractional binary numbers and fixed-point
- Floating point representation
- IEEE 754 standard
- Rounding, addition, multiplication
- Floating point in C
- Summary

Can We Represent Fractions in Binary?

- What does 10.01_2 mean?
  - C.f., Decimal

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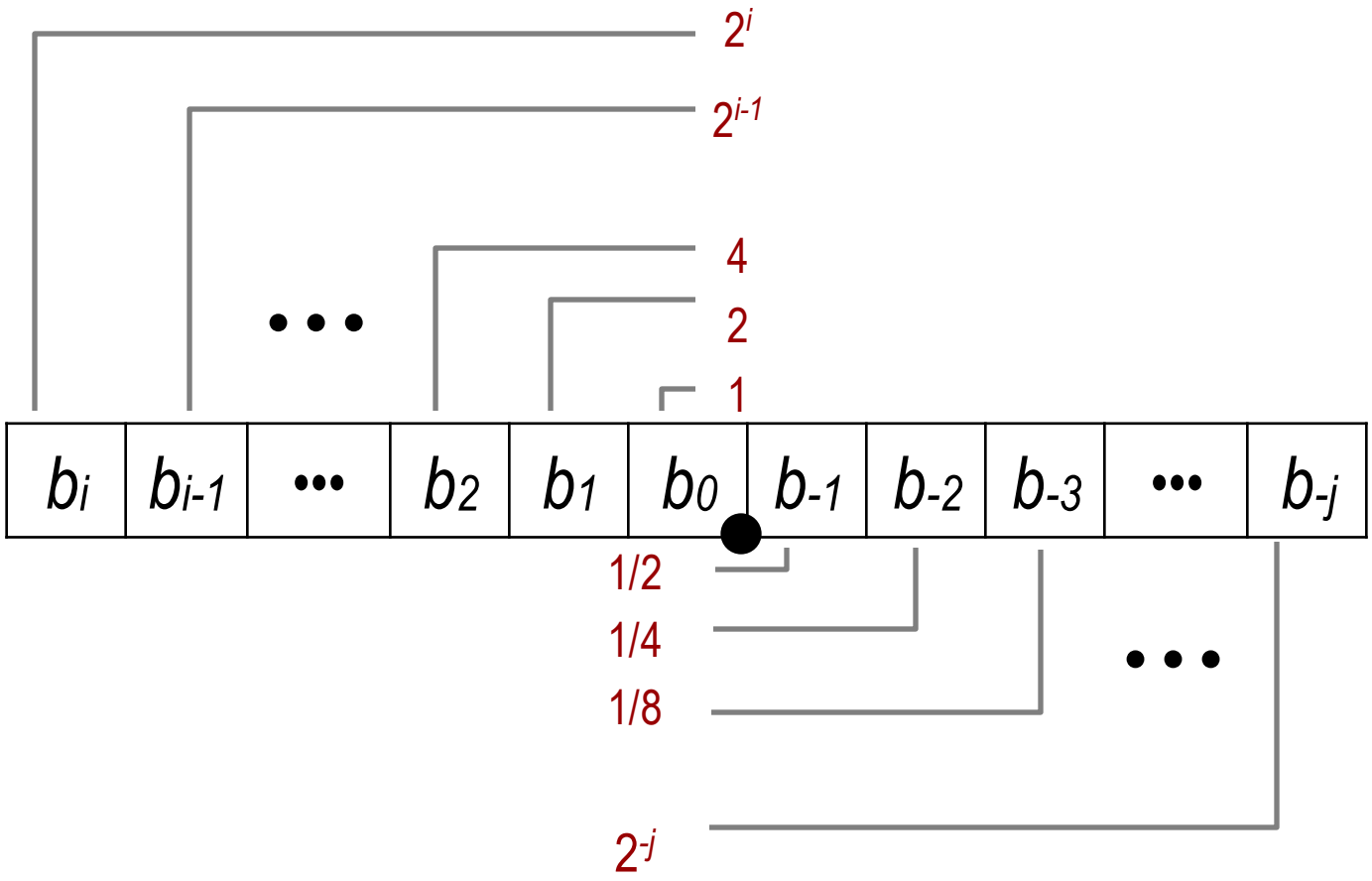
$$= 2.25_{10}$$

Fractional Binary Numbers



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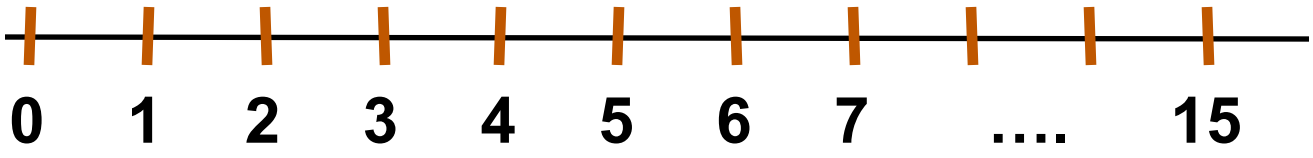

Decimal	Binary
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111

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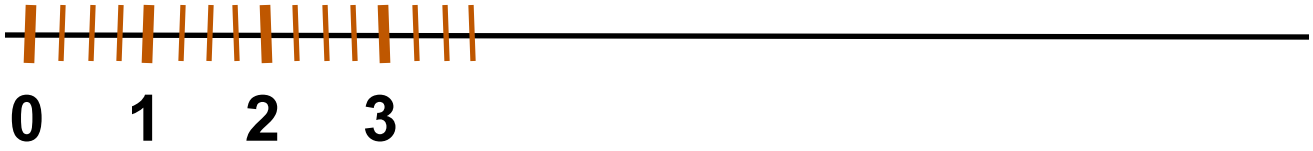

Decimal	Binary
0	00.00
0.25	00.01
0.5	00.10
0.75	00.11
1	01.00
1.25	01.01
1.5	01.10
1.75	01.11
2	10.00
2.25	10.01
2.5	10.10
2.75	10.11
3	11.00
3.25	11.01
3.5	11.10
3.75	11.11

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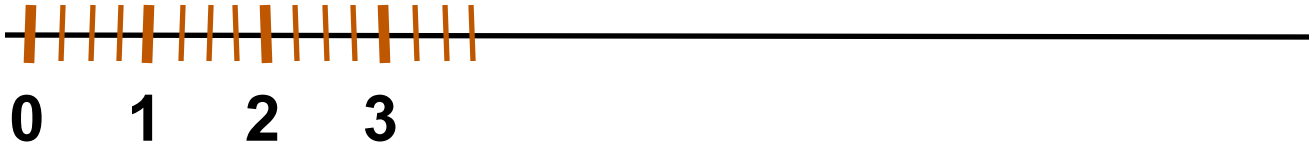


```
  01.10          1.50
+ 01.01        + 1.25
-------        ------
  10.11          2.75
```

Decimal	Binary
0	00.00
0.25	00.01
0.5	00.10
0.75	00.11
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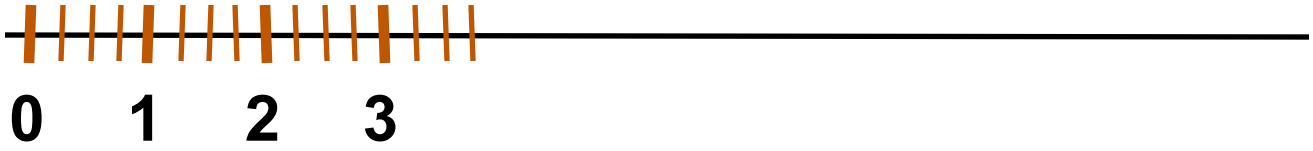


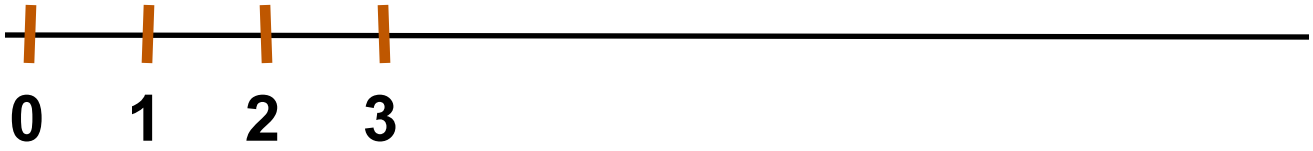
Integer Arithmetic Still Works!

```
  01.10          1.50
+ 01.01        + 1.25
-------        ------
  10.11          2.75
```

Decimal	Binary
0	00.00
0.25	00.01
0.5	00.10
0.75	00.11
1	01.00
1.25	01.01
1.5	01.10
1.75	01.11
2	10.00
2.25	10.01
2.5	10.10
2.75	10.11
3	11.00
3.25	11.01
3.5	11.10
3.75	11.11

Fixed-Point Representation

- Binary point stays fixed
- Fixed interval between two representable numbers as long as the binary point stays fixed
  - The interval in this example is 0.25_{10}
- Fixed-point representation of numbers
  - Integer is one special case of fixed-point



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0.75	00.11
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1.5	01.10
1.75	01.11
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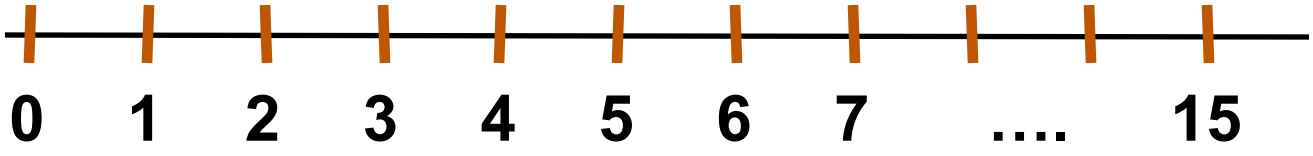

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5	0101.
6	0110.
7	0111.
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9	1001.
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11	1011.
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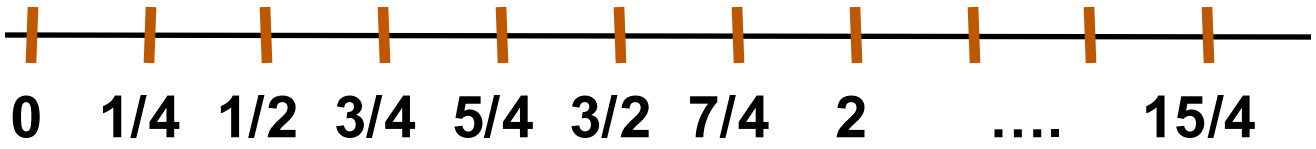

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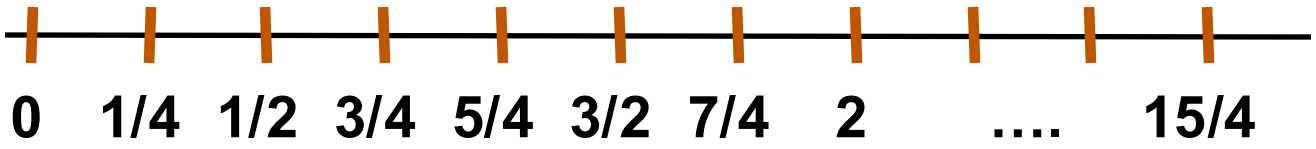

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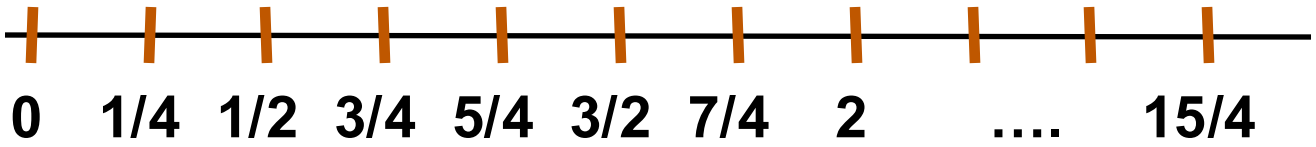
Decimal Value	Binary Representation
1/3	0.0101010101[01]…
1/5	0.001100110011[0011]…
1/10	0.0001100110011[0011]…



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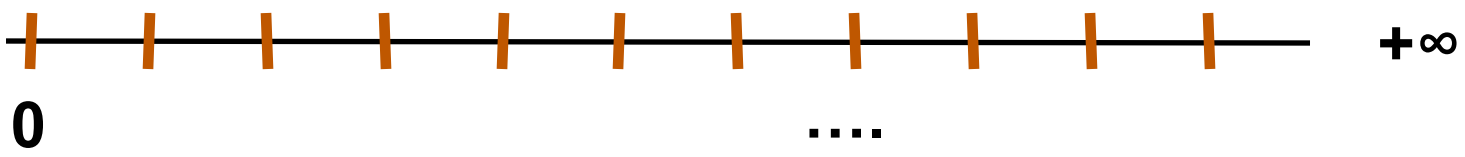

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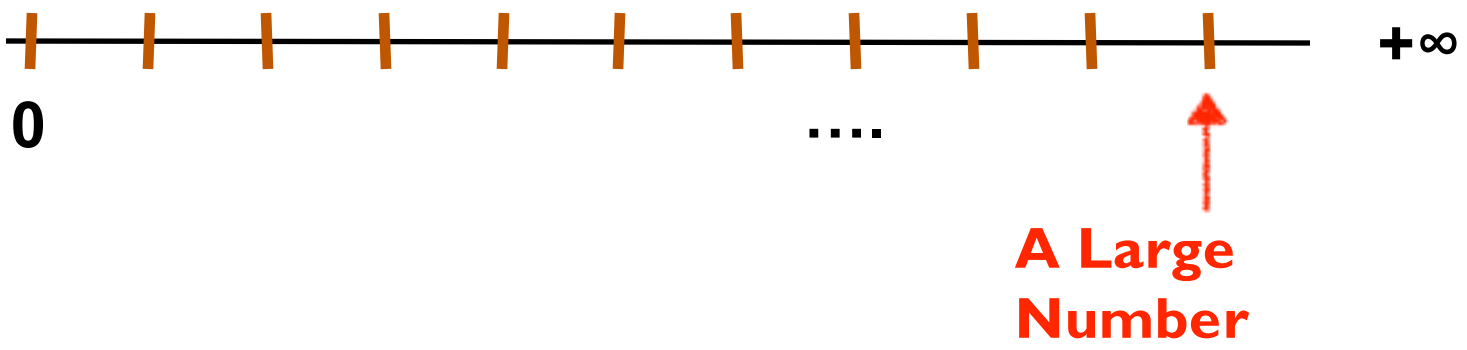

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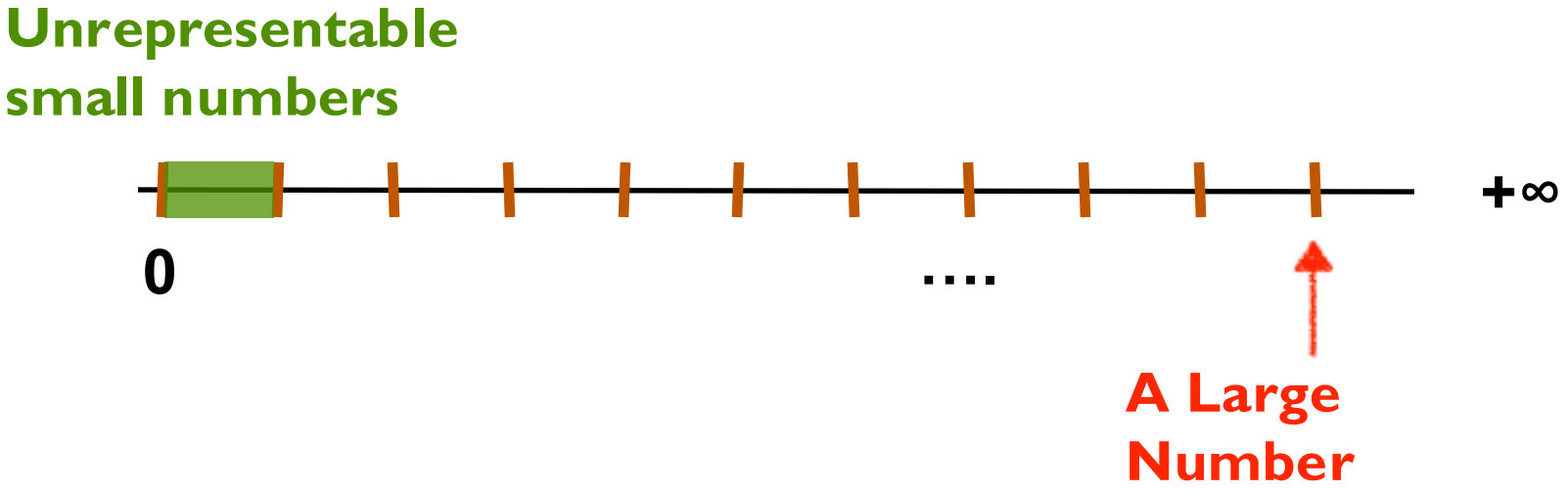

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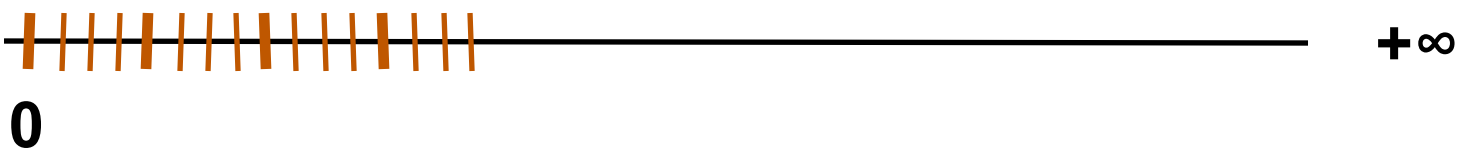

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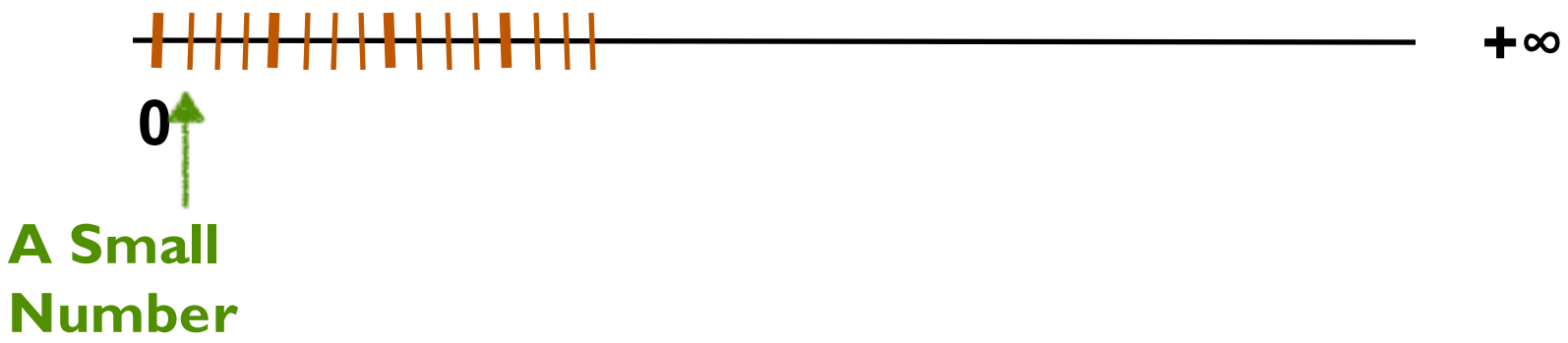

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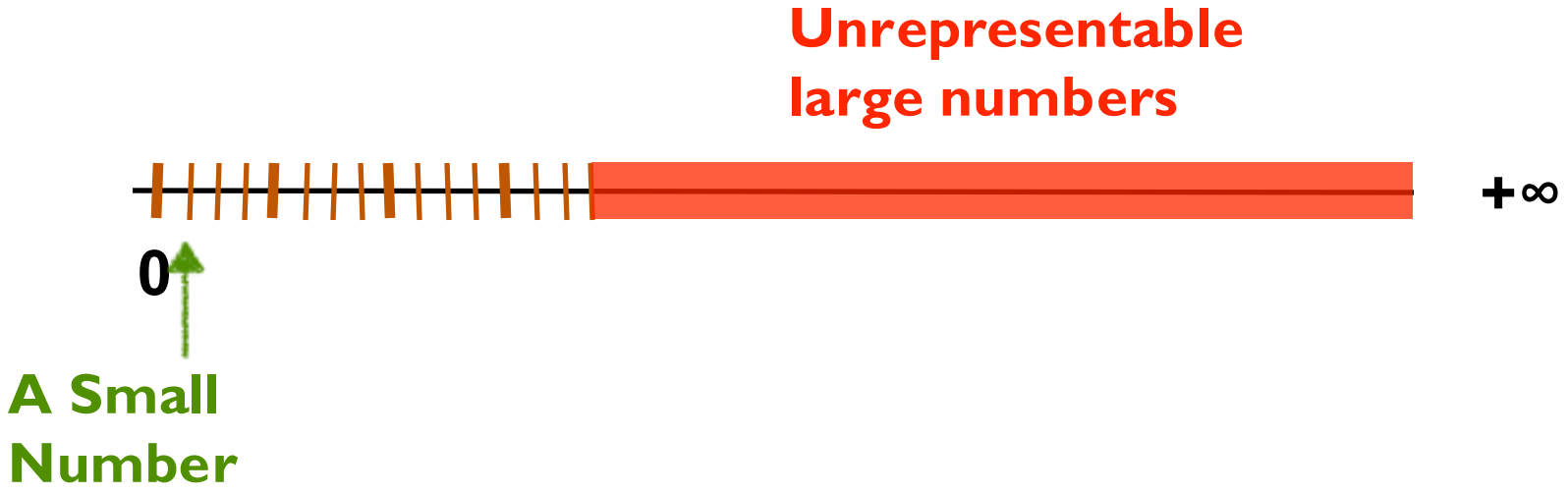

Today: Floating Point

- Background: Fractional binary numbers and fixed-point
- Floating point representation
- IEEE 754 standard
- Rounding, addition, multiplication
- Floating point in C
- Summary

Primer: (Normalized) Scientific Notation

- In decimal: $M \times 10^E$
  - *E* is an integer
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Significand

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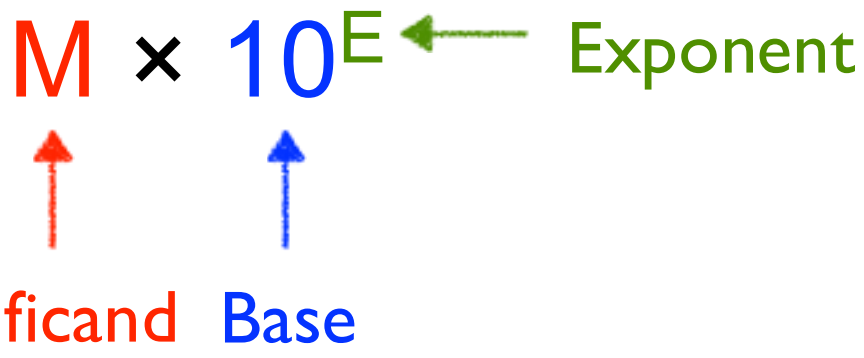

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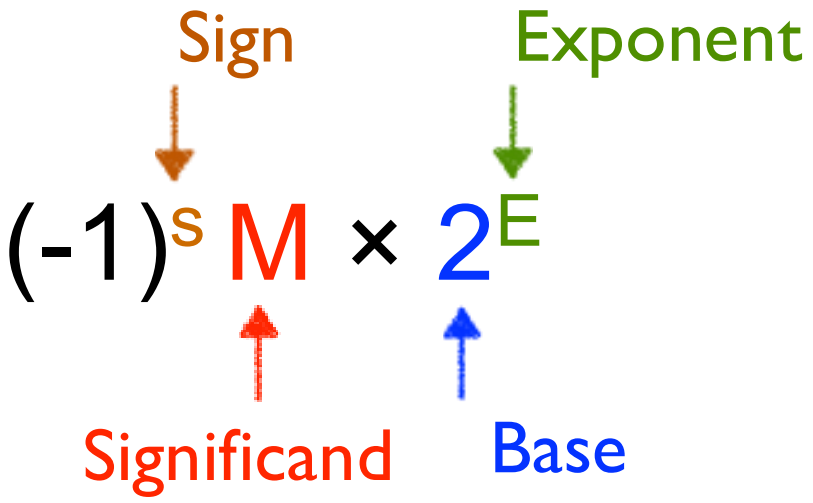
Fraction



Binary Value	Scientific Notation
1110110110110	$(-1)^0\ 1.1110110110110 \times 2^{12}$
-101.11	$(-1)^1\ 1.0111 \times 2^2$
0.00101	$(-1)^0\ 1.01 \times 2^{-3}$

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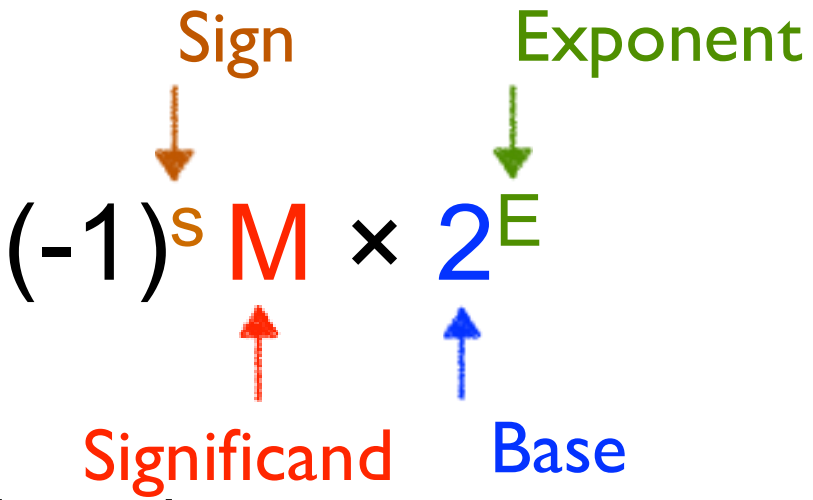

- If I tell you that there is a number where:
  - Fraction = 0101
  - s = 1
  - E = 10
  - You could reconstruct the number as $(-1)^1 1.0101 \times 2^{10}$

Primer: Floating Point Representation

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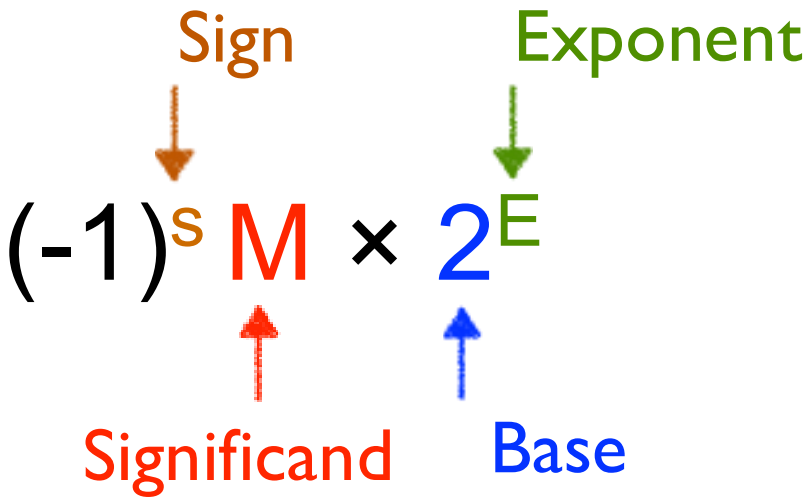

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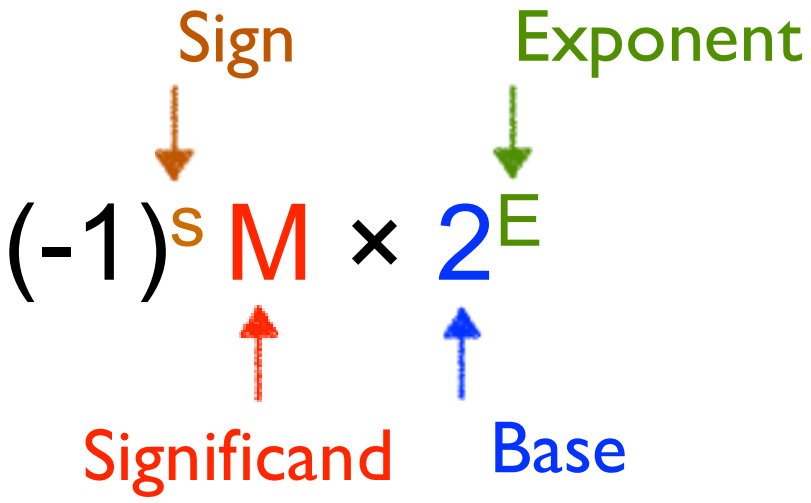

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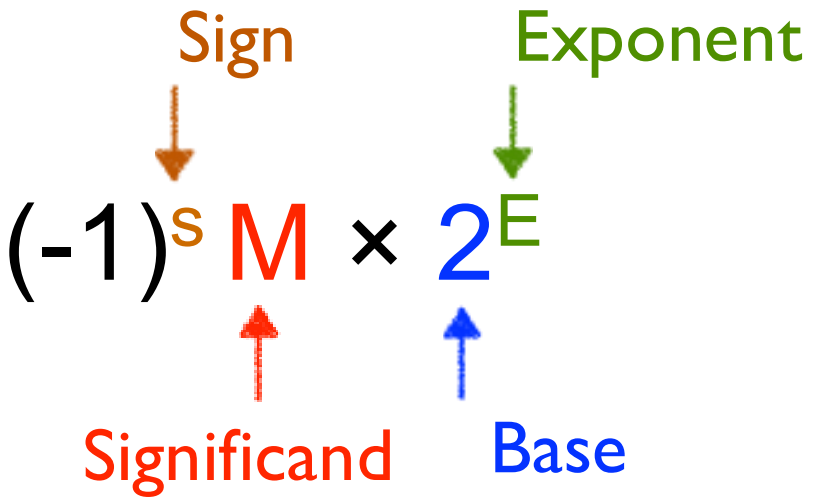

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Fraction

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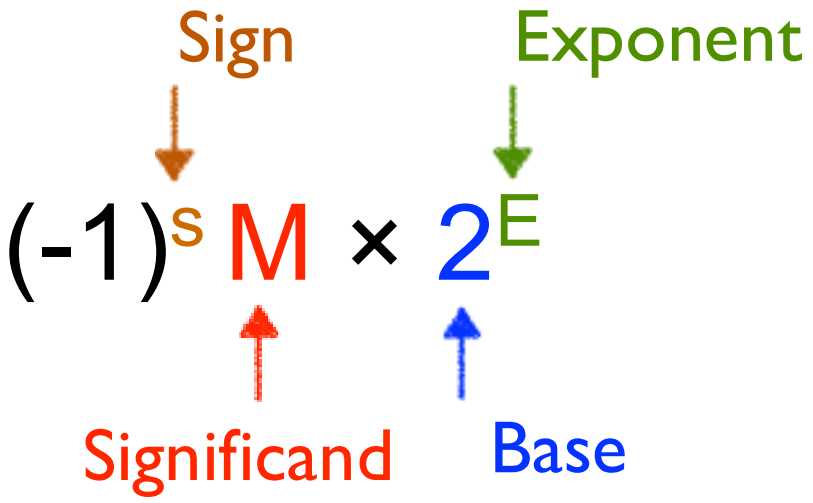

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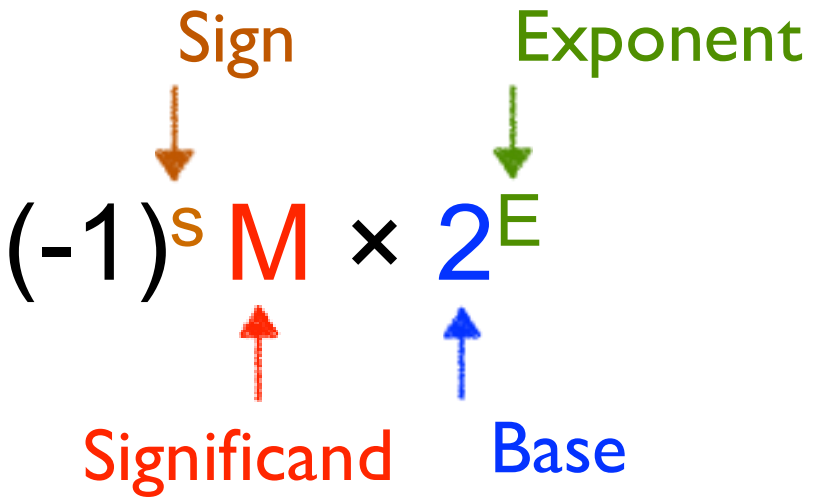

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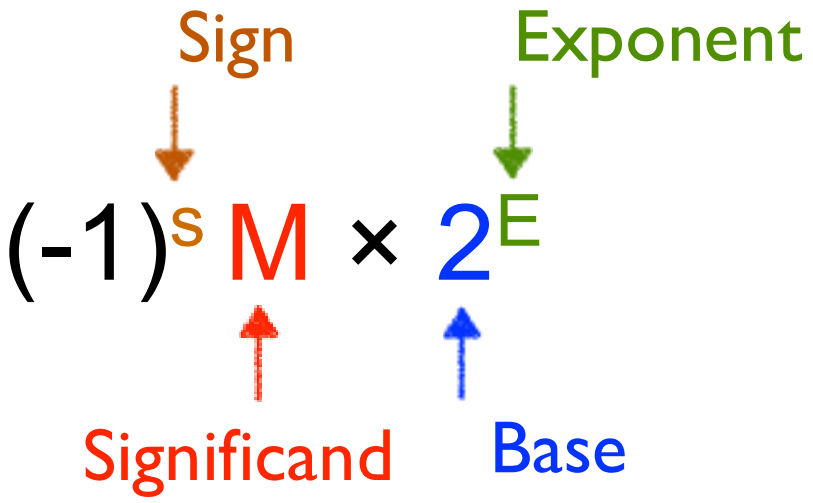

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6-bit Floating Point Example

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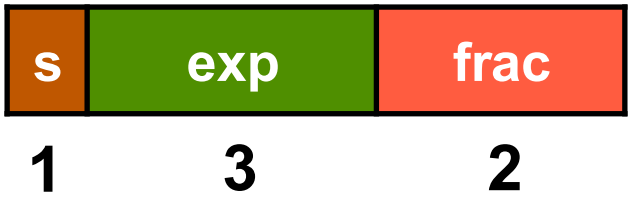

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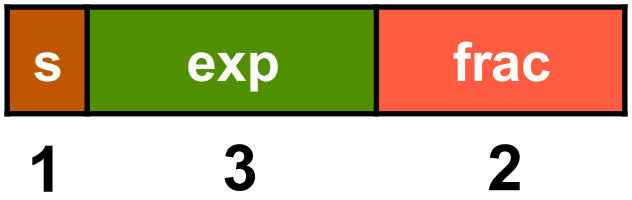

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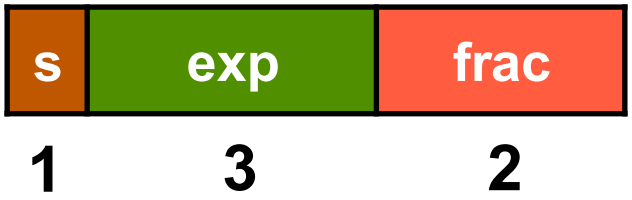

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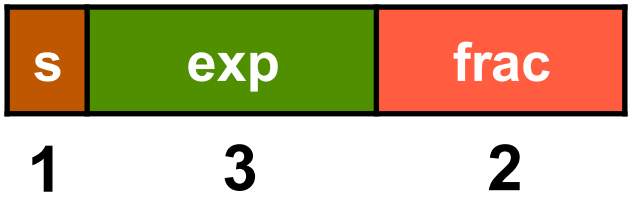

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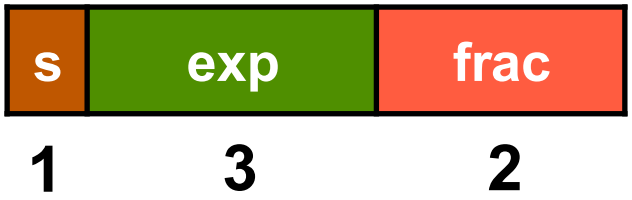

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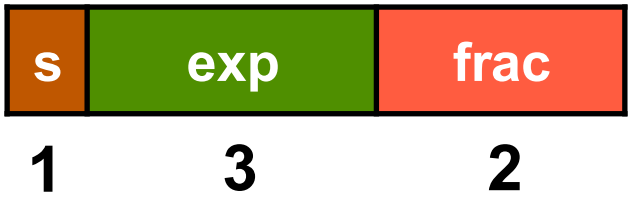

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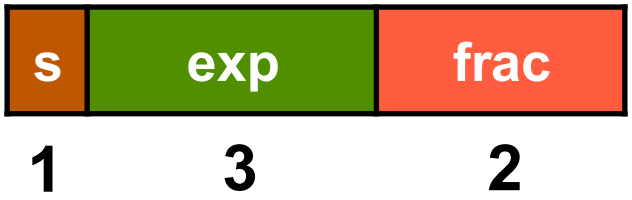

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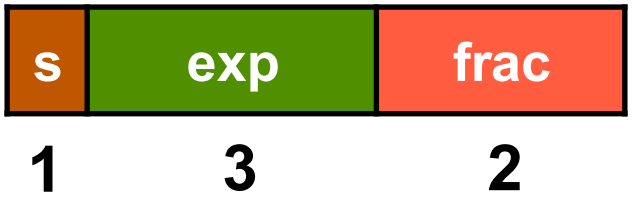

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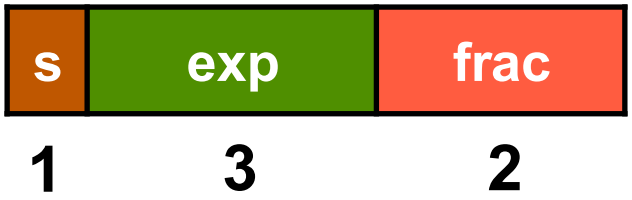

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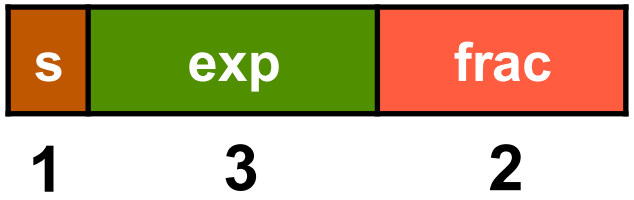

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E	exp
-3	000
-2	001
-1	010
0	011
1	100
2	101
3	110
4	111

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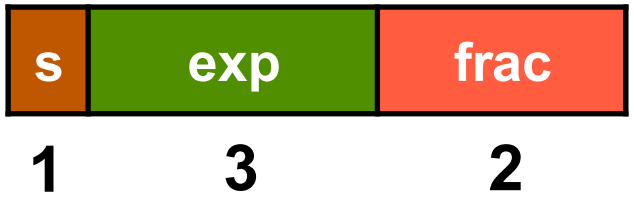

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  - Reserve 000 and 111 for other purposes (more on this later)

E	exp
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

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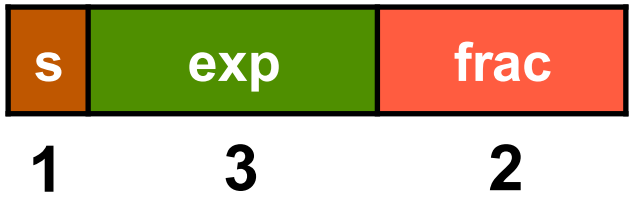

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  - If ***E*** = -2, ***exp*** is 1 (001_2)
  - Reserve 000 and 111 for other purposes (more on this later)
  - We can now represent exponents from **-2 (exp 001) to 3 (exp 110)**

E	exp
~~3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

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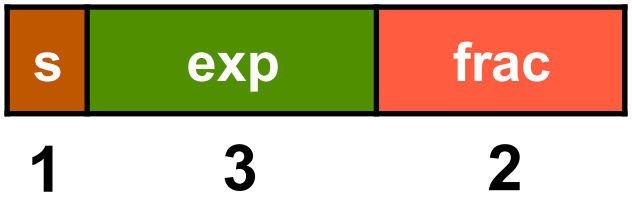

- *frac* has 2 bits, append them after “1.” to form M
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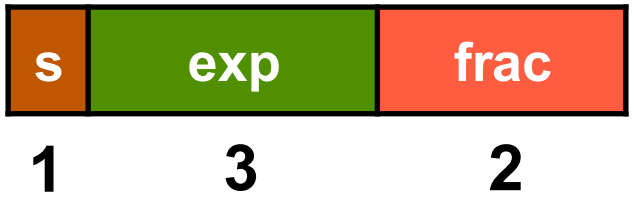

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- Putting it Together: An Example:

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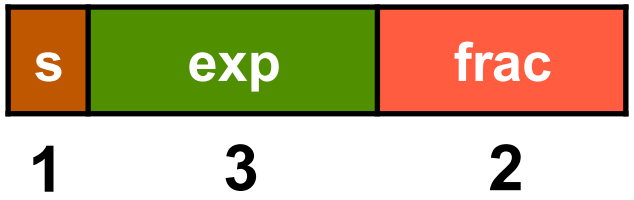
6-bit Floating Point Example

$v = (-1)^s M 2^E$



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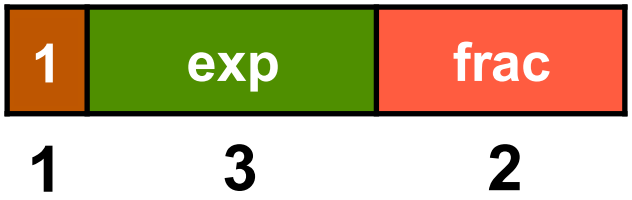

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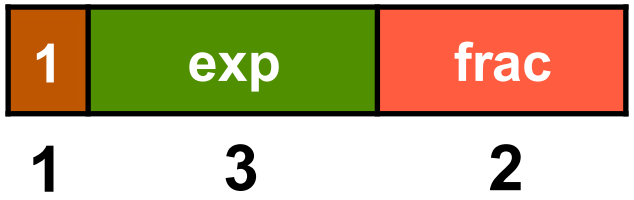

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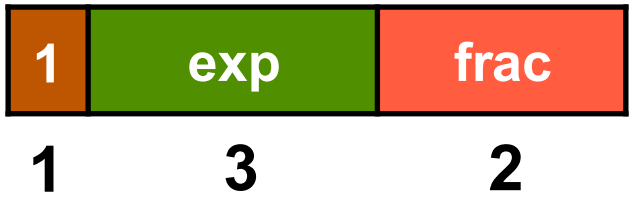

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3	110
~~4~~	~~111~~

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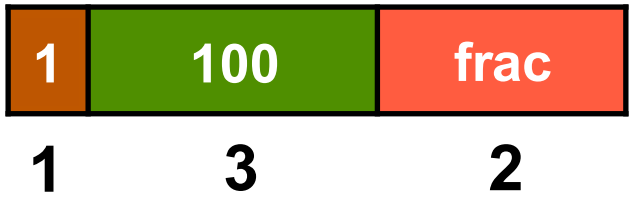

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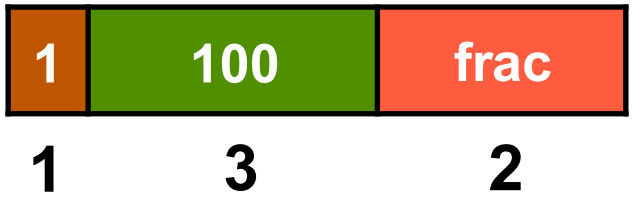

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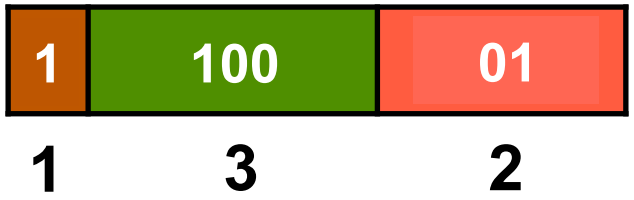

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