CSC 252: Computer Organization Spring 2018: Lecture 4

Instructor: Yuhao Zhu

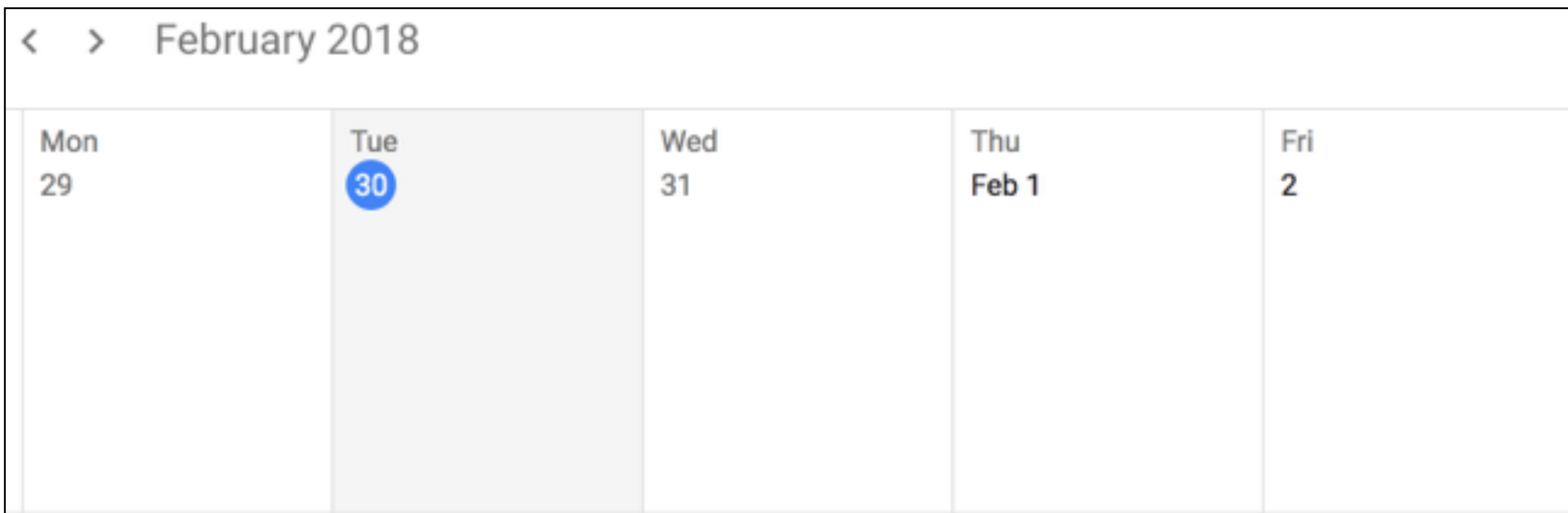
Department of Computer Science
University of Rochester

Action Items:

- **Assignment 1 due Feb. 2, midnight**

Announcement

- Programming Assignment 1 is out
  - Due on **Feb 2, 11:59 PM**
  - You have 3 slip days
  - Try to submit once to make sure you can submit
  - We count only the latest submission before the deadline



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- TAs are better positioned to answer questions regarding assignments

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Google "Hexspeak"

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Today: Floating Point

- Background: Fractional binary numbers and fixed-point
- Floating point representation
- IEEE 754 standard
- Rounding, addition, multiplication
- Floating point in C
- Summary

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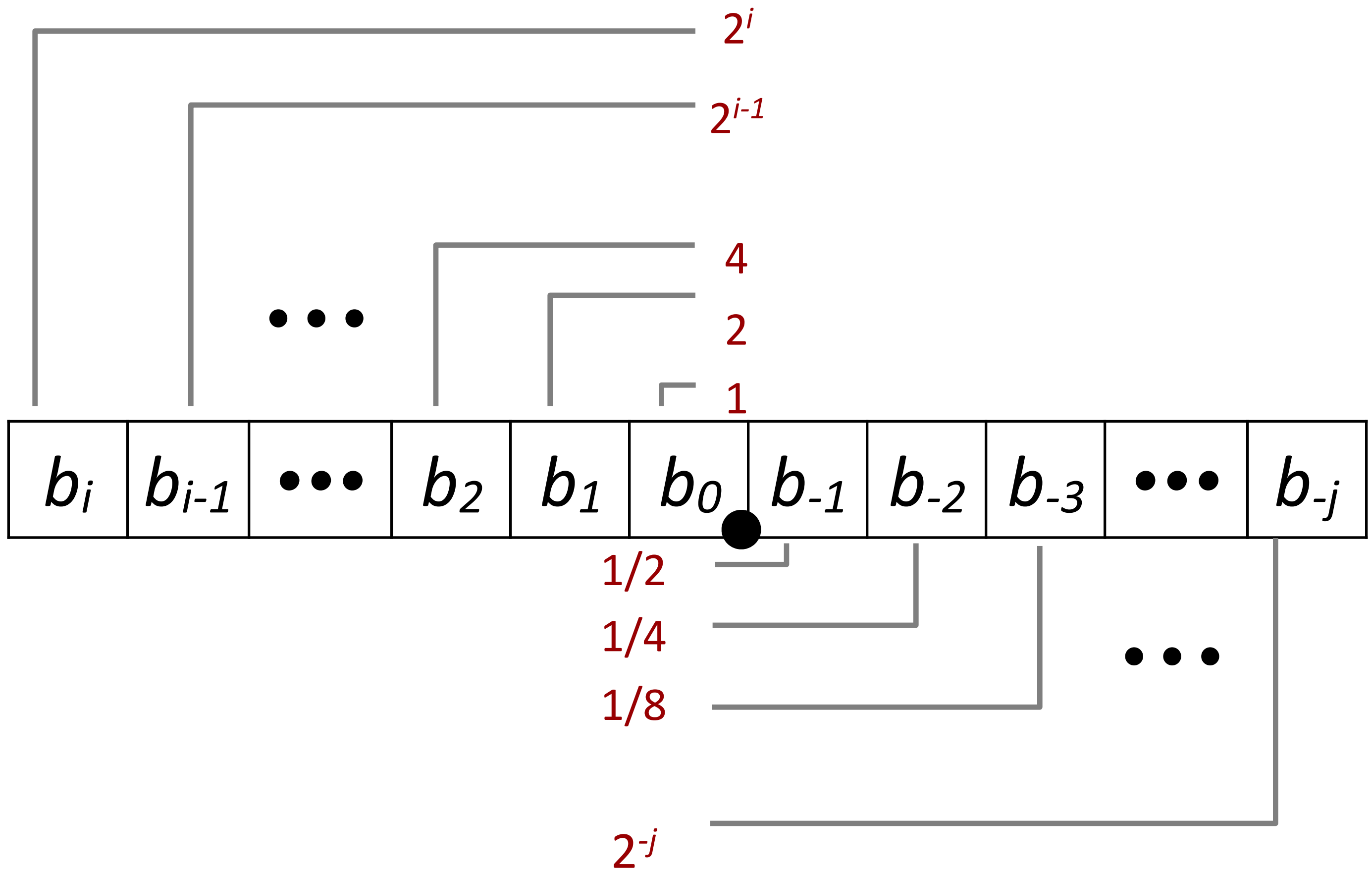
$$= 2.25_{10}$$

Fractional Binary Numbers



Fractional Binary Numbers: Examples

Decimal Value	**Binary Representation**
5 3/4	101.11
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Exact Same Raw Bit Stream!

- We would need to remember:
  - The raw bit stream
  - Where the binary point is
- Makes calculations (e.g. addition) hard, and not very elegant
  - Need to first align numbers according to the binary point

Fixed-Point Representation

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Decimal	Binary
0	0000.
1	0001.
2	0010.
3	0011.
4	0100.
5	0101.
6	0110.
7	0111.
8	1000.
9	1001.
10	1010.
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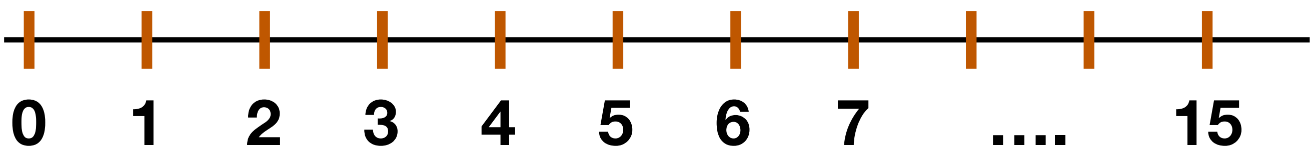

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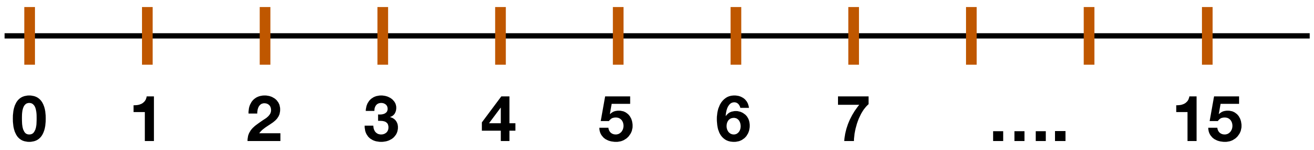

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0.25	00.01
0.5	00.10
0.75	00.11
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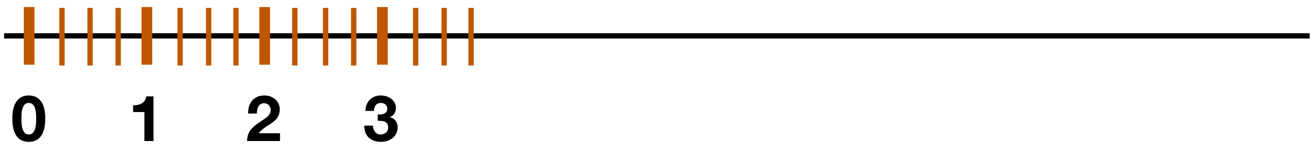

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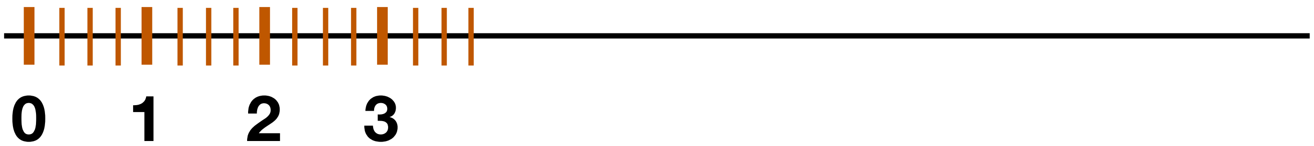
  - Each bit represents 0.25_{10}



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- Still need to remember the binary point, but just once for all numbers
- No need to align (already aligned)
- C uses fixed-point encoding only for integral data types (**`long`**, **`int`**, **`short`**, etc.)
  - Effectively, implicitly assumes the binary point is at the rightmost

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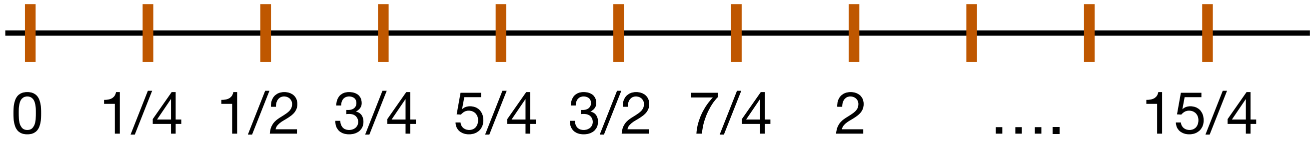

$b_3b_2.b_1b_0$

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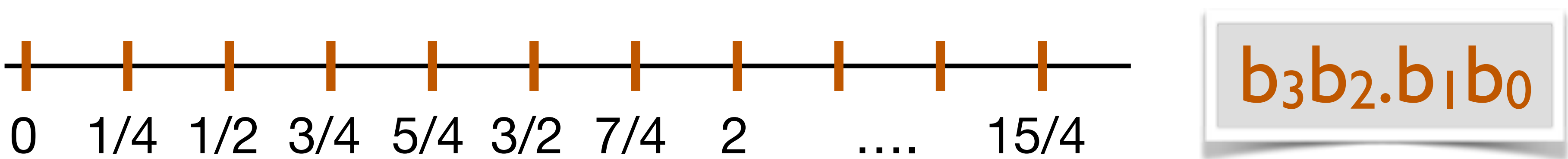
  - Other rational numbers have repeating bit representations



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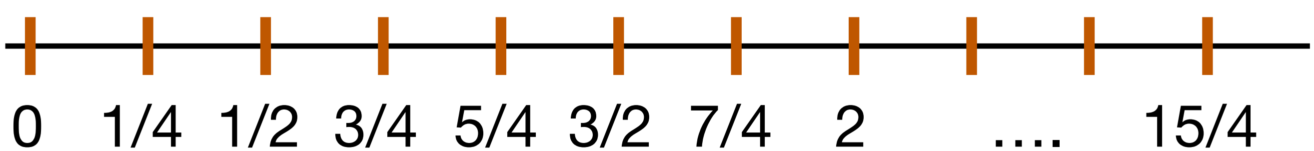

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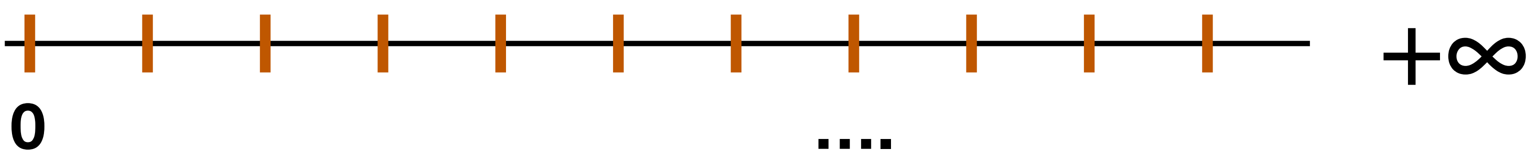

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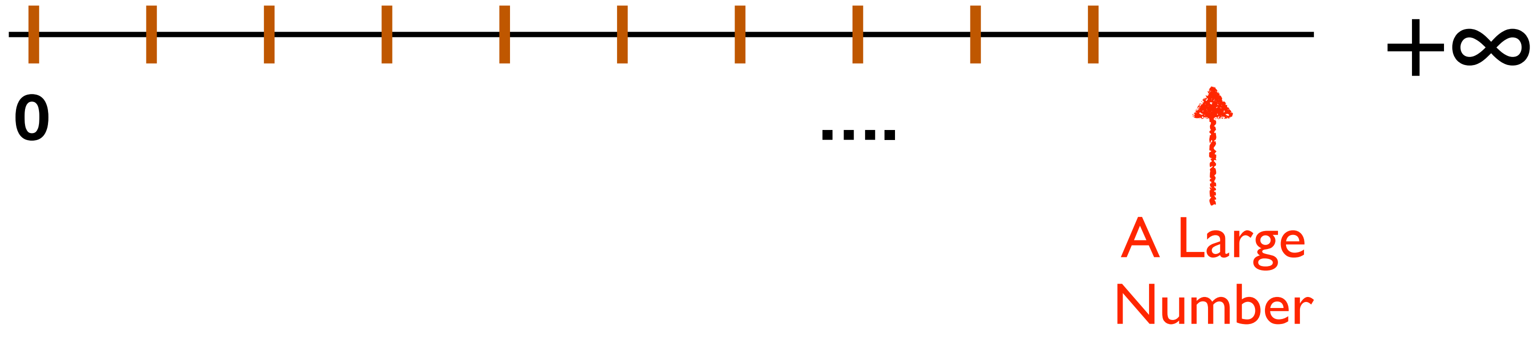

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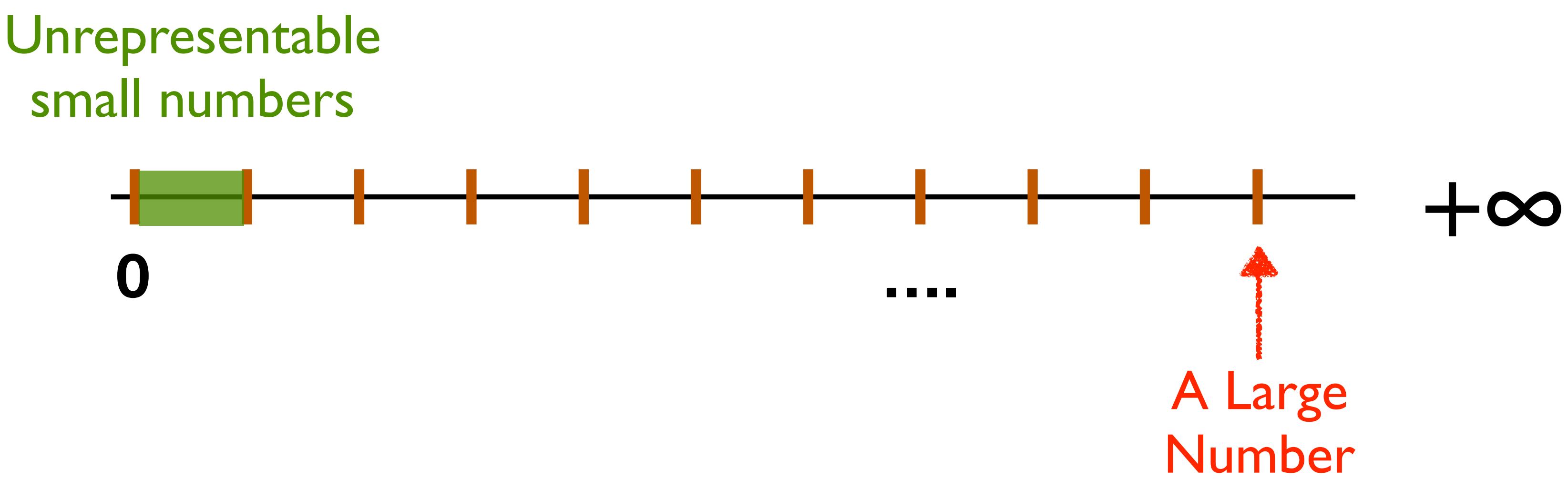

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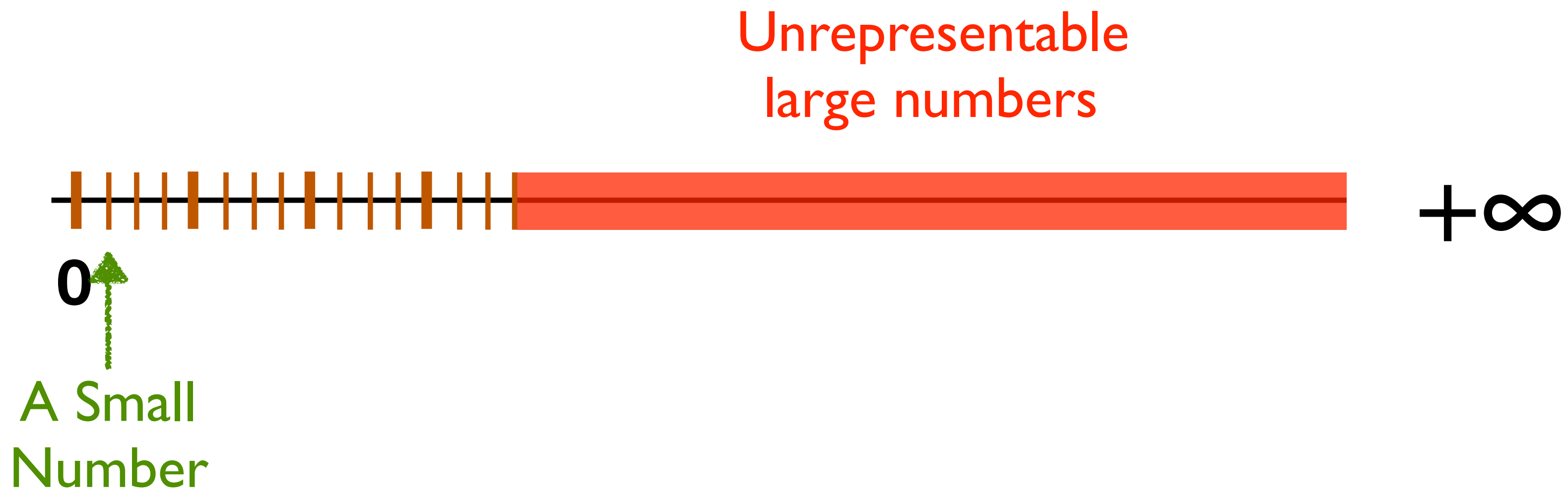

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- Background: Fractional binary numbers and fixed-point
- Floating point representation
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- Rounding, addition, multiplication
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-4,321.768	-4.321768×10^3
0.000 000 007 51	7.51×10^{-9}

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Primer: (Normalized) Scientific Notation

- In binary: $(-1)^s M\ 2^E$
- Normalized form:
  - $1 <= M < 10$
  - $M = 1.b_0b_1b_2b_3\ldots$

$$(-1)^s M \times 2^E$$

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    Fraction

Sign Exponent

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Significand Base

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$$(-1)^{\overset{\text{Sign}}{\downarrow} s}\ \underset{\text{Significand}}{\underset{\uparrow}{M}} \times \underset{\text{Base}}{\underset{\uparrow}{2}}^{\overset{\text{Exponent}}{\downarrow} E}$$

Binary Value	Scientific Notation
1110110110110	$(-1)^{0}$ 1.110110110110 x 2^{12}
-101.11	$(-1)^{1}$ 1.0111 x 2^{2}
0.00101	$(-1)^{0}$ 1.01 x 2^{-3}

Primer: Floating Point Representation

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- Encoding

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    Fraction
- Encoding
  - MSB *s* is sign bit s
  - *exp* field encodes Exponent (but not exactly the same as E)

Sign Exponent

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  - $1 <= M < 2$
  - $M = 1.\underbrace{b_0 b_1 b_2 b_3 \ldots}_{\text{Fraction}}$

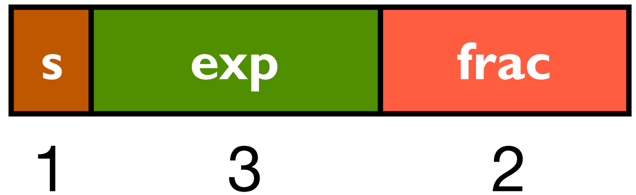
$$\overset{\text{Sign}}{(-1)^{s}}\ \underset{\text{Significand}}{M} \times \underset{\text{Base}}{2}^{\overset{\text{Exponent}}{E}}$$

- Encoding
  - MSB *s* is sign bit s
  - *exp* field encodes Exponent (but not exactly the same as E)
  - *frac* field encodes Fraction (but not exactly the same as Fraction)

s	**exp**	**frac**

6-bit Floating Point Example

$v = (-1)^s M\, 2^E$



6-bit Floating Point Example

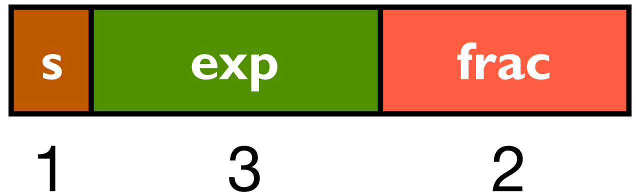
$$v = (-1)^s M 2^E$$

s	exp	frac
1	3	2

- *exp* has 3 bits, interpreted as an unsigned value

6-bit Floating Point Example

$$v = (-1)^s \, M \, 2^E$$



- *exp* has 3 bits, interpreted as an unsigned value
  - If *exp* were *E*, we could represent exponents from **0 to 7**

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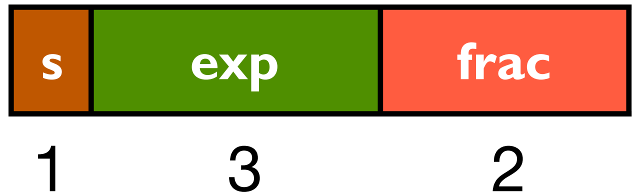

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  - bias is always $2^{k-1} - 1$, where k is number of exponent bits

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- Example when *k* = 3:
  - bias = 3
  - If *E* = -2, *exp* is 1 (001_2)

6-bit Floating Point Example

$v = (-1)^s M\, 2^E$

s	exp	frac
1	3	2

- *exp* has 3 bits, interpreted as an unsigned value
  - If *exp* were *E*, we could represent exponents from **0 to 7**
  - How about negative exponent?
  - Add a bias term: *E = exp - bias* (i.e., exp = E + bias)
  - bias is always $2^{k-1} - 1$, where k is number of exponent bits
- Example when $k = 3$:
  - bias = 3
  - If $E = -2$, *exp* is 1 (001_2)

E	exp
-3	000
-2	001
-1	010
0	011
1	100
2	101
3	110
4	111

6-bit Floating Point Example

$$v = (-1)^s \, M \, 2^E$$

s	exp	frac
1	3	2

- *exp* has 3 bits, interpreted as an unsigned value
  - If *exp* were *E*, we could represent exponents from **0 to 7**
  - How about negative exponent?
  - Add a bias term: *E = exp - bias* (i.e., exp = E + bias)
  - bias is always $2^{k-1} - 1$, where k is number of exponent bits
- Example when $k = 3$:
  - bias = 3
  - If $E = -2$, *exp* is 1 (001_2)
  - Reserve 000 and 111 for other purposes (more on this later)

E	exp
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

$$v = (-1)^s M\, 2^E$$

s	exp	frac
1	3	2

- *exp* has 3 bits, interpreted as an unsigned value
  - If *exp* were *E*, we could represent exponents from **0 to 7**
  - How about negative exponent?
  - Add a bias term: *E = exp - bias* (i.e., exp = E + bias)
  - bias is always 2^{k-1} - 1, where k is number of exponent bits
- Example when *k* = 3:
  - bias = 3
  - If *E* = -2, *exp* is 1 (001_2)
  - Reserve 000 and 111 for other purposes (more on this later)
  - We can now represent exponents from **-2 (exp 001) to 3 (exp 110)**

E	exp
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

$v = (-1)^s \, M \, 2^E$

s	exp	frac
1	3	2

6-bit Floating Point Example

$v = (-1)^s M 2^E$

- *frac* has 2 bits, append them after “1.” to form M
  - *frac* = 10 implies M = 1.10

6-bit Floating Point Example

$$v = (-1)^s M 2^E$$

s	exp	frac
1	3	2

- *frac* has 2 bits, append them after "1." to form M
  - *frac* = 10 implies M = 1.10
- Putting it Together: An Example:

$$-10.1_2 = (-1)^1\ 1.01 \times 2^1$$

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1	3	2

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1	exp	frac
1	3	2

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6-bit Floating Point Example

$v = (-1)^s M\, 2^E$

1	exp	frac
1	3	2

- *frac* has 2 bits, append them after “1.” to form M
  - *frac* = 10 implies M = 1.10
- Putting it Together: An Example:

$$-10.1_2 = (-1)^1\ 1.01 \times 2^1$$

E	exp
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

$v = (-1)^s M\, 2^E$

1	100	frac
1	3	2

- *frac* has 2 bits, append them after “1.” to form M
  - *frac* = 10 implies M = 1.10
- Putting it Together: An Example:

$$-10.1_2 = (-1)^1\ 1.01 \times 2^1$$

E	exp
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

$v = (-1)^s M 2^E$

1	100	frac
1	3	2

- *frac* has 2 bits, append them after “1.” to form M
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E	exp
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

$$v = (-1)^s\, M\, 2^E$$

1	100	01
1	3	2

- *frac* has 2 bits, append them after "1." to form M
  - *frac* = 10 implies M = 1.10
- Putting it Together: An Example:

$$-10.1_2 = (-1)^1\ 1.01 \times 2^1$$

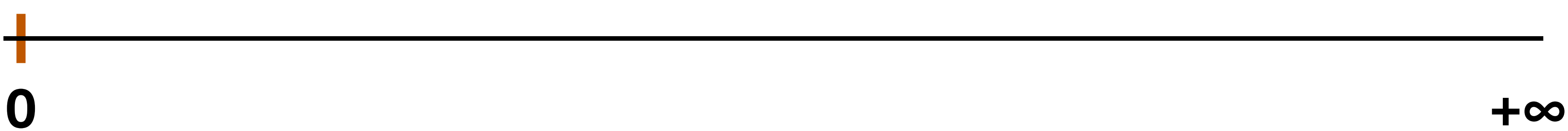
E	exp
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

s	exp	frac

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

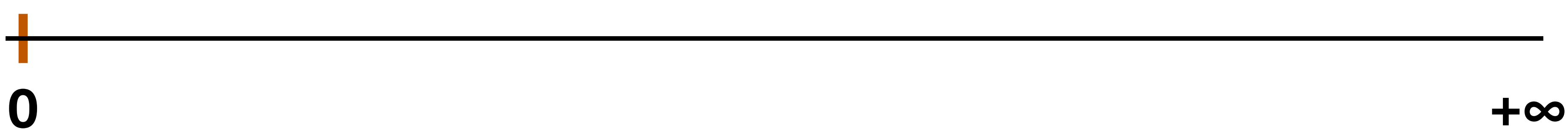


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	exp	frac

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

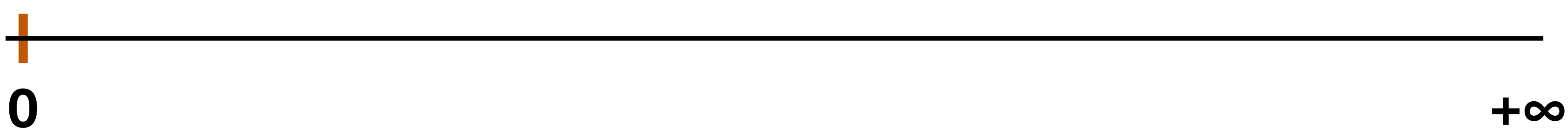


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	exp	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

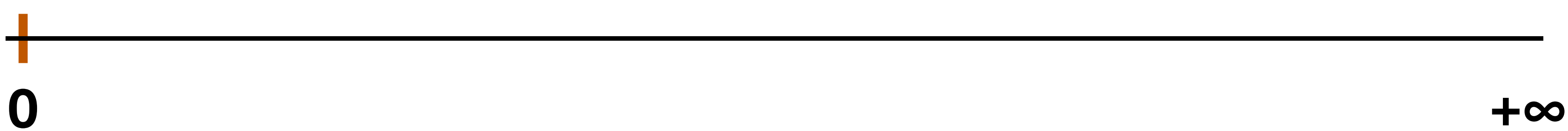


Representable Numbers (Positive Only)

$$v = (-1)^s \, M \, 2^E$$

0	001	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~



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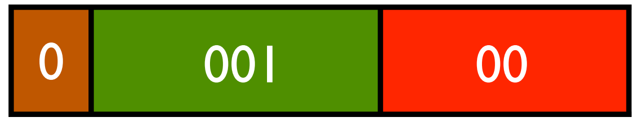


E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

Representable Numbers (Positive Only)

$v = (-1)^s M\, 2^E$

0	010	00

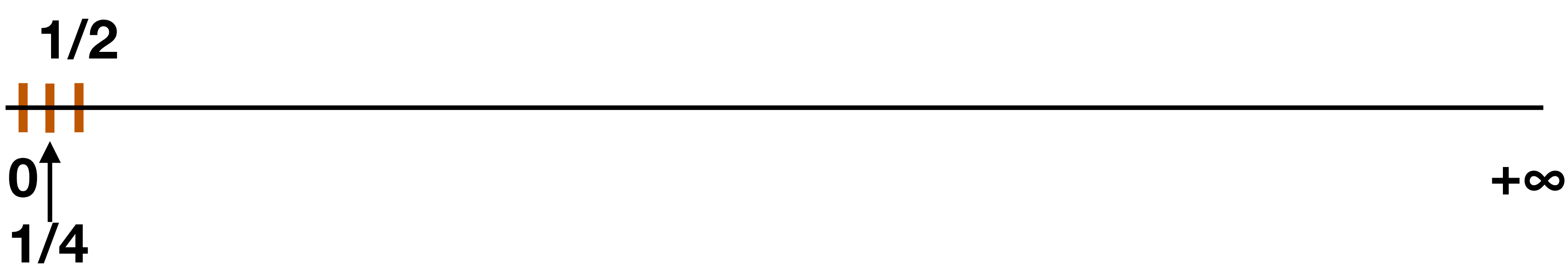
E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	010	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

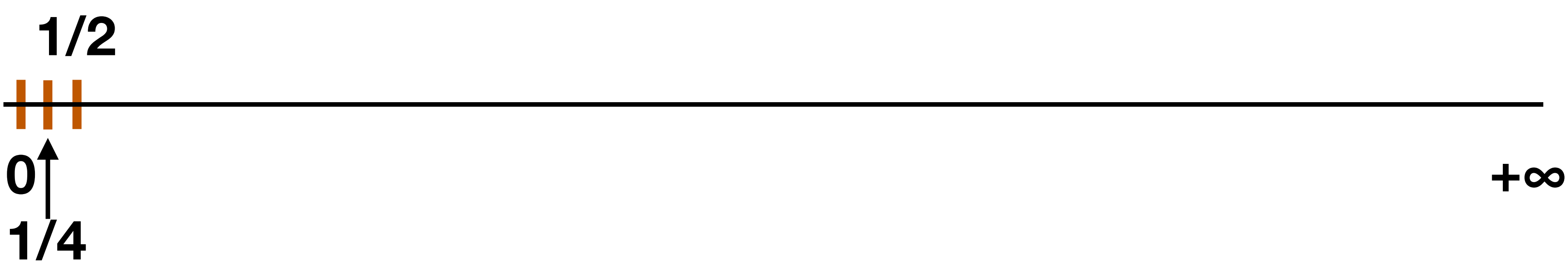


Representable Numbers (Positive Only)

$v = (-1)^s M\, 2^E$

0	011	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

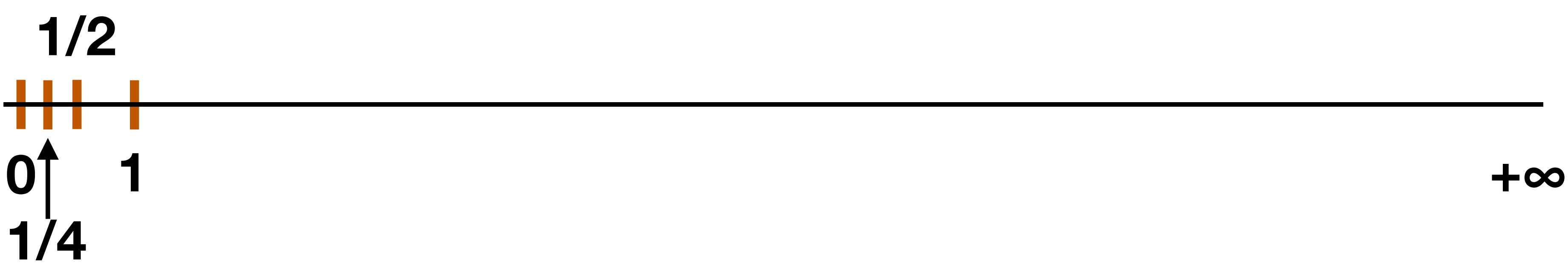


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	011	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

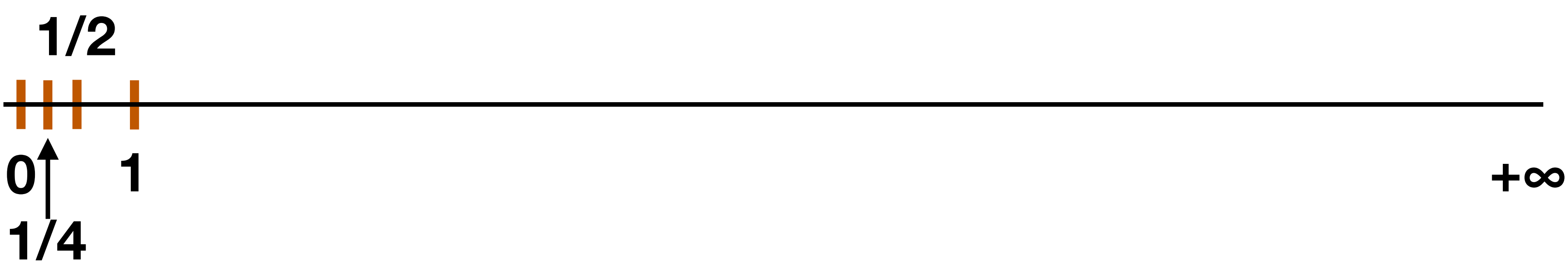


Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	100	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

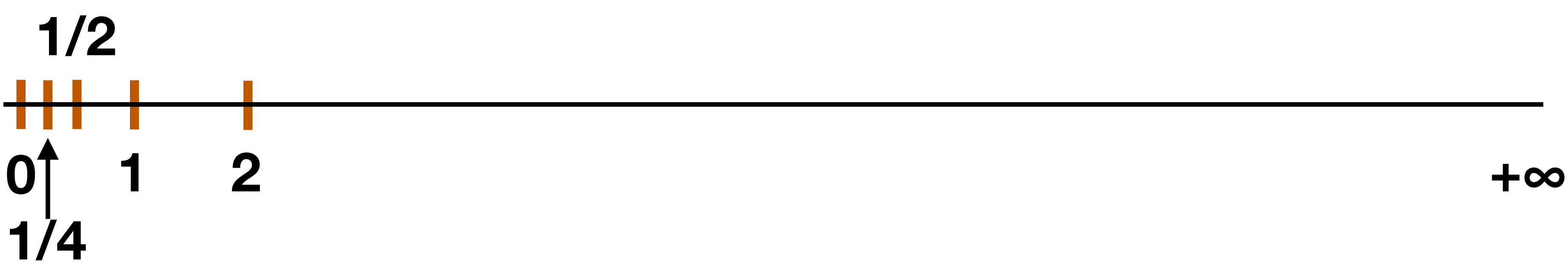


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	100	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

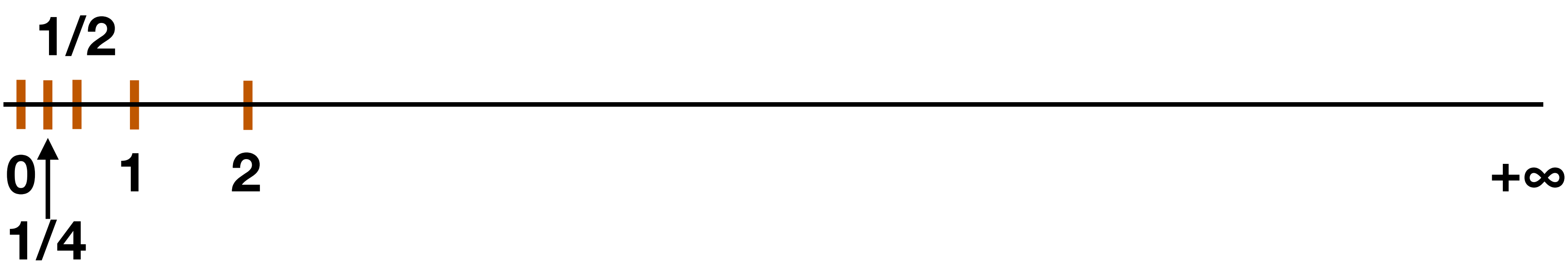


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	101	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

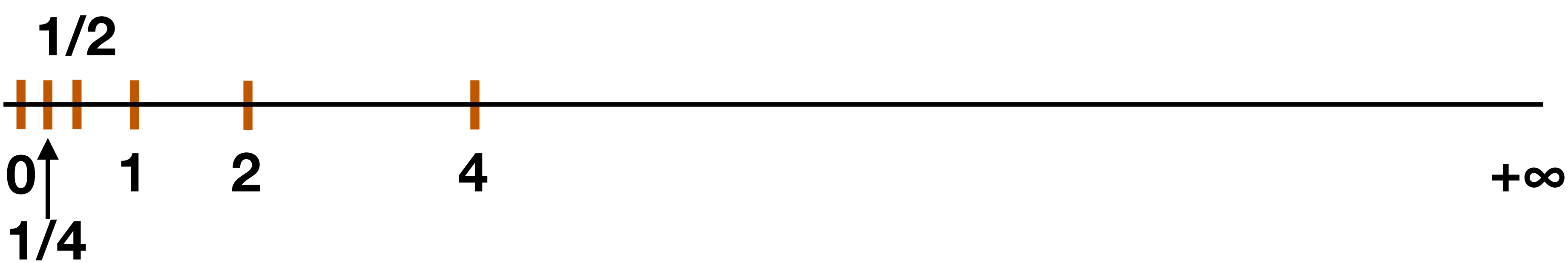


Representable Numbers (Positive Only)

$v = (-1)^s M\, 2^E$

0	101	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

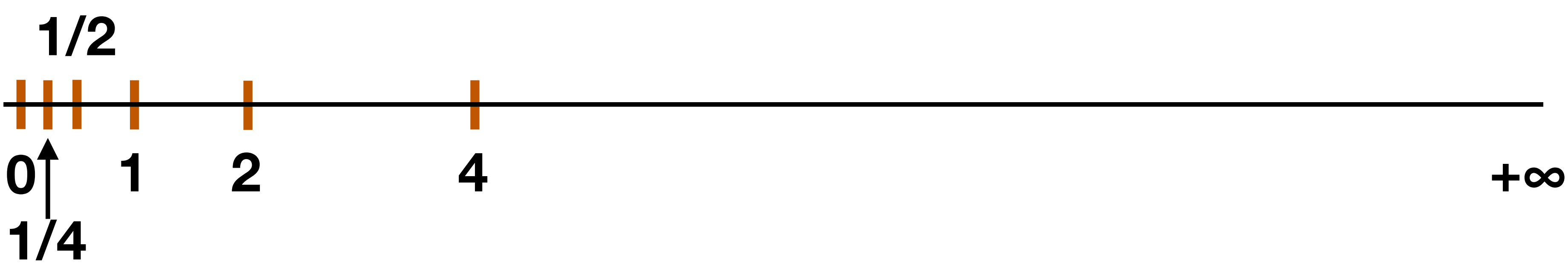


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	110	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

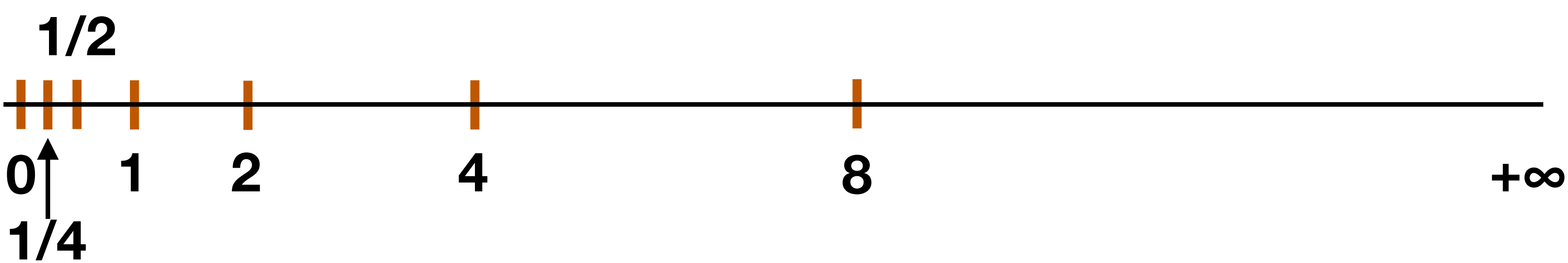


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	110	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

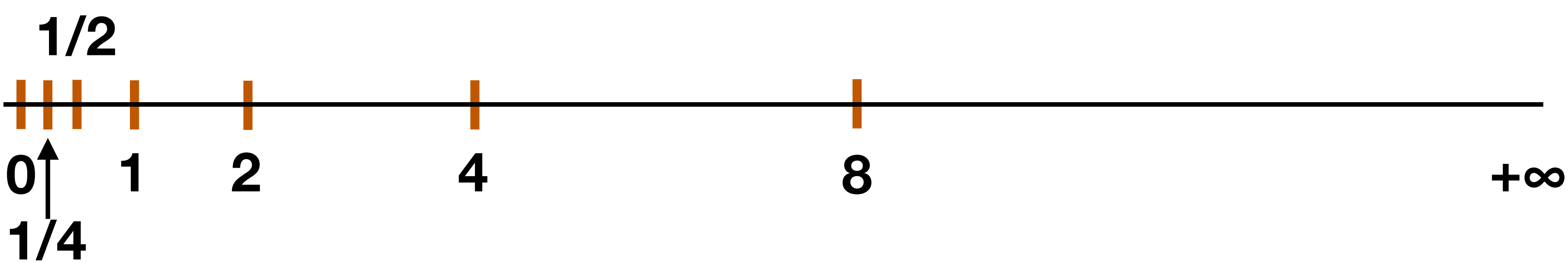


Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	110	01

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~



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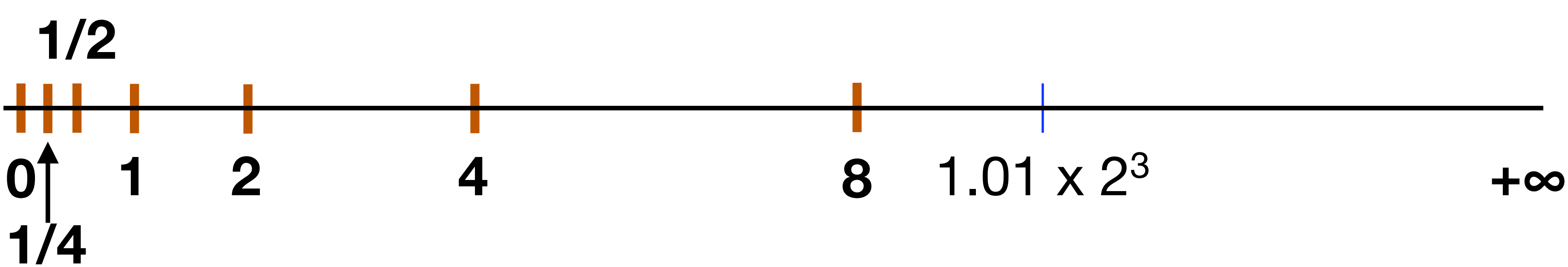
0	110	01

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

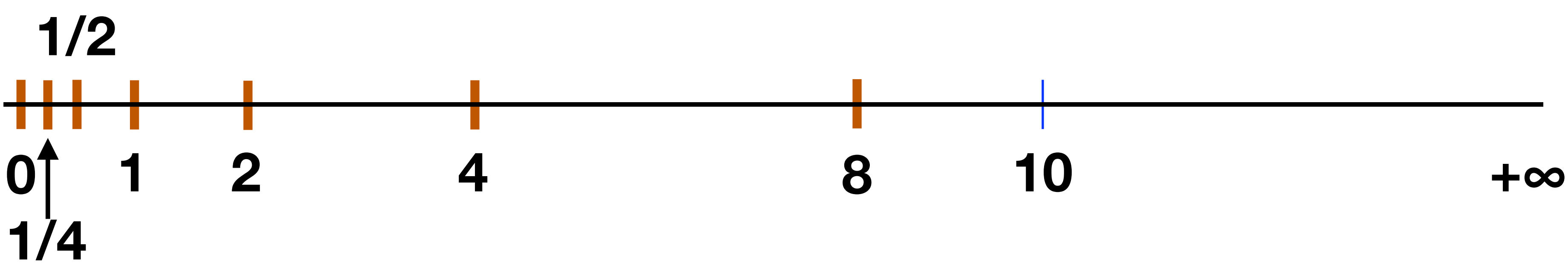


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	110	01

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

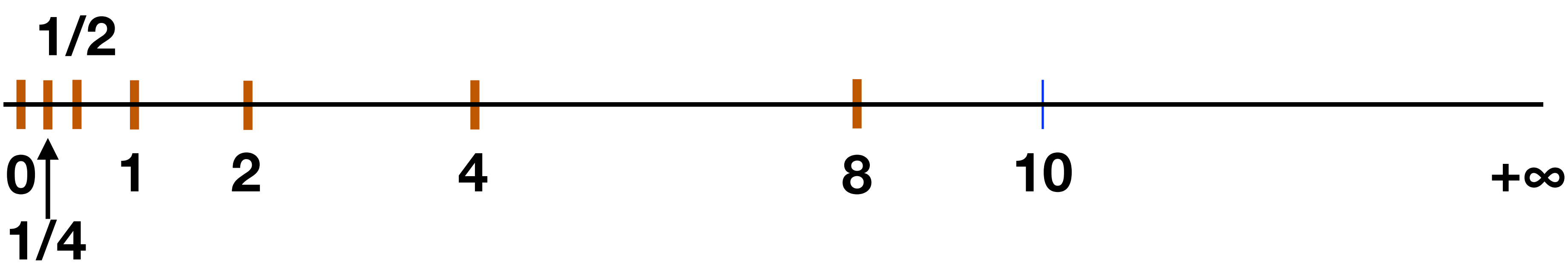


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	110	10

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

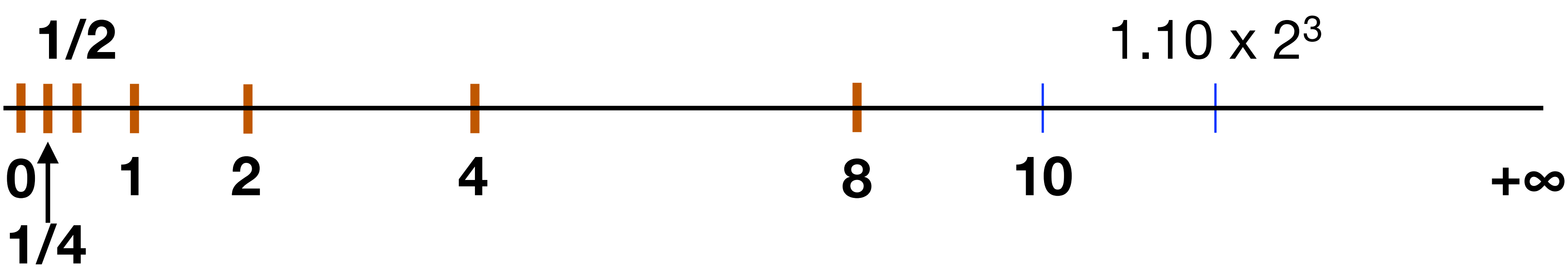


Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	110	10

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

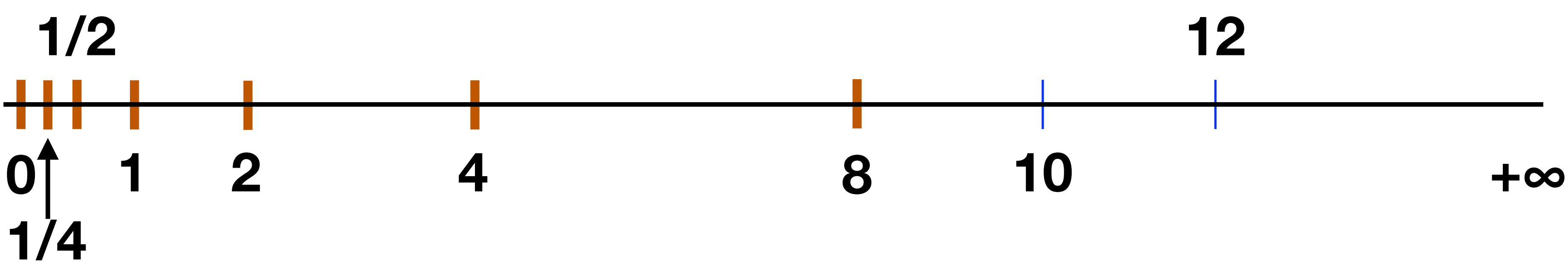


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	110	10

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

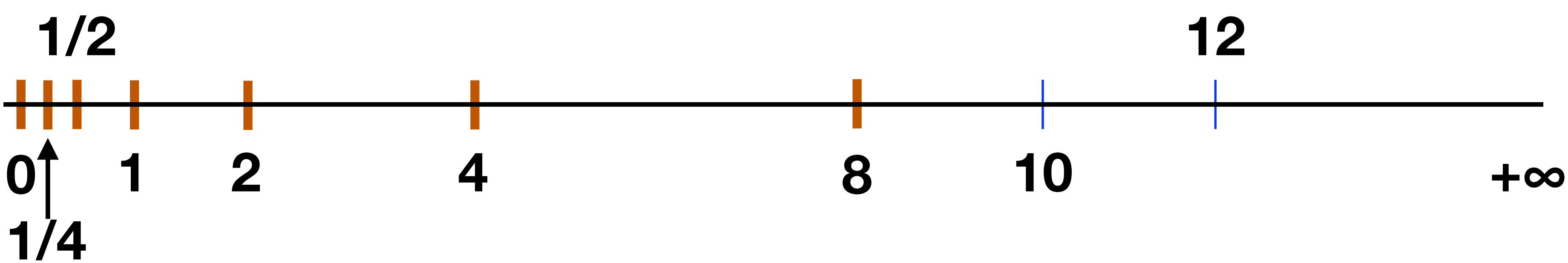


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	110	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

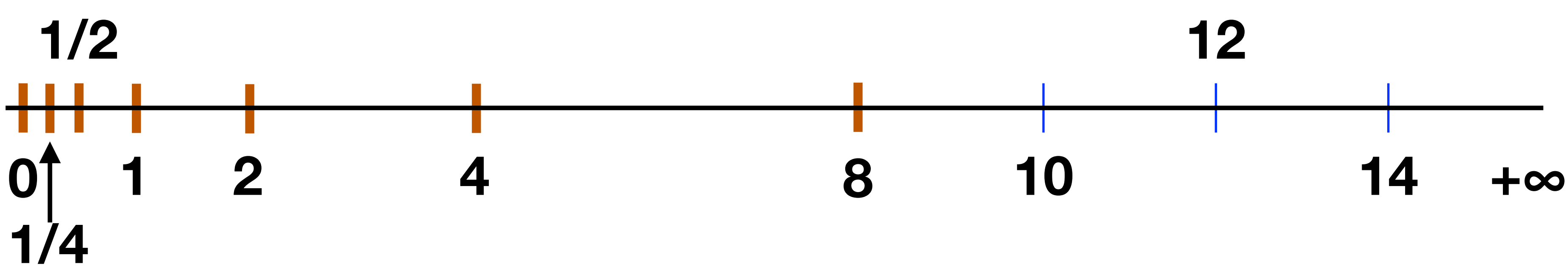


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	110	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

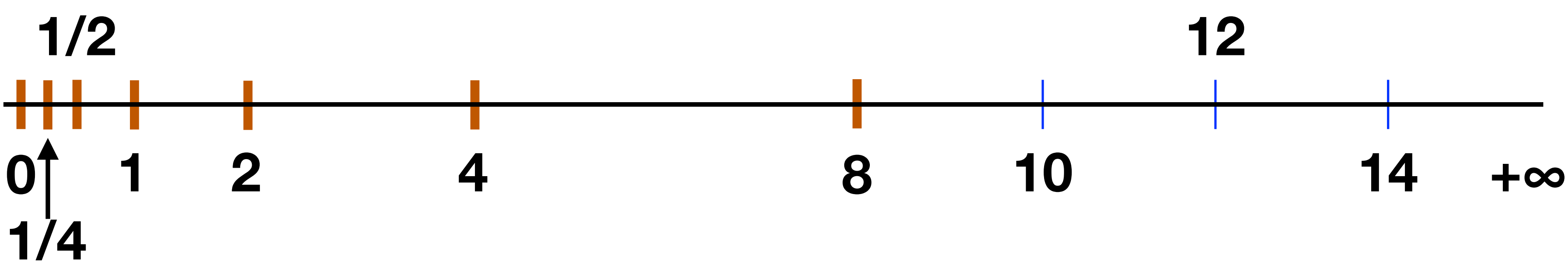


Representable Numbers (Positive Only)

$v = (-1)^s\ M\ 2^E$

0	101	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

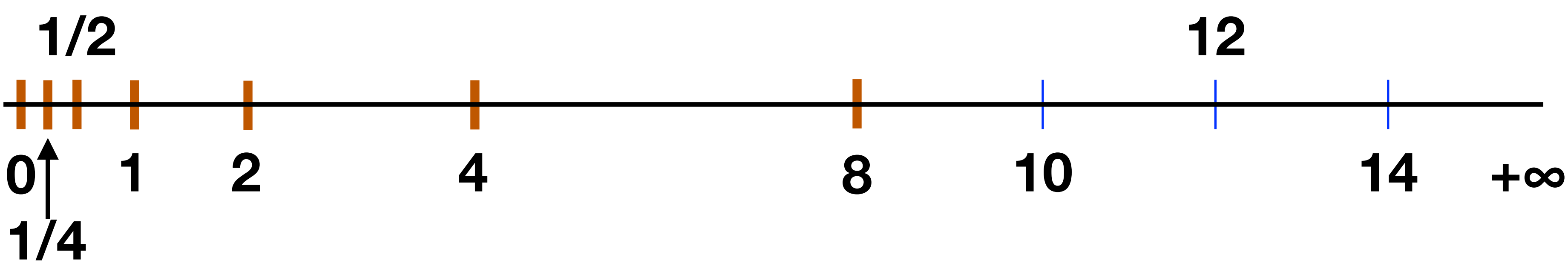


Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	101	01

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

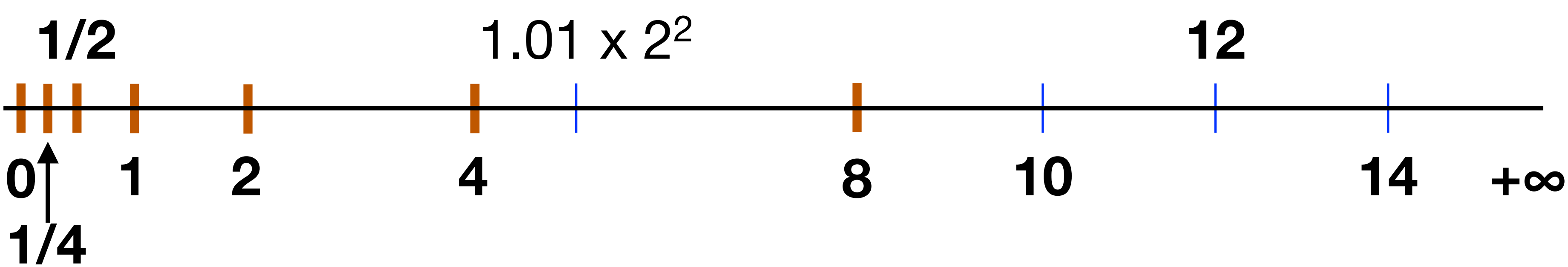


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

s	exp	frac
0	101	01

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

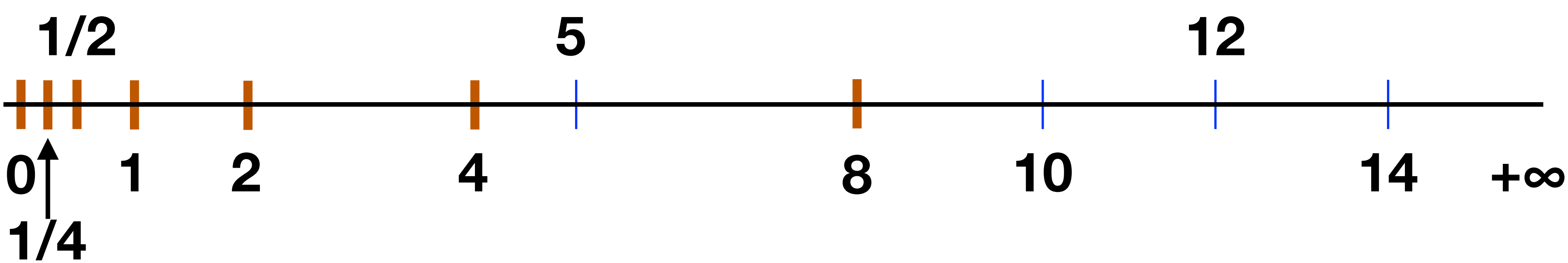


Representable Numbers (Positive Only)

$v = (-1)^s M\ 2^E$

0	101	01

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

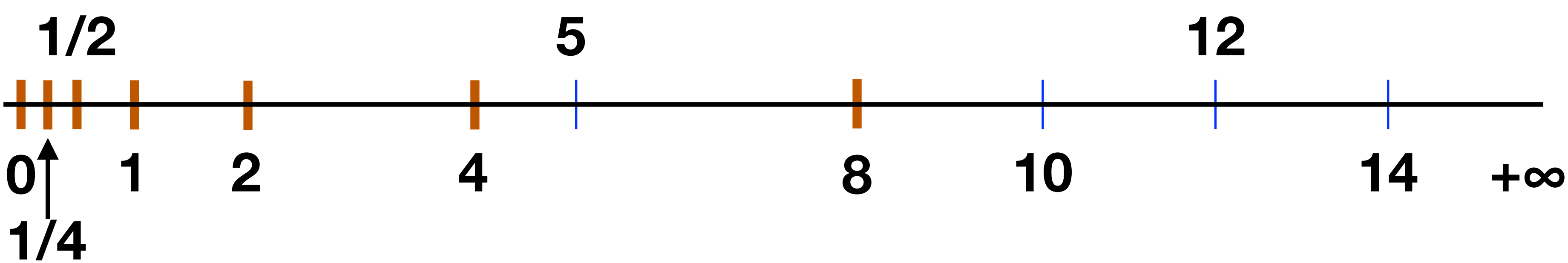


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	101	10

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

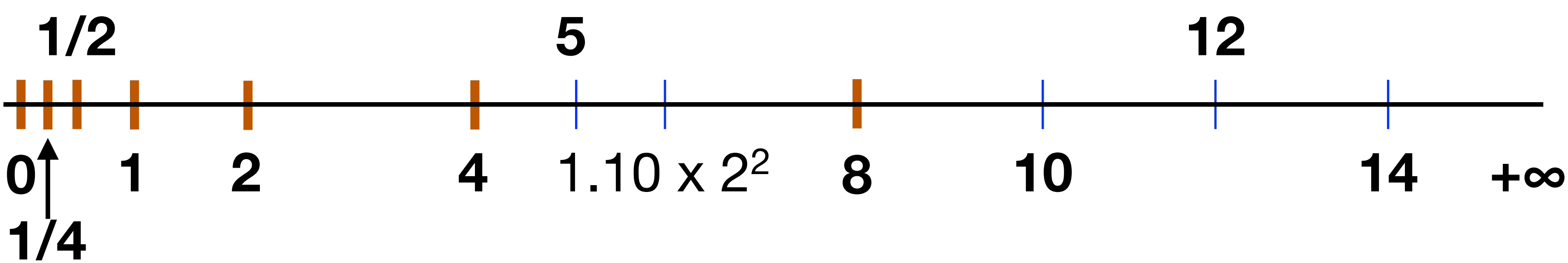


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	101	10

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

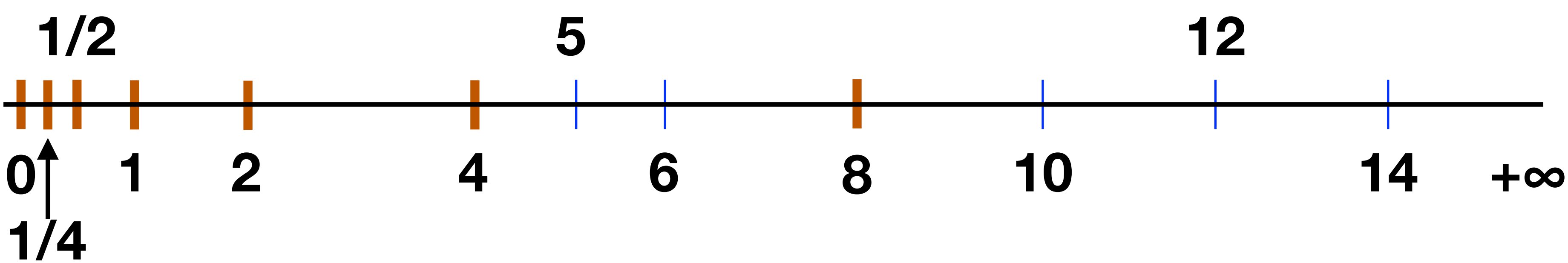


Representable Numbers (Positive Only)

$$v = (-1)^s M 2^E$$

0	101	10

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

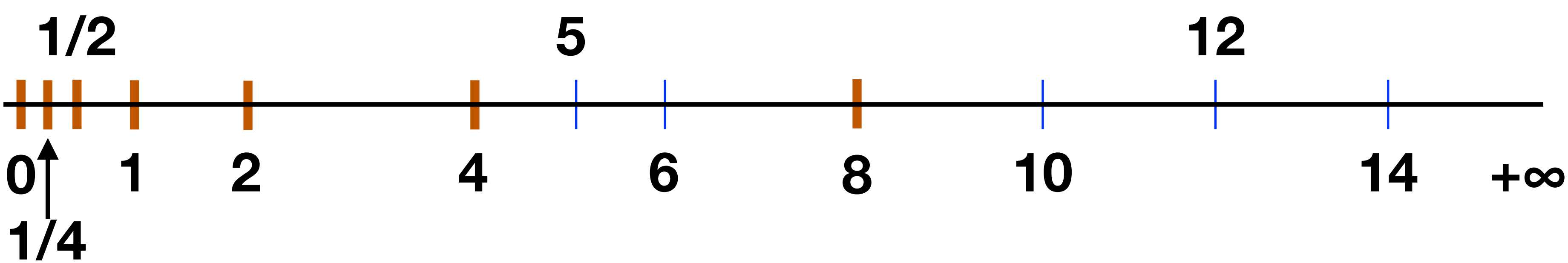


Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	101	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

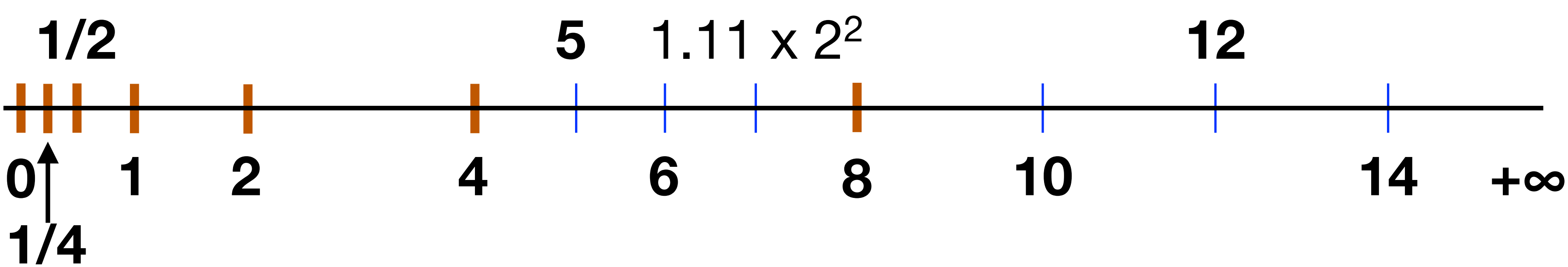


Representable Numbers (Positive Only)

$v = (-1)^s M \, 2^E$

0	101	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

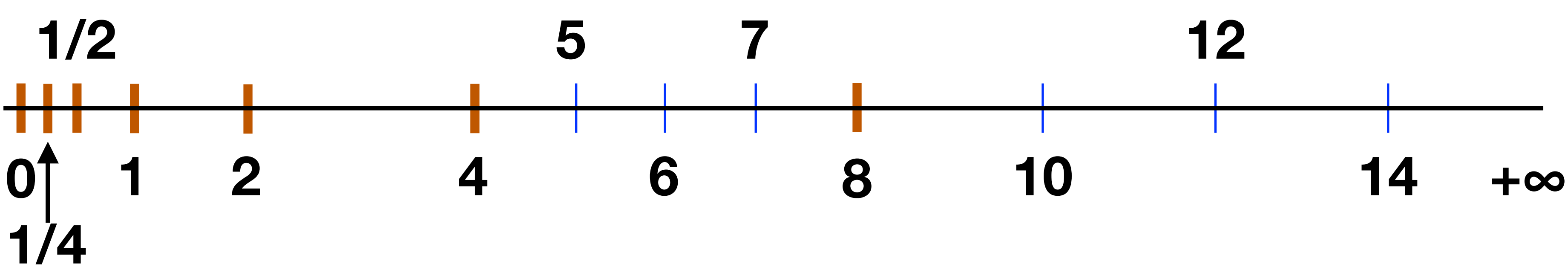


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	101	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

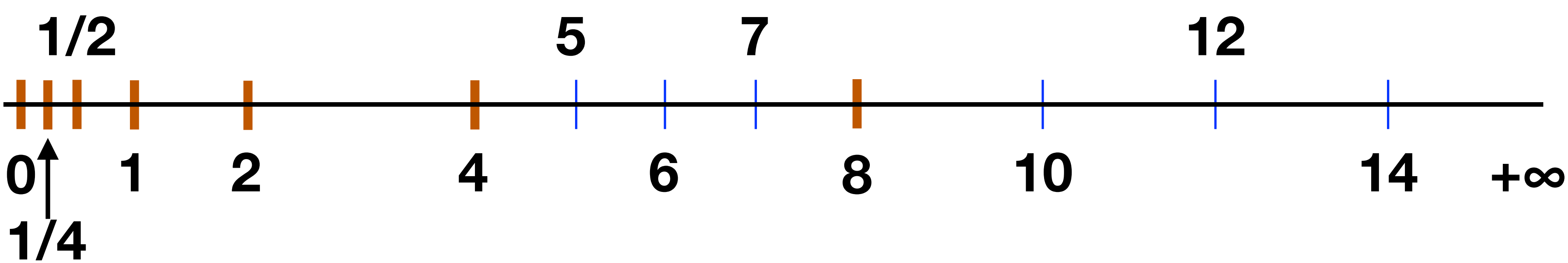


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	100	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

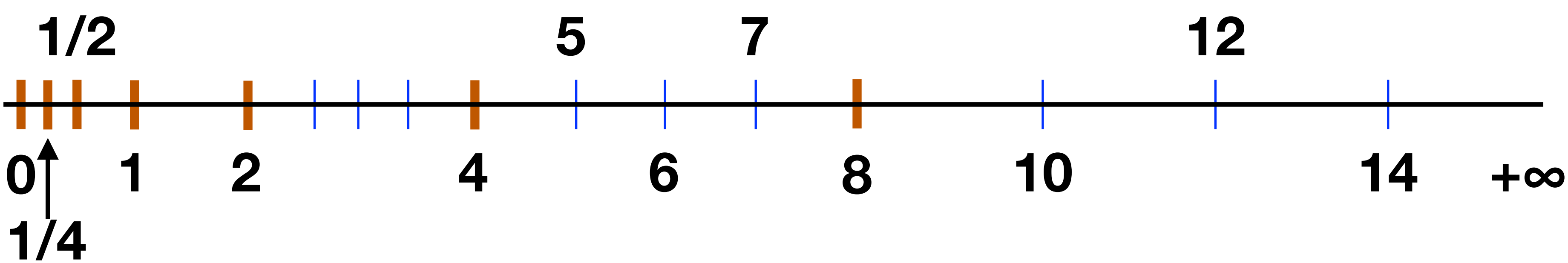


Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	100	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

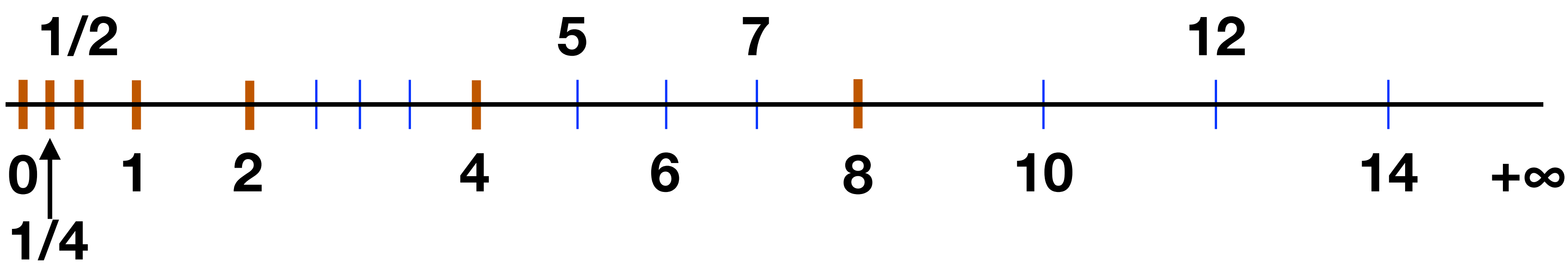


Representable Numbers (Positive Only)

$$v = (-1)^s \, M \, 2^E$$

0	011	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

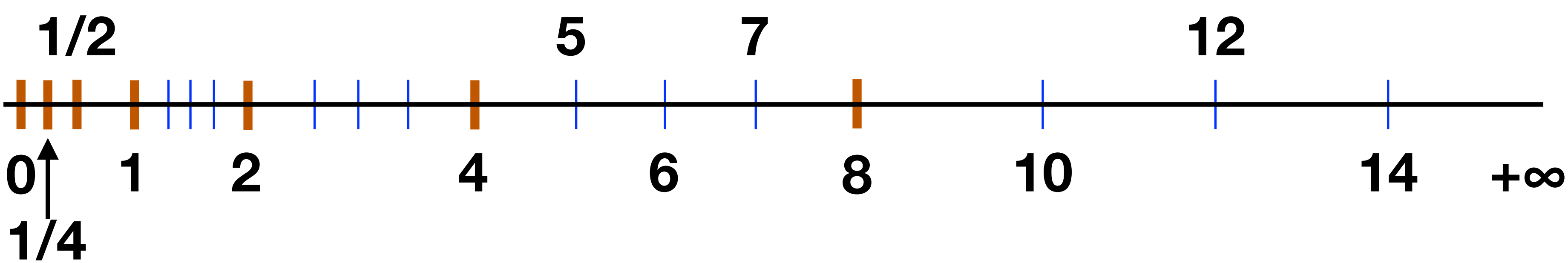


Representable Numbers (Positive Only)

$v = (-1)^s M\, 2^E$

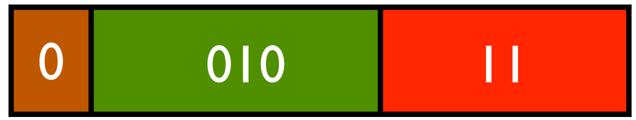
0	011	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

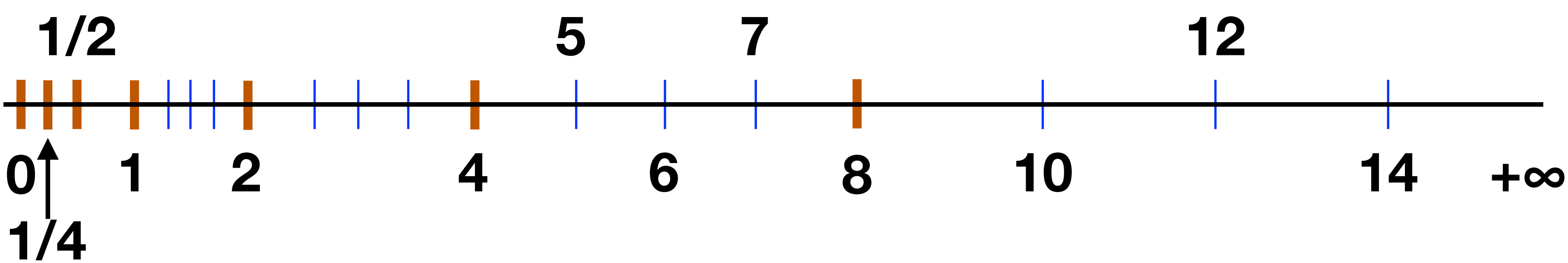


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$



E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

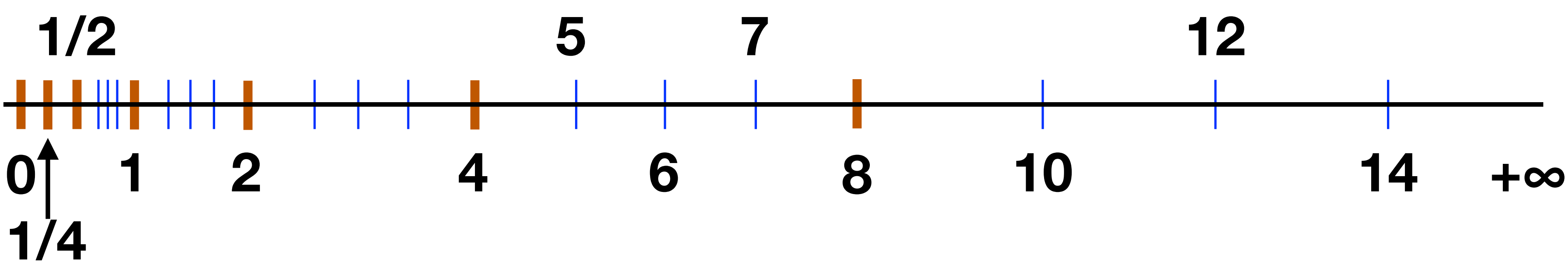


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	010	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

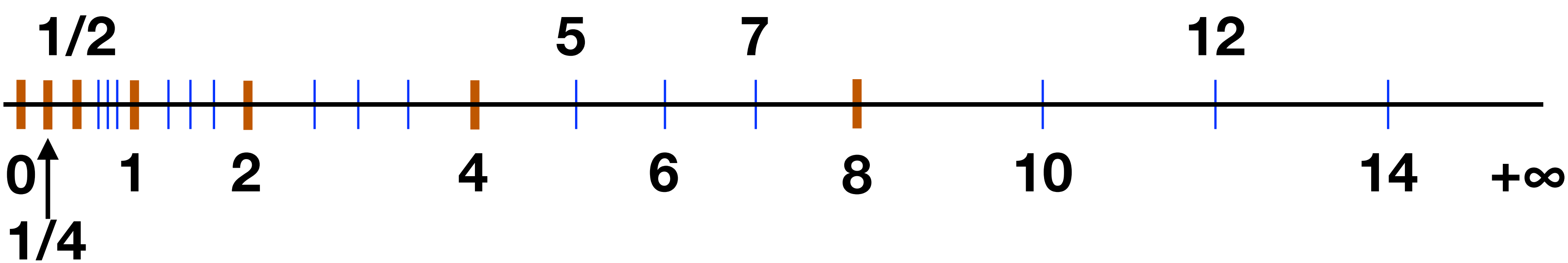


Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	001	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

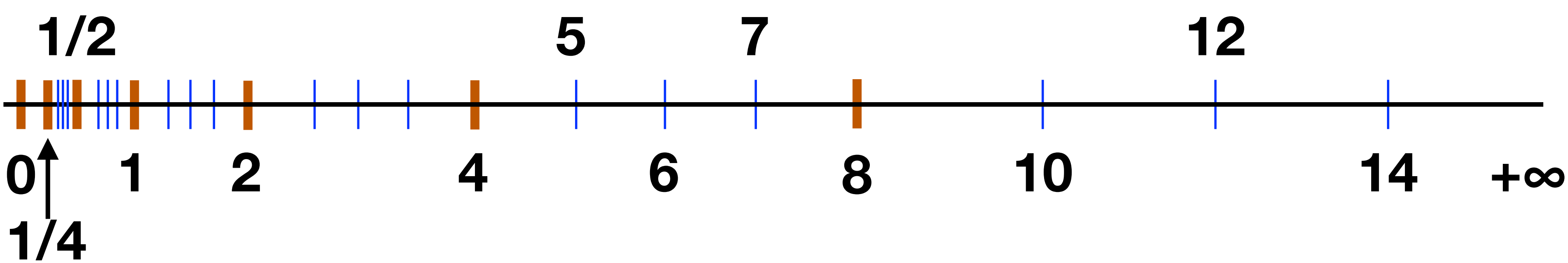


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	001	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~



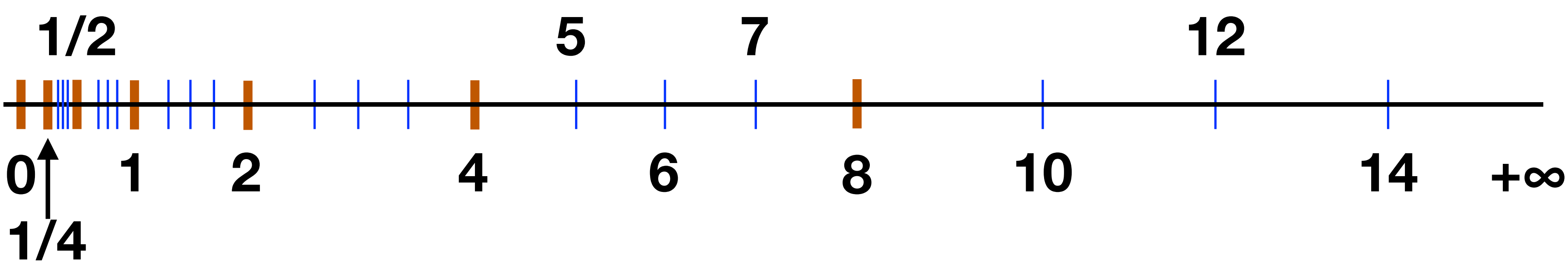
Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	001	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Uneven interval (c.f., fixed interval in fixed-point)
  - More dense toward 0, sparser toward infinite
  - Allow encoding small and large numbers at the same time



Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	001	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

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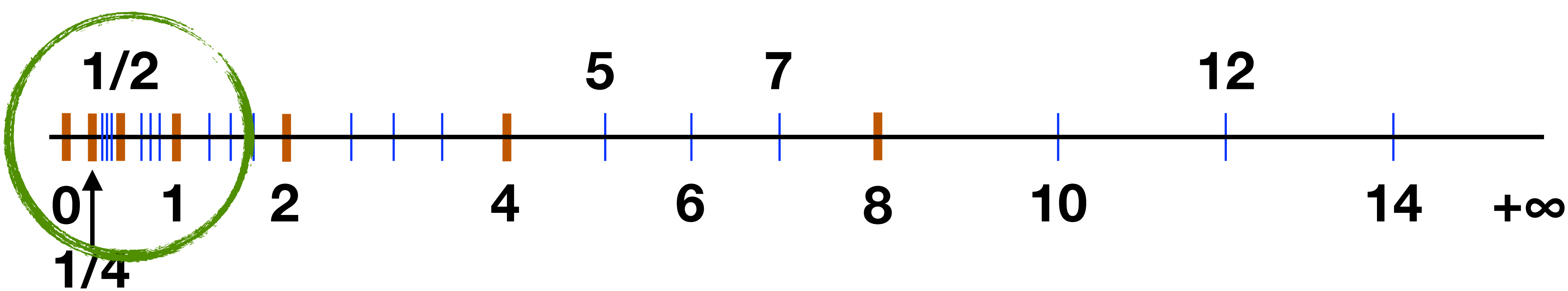
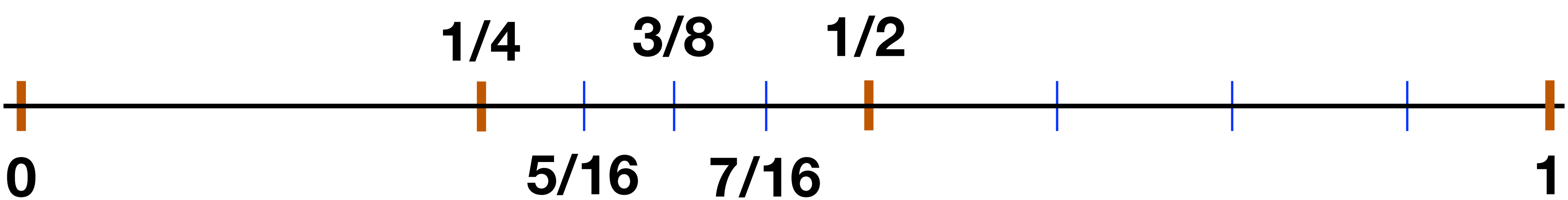


Representable Numbers (Positive Only)

$v = (-1)^s M\, 2^E$

s	exp	frac

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

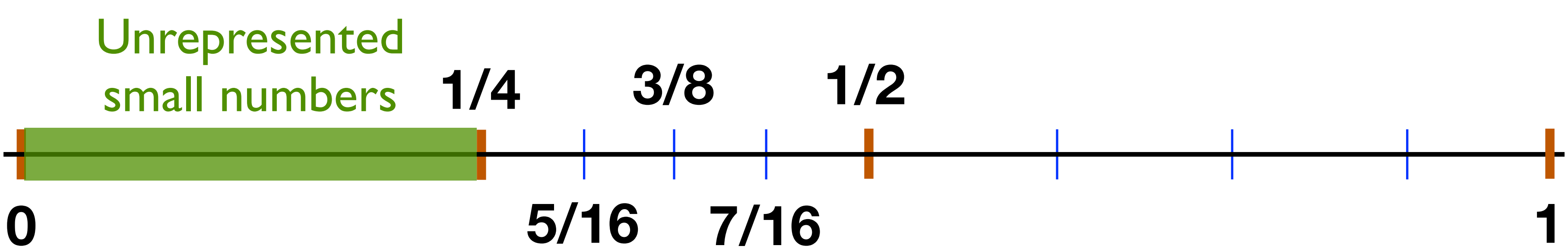


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

s	exp	frac

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

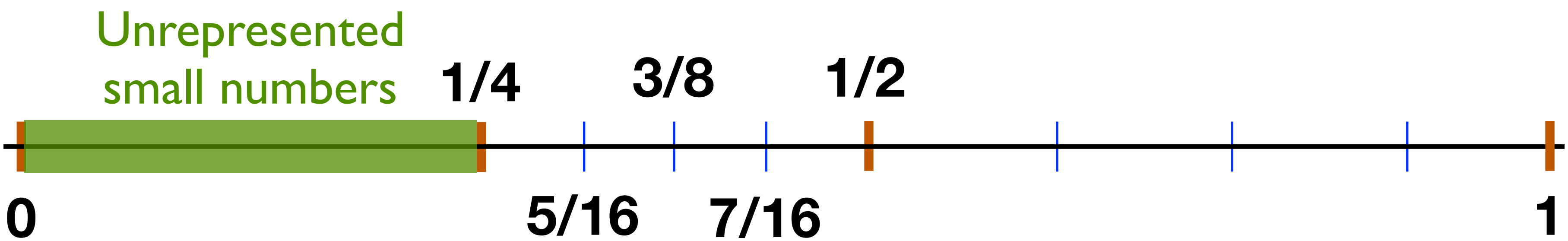


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Underflow: always round to 0 is inelegant



Representable Numbers (Positive Only)

$$v = (-1)^s \, M \, 2^E$$

s	exp	frac

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

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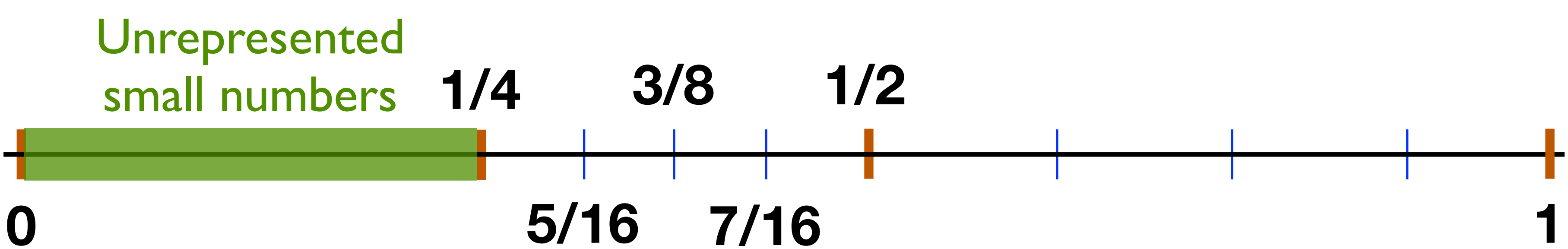

Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

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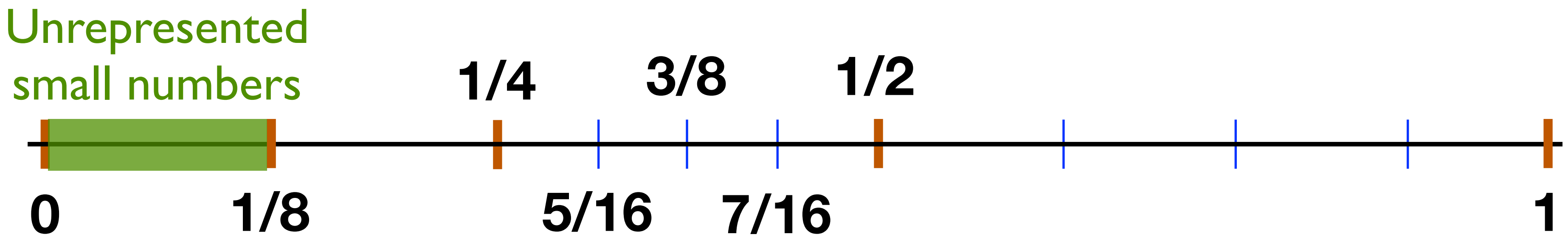


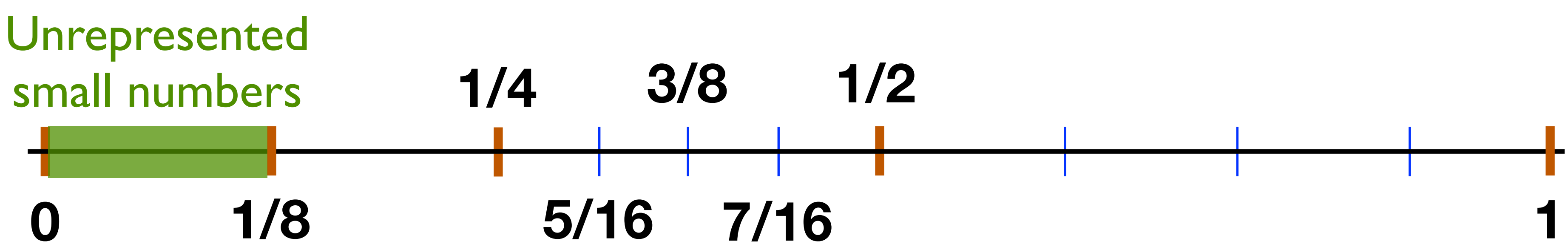
Representable Numbers (Positive Only)

$$v = (-1)^s M\, 2^E$$

s	exp	frac

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Underflow: always round to 0 is inelegant
- Using 000 for *exp* would only postpone the problem rather than solving it

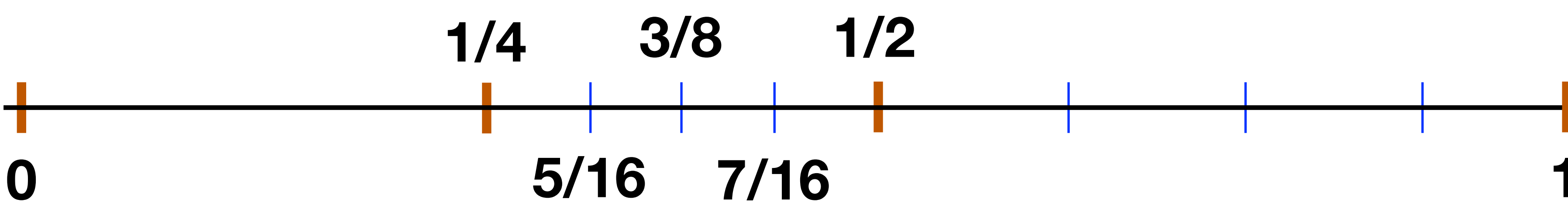


Subnormal (De-normalized) Numbers

$v = (-1)^s M 2^E$

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Idea: Evenly divide between 0 and 1/4 rather than exponentially decreasing **when *exp* = 0** (subnormal numbers)



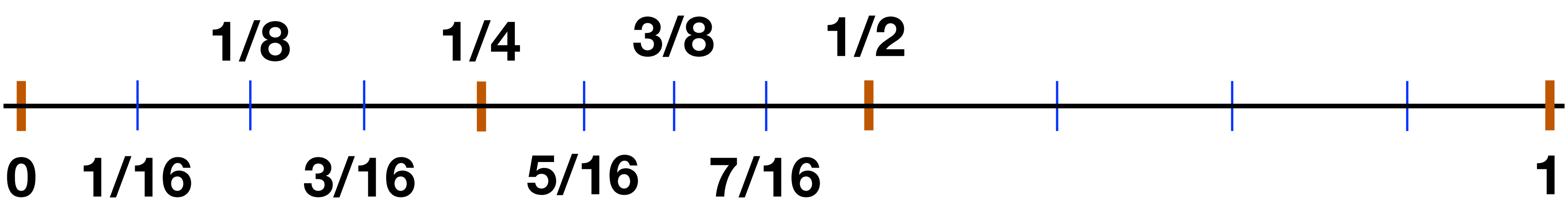
Subnormal (De-normalized) Numbers

$$v = (-1)^s \, M \, 2^E$$

s	exp	frac

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Idea: Evenly divide between 0 and 1/4 rather than exponentially decreasing **when *exp* = 0** (subnormal numbers)



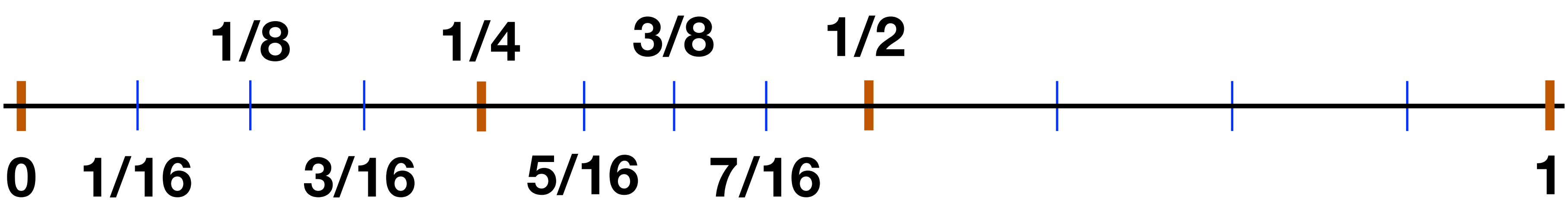
Subnormal (De-normalized) Numbers

$v = (-1)^s M \, 2^E$

s	exp	frac

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Idea: Evenly divide between 0 and 1/4 rather than exponentially decreasing **when *exp* = 0** (subnormal numbers)
- $E = (exp + 1) - \text{bias}$ (instead of *exp* - *bias*)
- $M = 0.frac$ (instead of 1.frac)



Subnormal (De-normalized) Numbers

$v = (-1)^s \, M \, 2^E$

s	exp	frac

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Idea: Evenly divide between 0 and 1/4 rather than exponentially decreasing **when *exp* = 0** (subnormal numbers)
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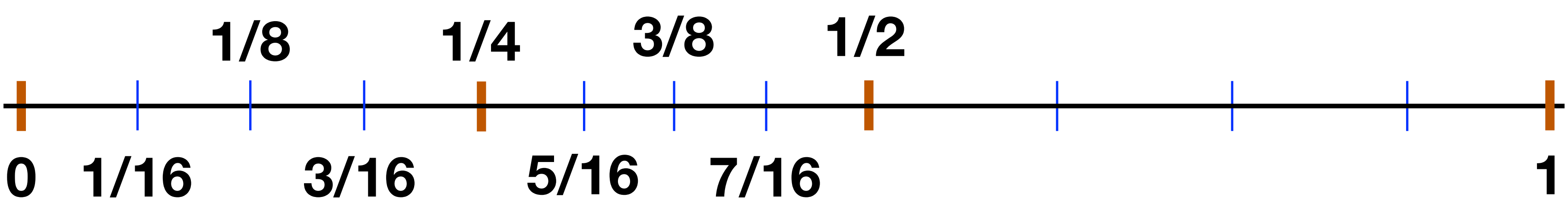


0	000	01

$= (-1)^0 \, 0.01 \times 2^{(0+1-3)} = 1/16$

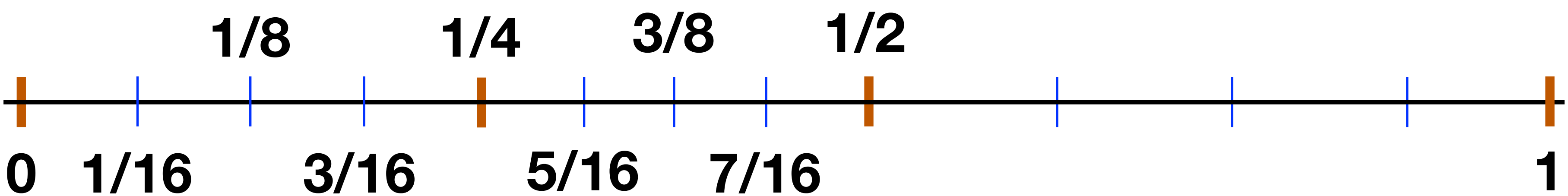
Subnormal (De-normalized) Numbers

$v = (-1)^s\, M\, 2^E$

s	exp	frac

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Idea: Evenly divide between 0 and 1/4 rather than exponentially decreasing **when *exp* = 0** (subnormal numbers)
- $E = (exp + 1) - \text{bias}$ (instead of *exp* - *bias*)
- $M = 0.frac$ (instead of 1.frac)
- Subnormal numbers allow graceful underflow



0	000	01

$= (-1)^0\, 0.01 \times 2^{(0+1-3)} = 1/16$

Special Values

$$v = (-1)^s M \, 2^E$$

E	exp	E	exp
-2	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

Special Values

$$v = (-1)^s \, M \, 2^E$$

s	exp	frac

E	exp	E	exp
-2	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- There are many special values in scientific computing
  - +/- ∞, NaNs (0 / 0, 0 / ∞, ∞ / ∞, …), etc.

Special Values

$v = (-1)^s \, M \, 2^E$

E	exp	E	exp
-2	000	1	100
-2	001	2	101
-1	010	3	110
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  - +/- ∞, NaNs (0 / 0, 0 / ∞, ∞ / ∞, …), etc.
- exp = 111 is reserved to represent these numbers

Special Values

$$v = (-1)^s M 2^E$$

E	exp	E	exp
-2	000	1	100
-2	001	2	101
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0	011		111

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  - +/- ∞, NaNs (0 / 0, 0 / ∞, ∞ / ∞, …), etc.
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Special Values

$v = (-1)^s M \, 2^E$

s	exp	frac

E	exp	E	exp
-2	000	1	100
-2	001	2	101
-1	010	3	110
0	011		111

- There are many special values in scientific computing
  - +/- ∞, NaNs (0 / 0, 0 / ∞, ∞ / ∞, …), etc.
- exp = 111 is reserved to represent these numbers
- exp = 111, frac = 000
  - +/- ∞ (depending on the *s* bit). Overflow results.
  - Arithmetic on ∞ is exact: 1.0/0.0 = –1.0/–0.0 = + ∞, 1.0/–0.0 = -∞

Special Values

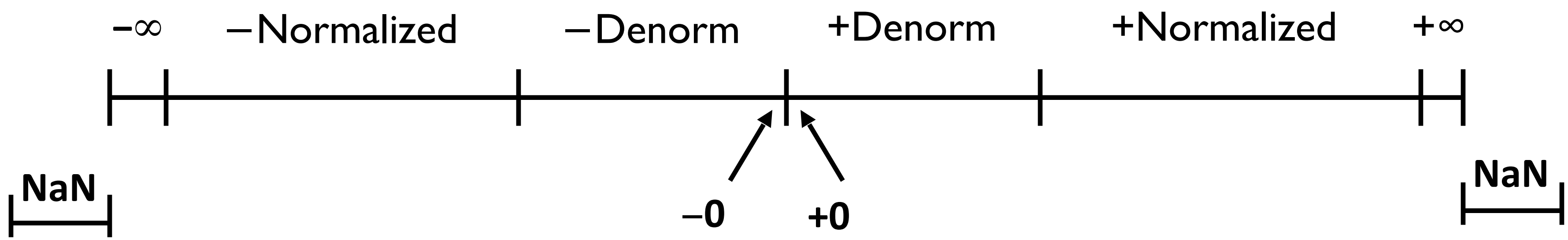
$$v = (-1)^s\, M\, 2^E$$

s	exp	frac

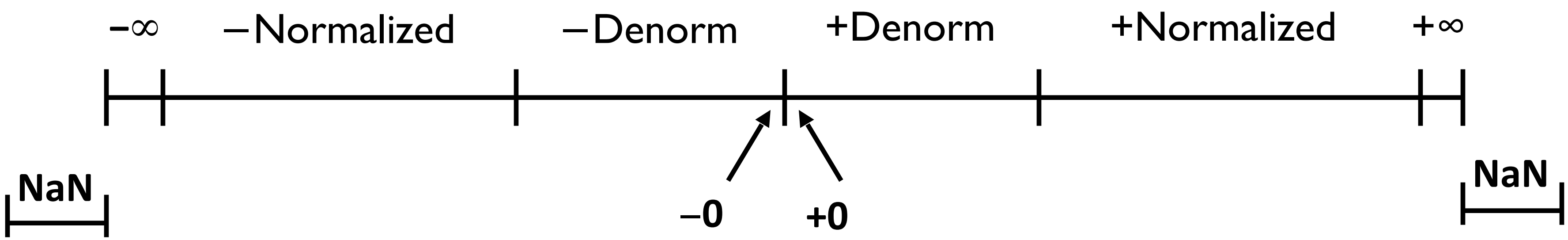
E	exp	E	exp
-2	000	1	100
-2	001	2	101
-1	010	3	110
0	011		111

- There are many special values in scientific computing
  - +/- ∞, NaNs (0 / 0, 0 / ∞, ∞ / ∞, …), etc.
- exp = 111 is reserved to represent these numbers
- exp = 111, frac = 000
  - +/- ∞ (depending on the *s* bit). Overflow results.
  - Arithmetic on ∞ is exact: 1.0/0.0 = –1.0/–0.0 = + ∞, 1.0/–0.0 = -∞
- exp = 111, frac != 000
  - Represent Not-a-Numbers (e.g., sqrt(–1), ∞ - ∞, ∞ x 0)

Visualization: Floating Point Encodings

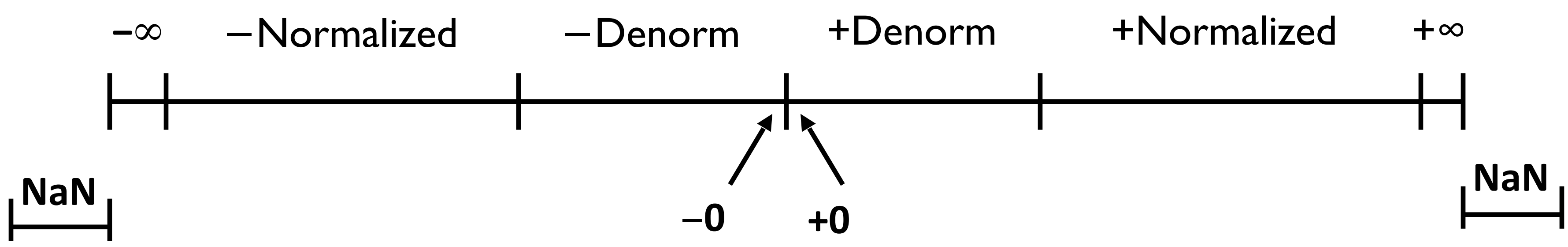


Visualization: Floating Point Encodings

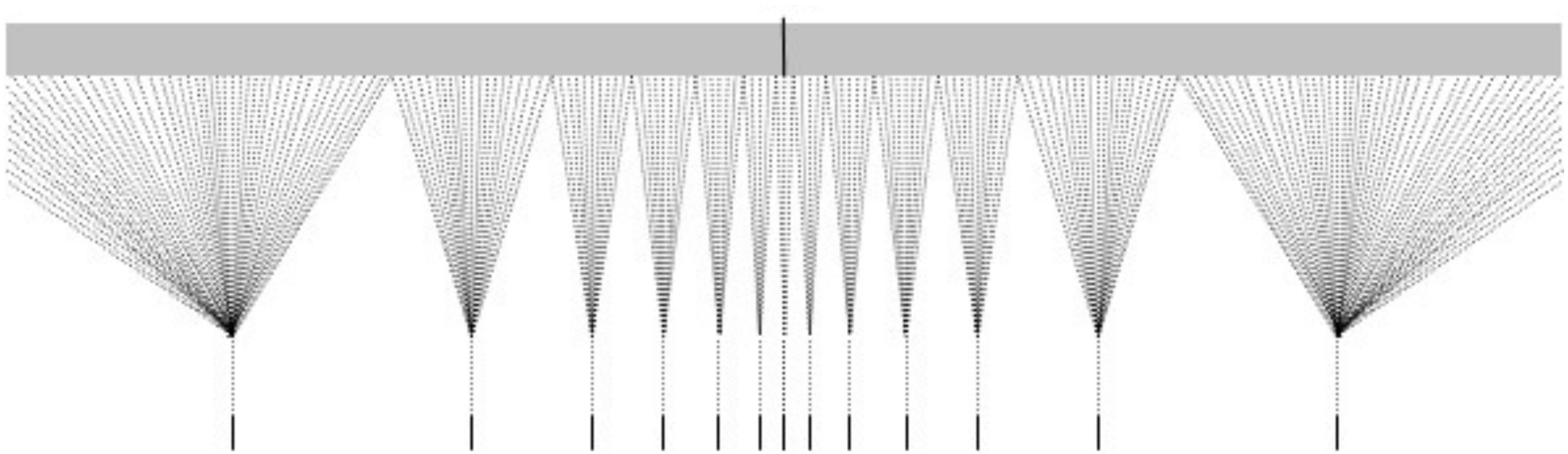


Infinite Amount of Real Numbers

Visualization: Floating Point Encodings

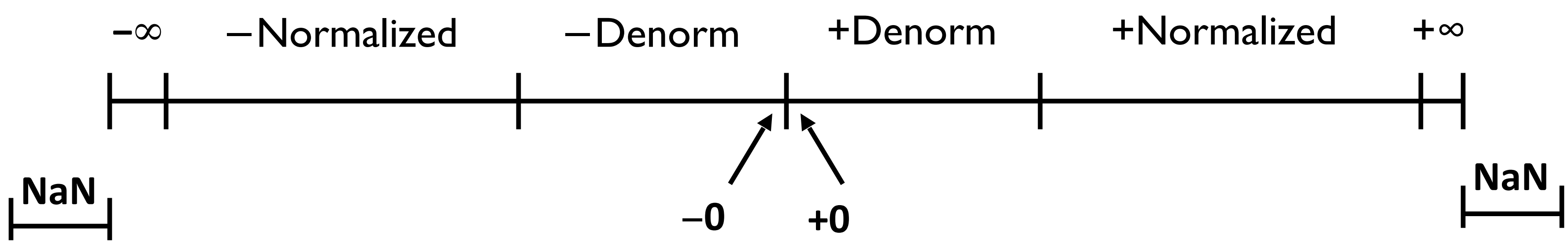


Infinite Amount of Real Numbers

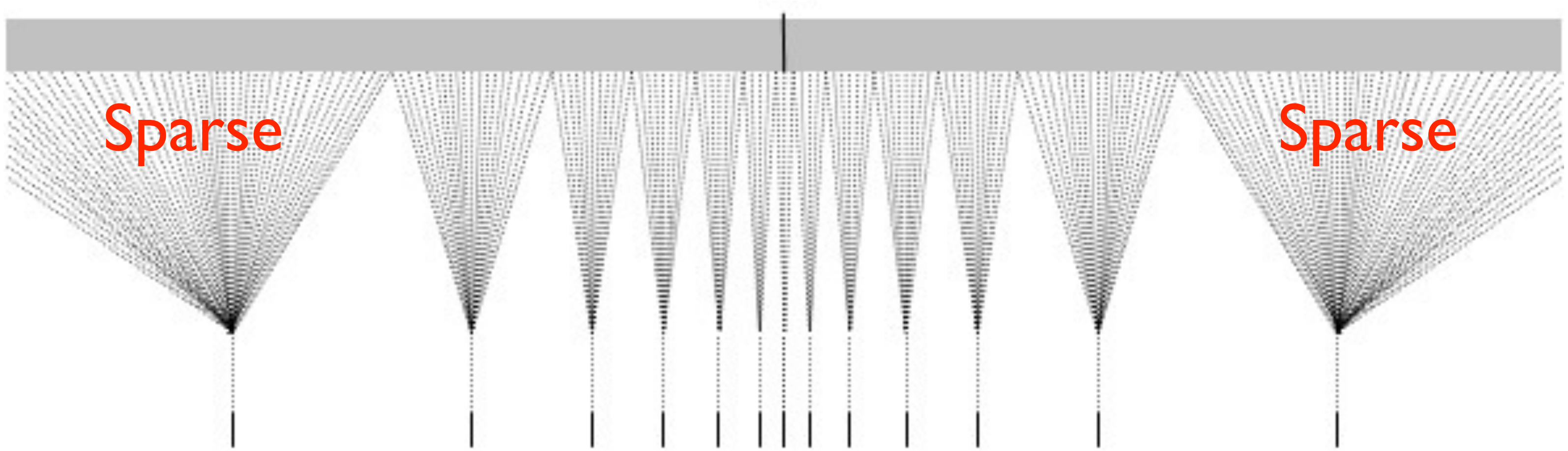


Finite Amount of Floating Point Numbers

Visualization: Floating Point Encodings

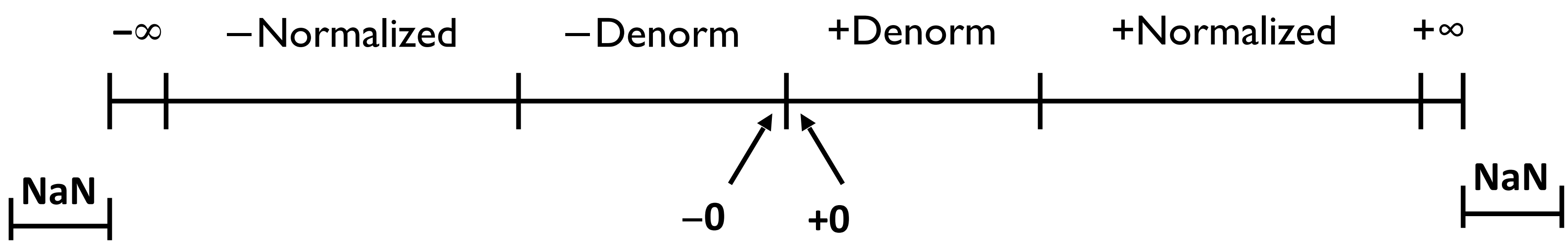


Infinite Amount of Real Numbers

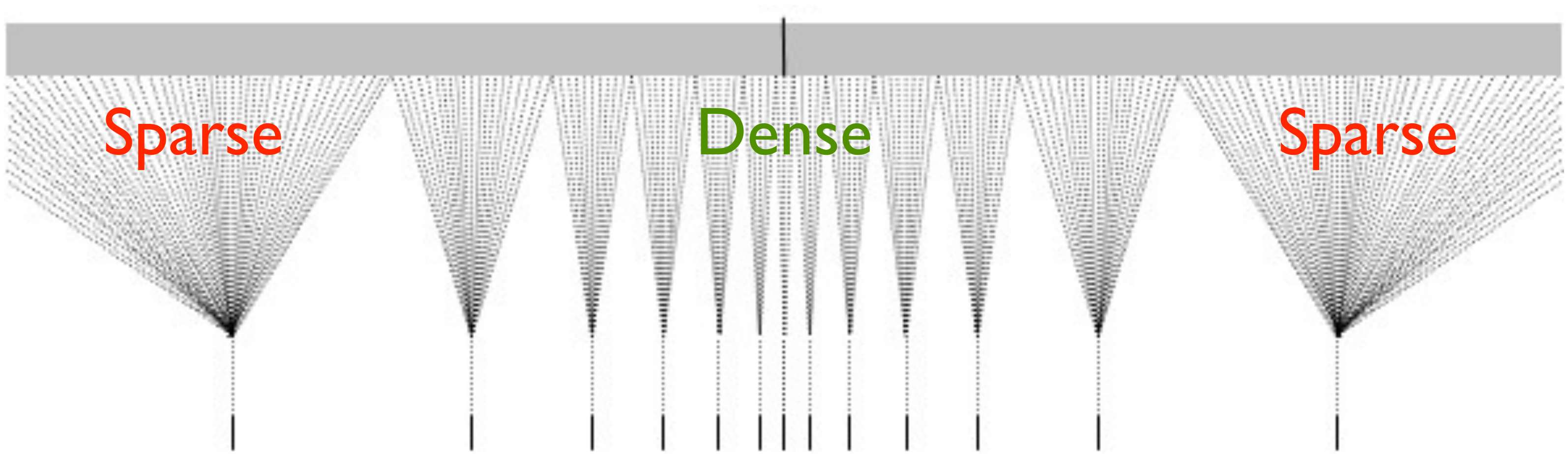


Finite Amount of Floating Point Numbers

Visualization: Floating Point Encodings

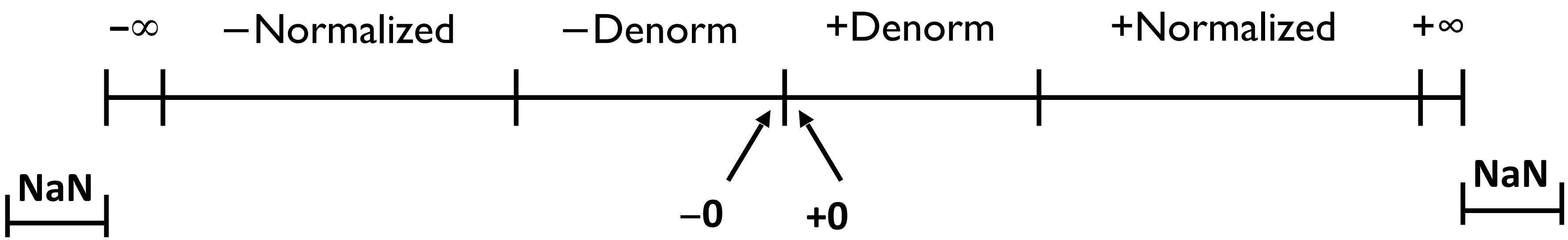


Infinite Amount of Real Numbers

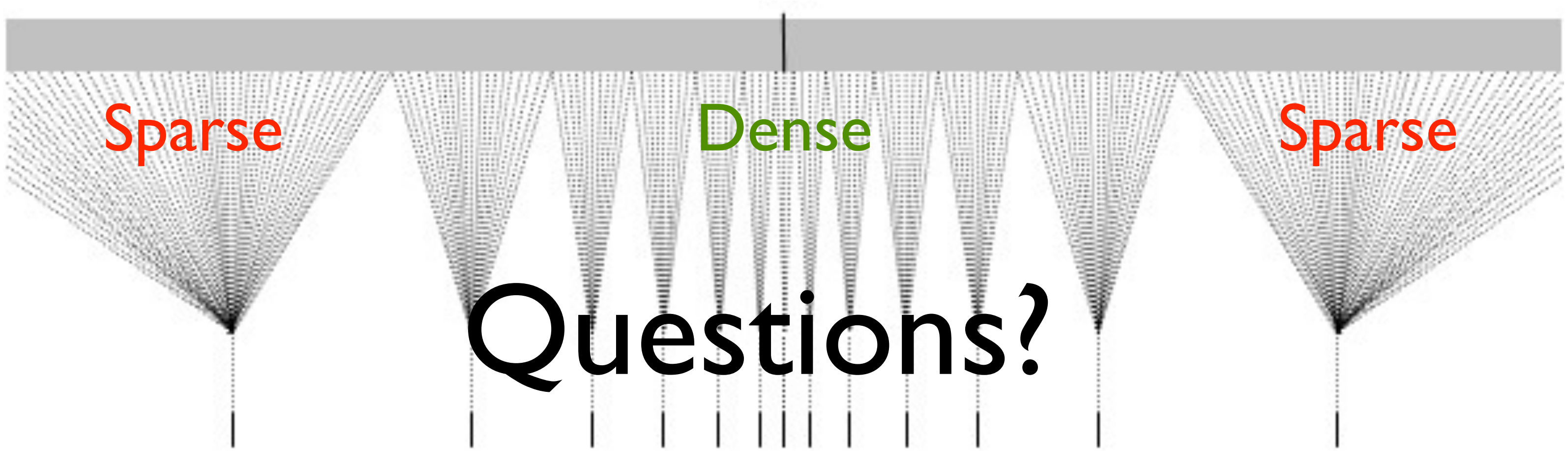


Finite Amount of Floating Point Numbers

Visualization: Floating Point Encodings



Infinite Amount of Real Numbers



Questions?

Finite Amount of Floating Point Numbers

Today: Floating Point

- Background: Fractional binary numbers and fixed-point
- Floating point representation
- IEEE 754 standard
- Rounding, addition, multiplication
- Floating point in C
- Summary

IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs (and even GPUs and other processors)

- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

IEEE 754 Standard Precision Options

- Single precision: 32 bits

- Double precision: 64 bits

Single Precision (32-bit) Example

$v = (-1)^s \, M \, 2^E$

bias = $2^{(8-1)}-1 = 127$

s	exp	frac
1	8-bit	23-bit

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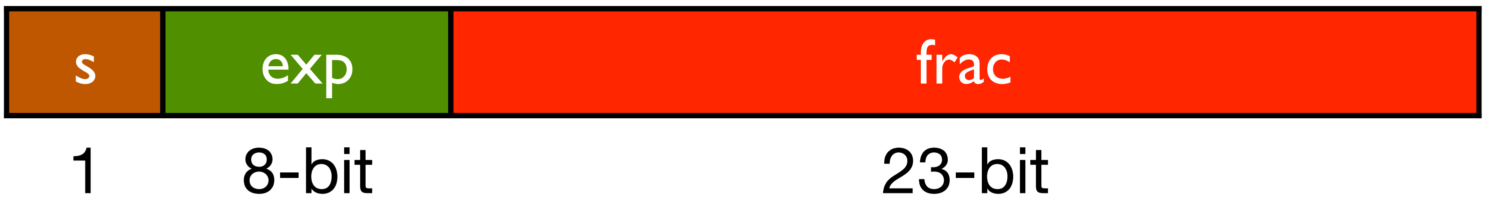

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Today: Floating Point

- Background: Fractional binary numbers and fixed-point
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Floating Point Operations: Basic Idea

- Basic idea
  - We perform the operation & produce the infinitely precise result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into frac

- $x +_f y = \mathrm{Round}(x + y)$

- $x \times_f y = \mathrm{Round}(x \times y)$

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Rounding Mode	**1.40**	**1.60**	**1.50**	**2.50**	**-1.50**
Towards zero	1	1	1	2	-1
Round down (-∞)	1	1	1	2	-2
Round up (+∞)	2	2	2	3	-1
Nearest even (default)	1	2	2	2	-2

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1.000110	1.001	1.001 is the nearest (up)
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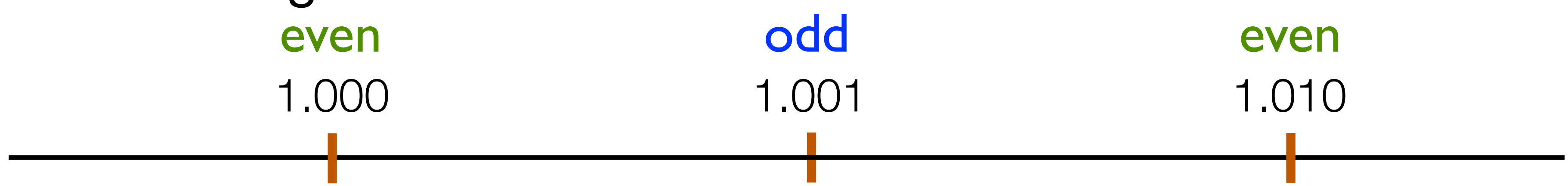

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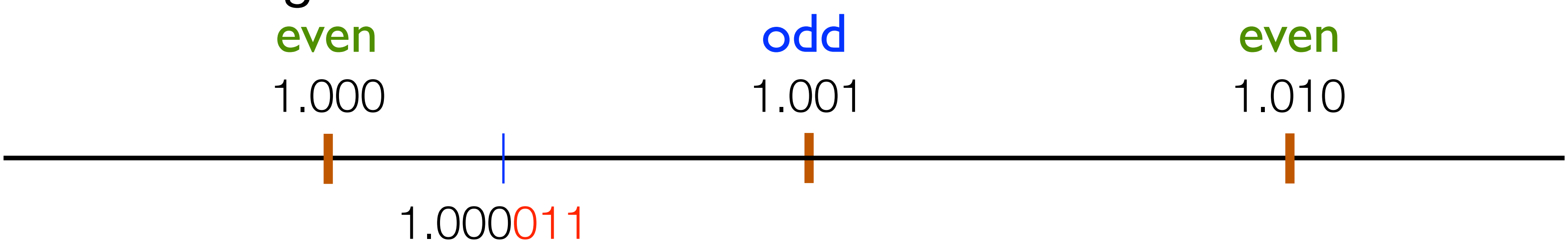

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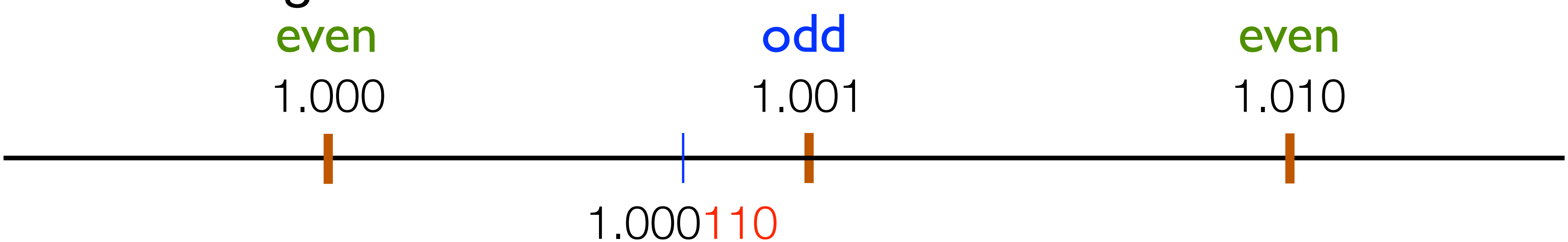

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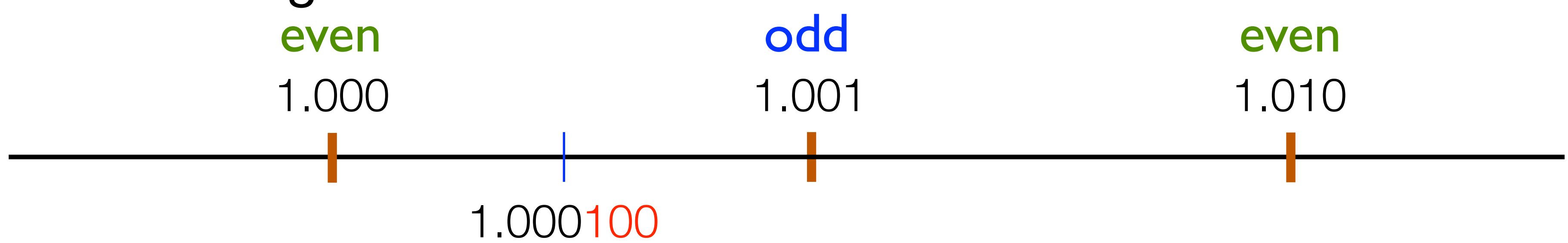

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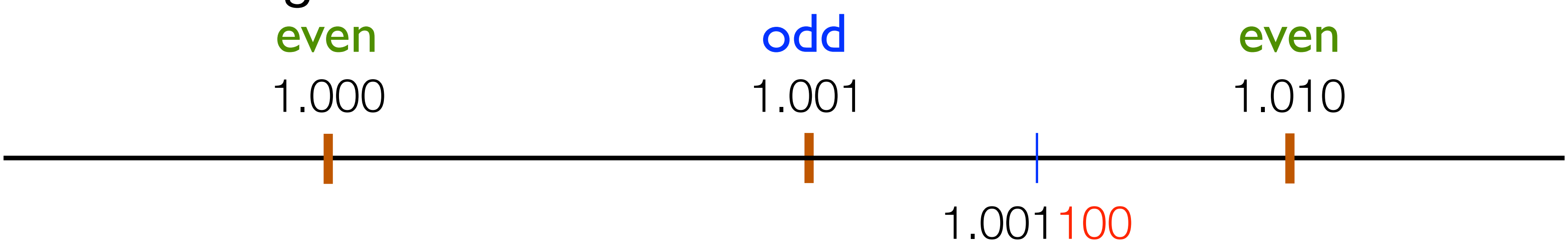

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    - Assume E1 > E2
- Fixing
  - If $M \geq 2$, shift M right, increment E
  - If $M < 1$, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit *frac* precision

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  - Significand *M*: $M1 \text{ x } M2$
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- Implementation
  - Biggest chore is multiplying significands

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Floating Point in C

64-bit Machine

	C Data Type	Bits	Max Value	Max Value (Decimal)
Fixed point (implicit binary point)	`char`	8	$2^7 - 1$	127
	`short`	16	$2^{15} - 1$	32767
	`int`	32	$2^{31} - 1$	2147483647
	`long`	64	$2^{31} - 1$	$\sim 9.2 \times 10^{18}$
SP floating point	`float`	32	$(2 - 2^{-23}) \times 2^{127}$	$\sim 3.4 \times 10^{38}$
DP floating point	`double`	64	$(2 - 2^{-52}) \times 2^{1023}$	$\sim 1.8 \times 10^{308}$

Floating Point in C

- C Guarantees Two Levels
  - `float` single precision
  - `double` double precision
- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`**/**`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion, as long as int has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode