CSC 252: Computer Organization
Spring 2022: Lecture 4

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Department of Computer Science
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Announcement

- Programming Assignment 1 is out
  - Details: https://www.cs.rochester.edu/courses/252/spring2022/labs/assignment1.html
  - Due on Jan. 28, 11:59 PM
  - You have 3 slip days

16	17	18	19	20	21	22
23	24	25 **Today**	26	27	28 **Due**	29

Announcement

- Programming assignment 1 is in C language. Seek help from TAs.
- TAs are best positioned to answer your questions about programming assignments!!!
- Programming assignments do NOT repeat the lecture materials. They ask you to synthesize what you have learned from the lectures and work out something new.

Multiplication

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- Goal: Computing Product of w-bit numbers x, y

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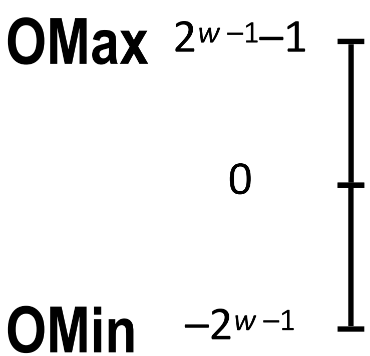
Original Number (w bits)



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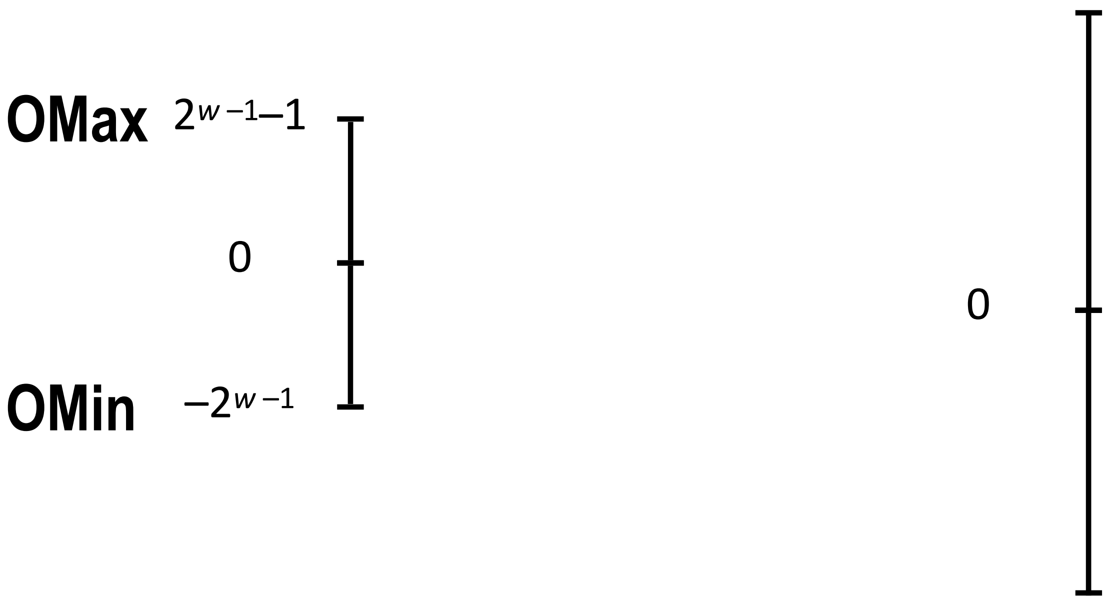

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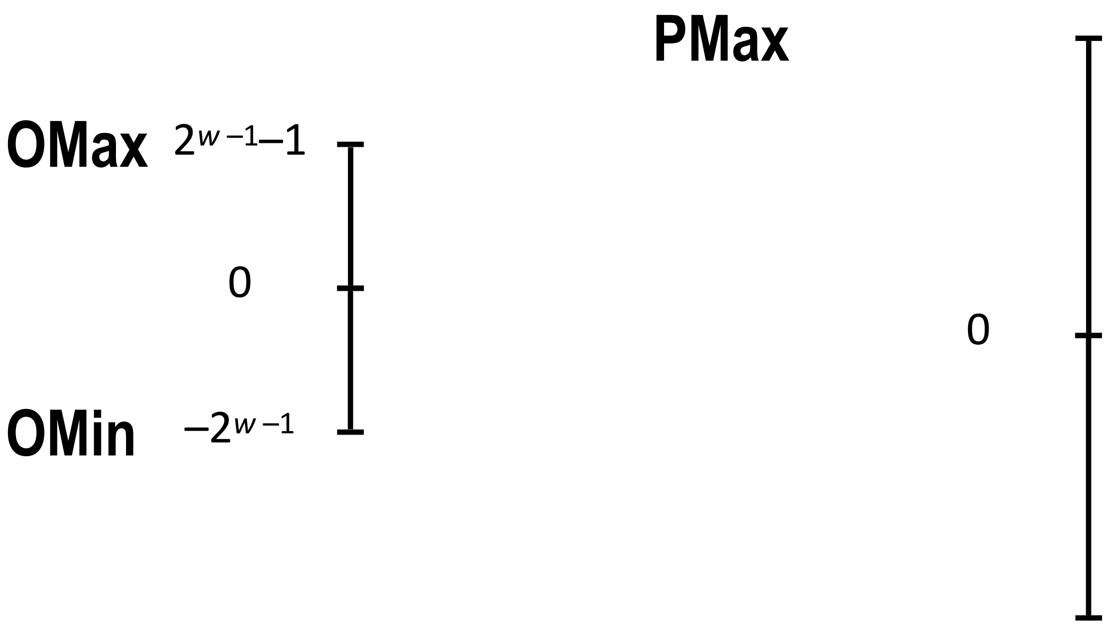

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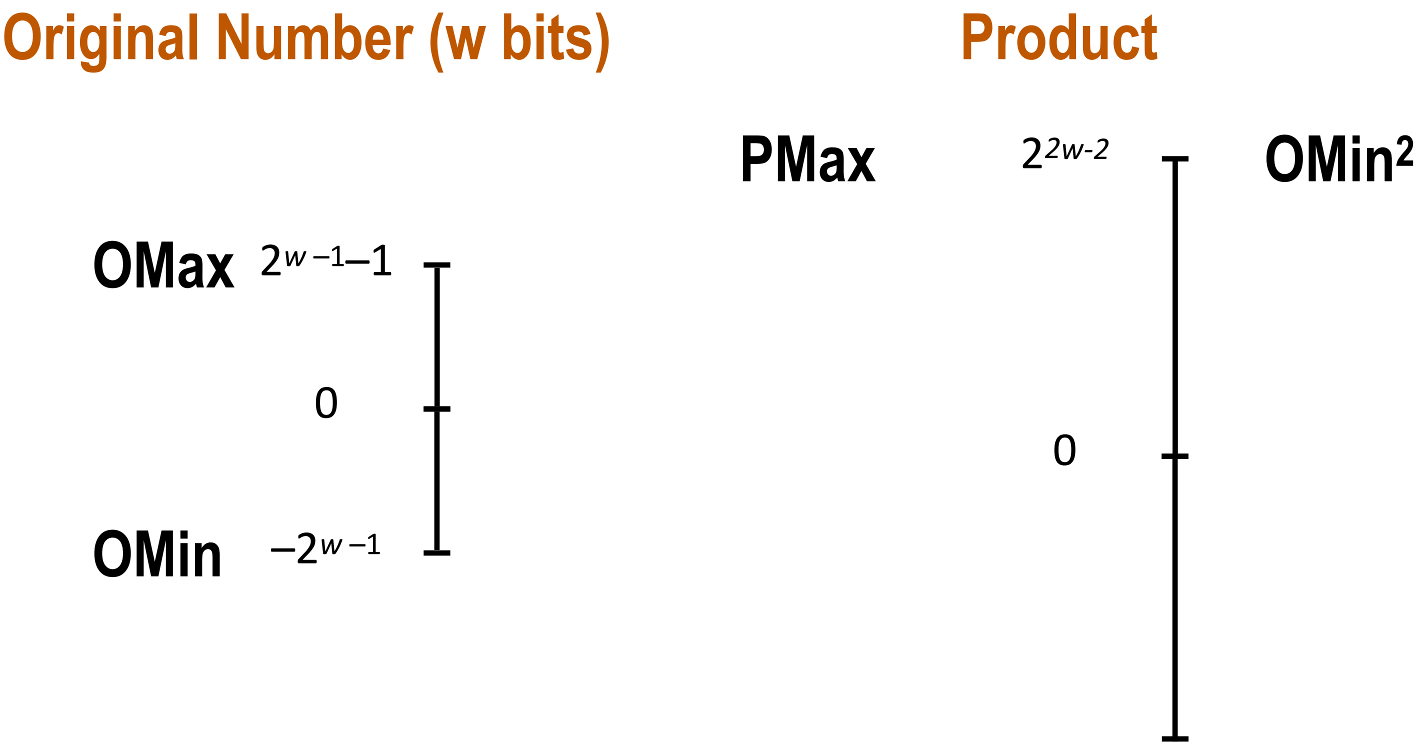

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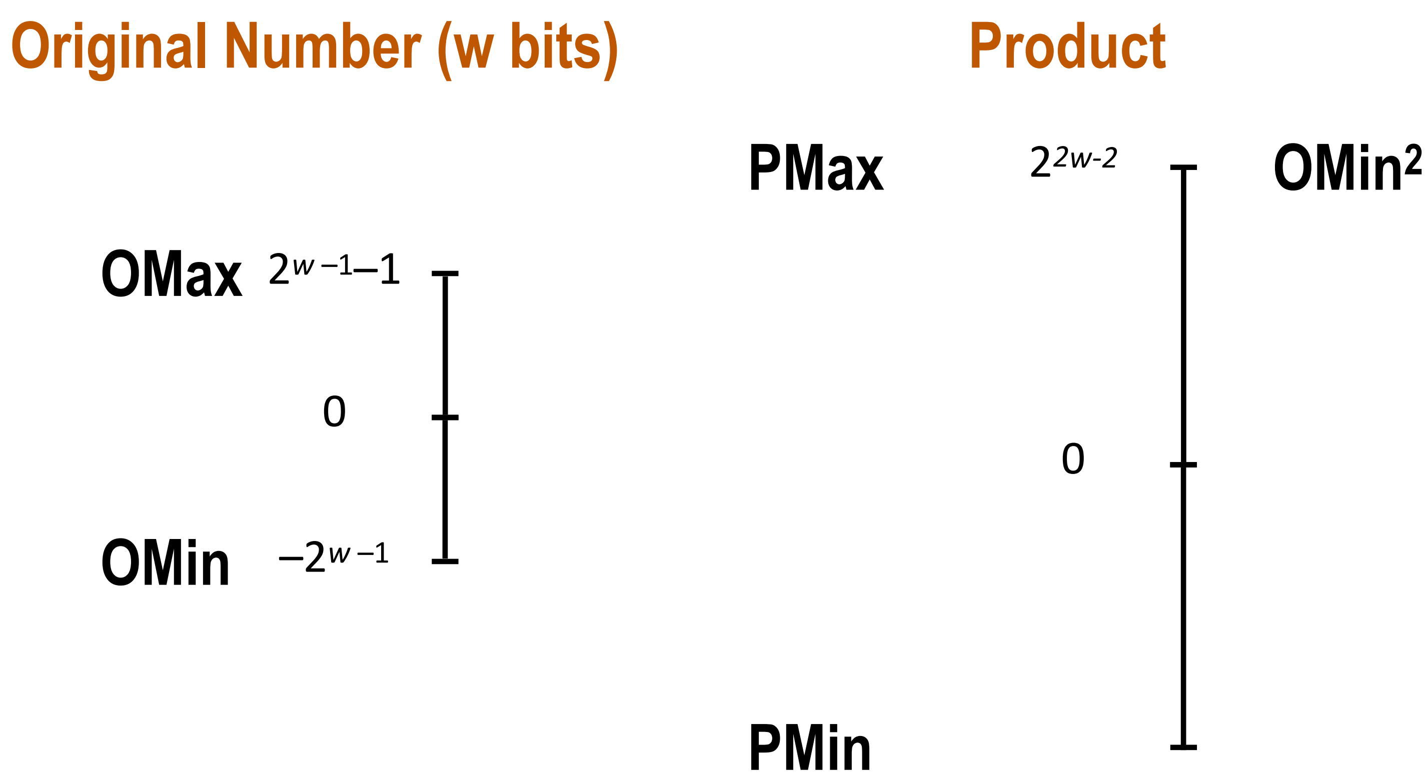

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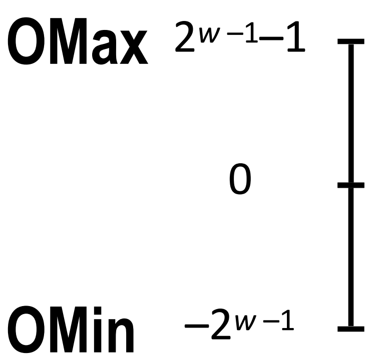

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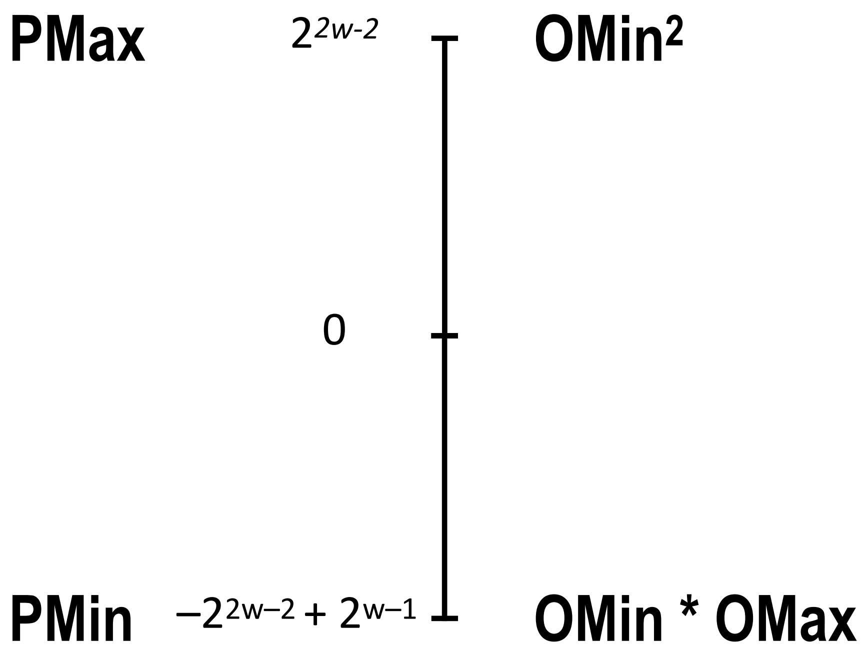

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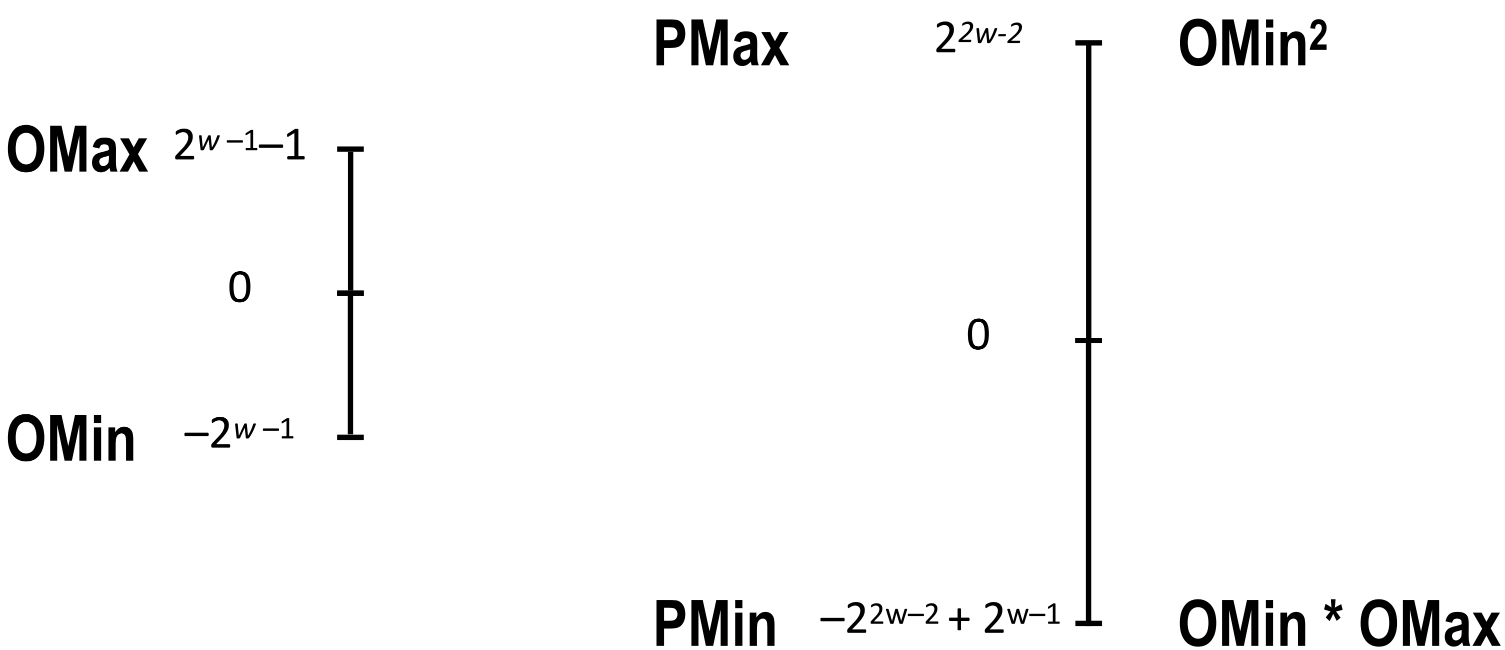
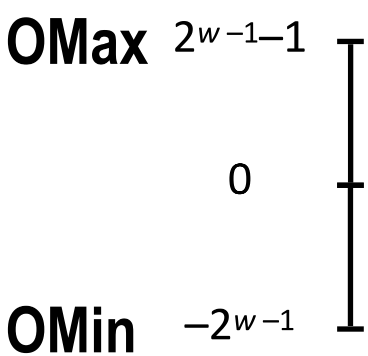
Original Number (w bits) **Product (2w bits)**



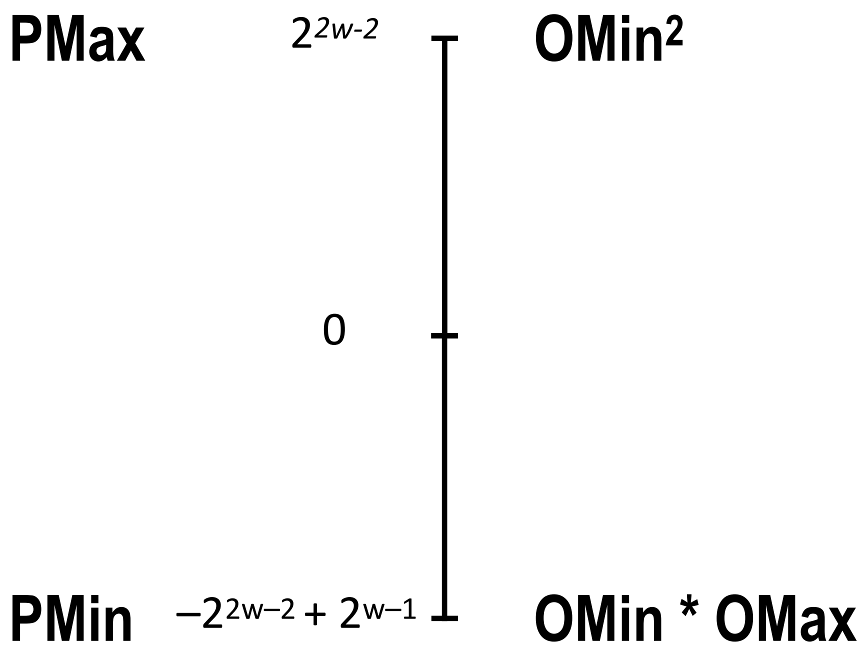
Multiplication

- Goal: Computing Product of w-bit numbers x, y
- Exact results can be bigger than w bits
  - Up to $2w$ bits (both signed and unsigned)

Original Number (w bits)



Product (2w bits)

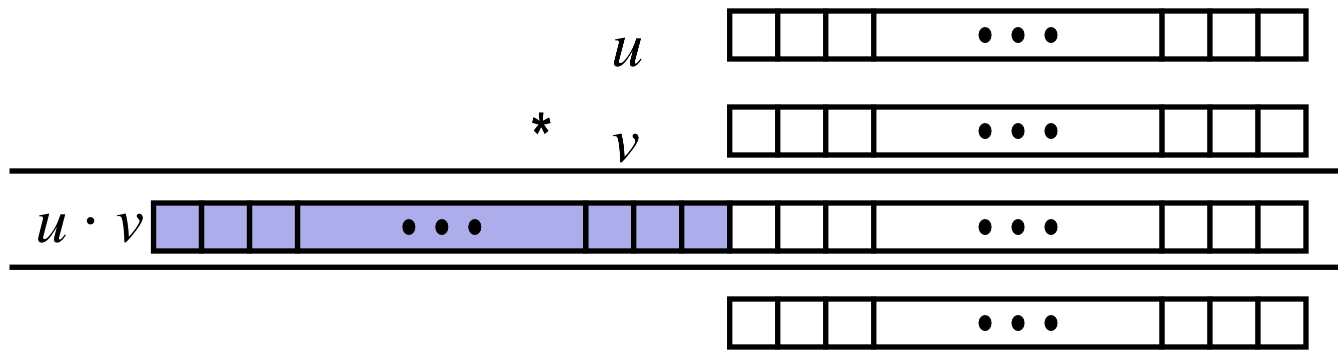


Unsigned Multiplication in C

Operands: *w* bits

True Product: 2*w bits

Discard *w* bits: *w* bits



- Standard Multiplication Function
  - Ignores high order *w* bits
- Effectively Implements the following:

$\mathrm{UMult}_w(u, v) = u \cdot v \bmod 2^w$

Today: Floating Point

- Background: Fractional binary numbers and fixed-point
- Floating point representation
- IEEE 754 standard
- Rounding, addition, multiplication
- Floating point in C
- Summary

Recall: Represent Fractions in Binary

- What does 10.01_2 mean?
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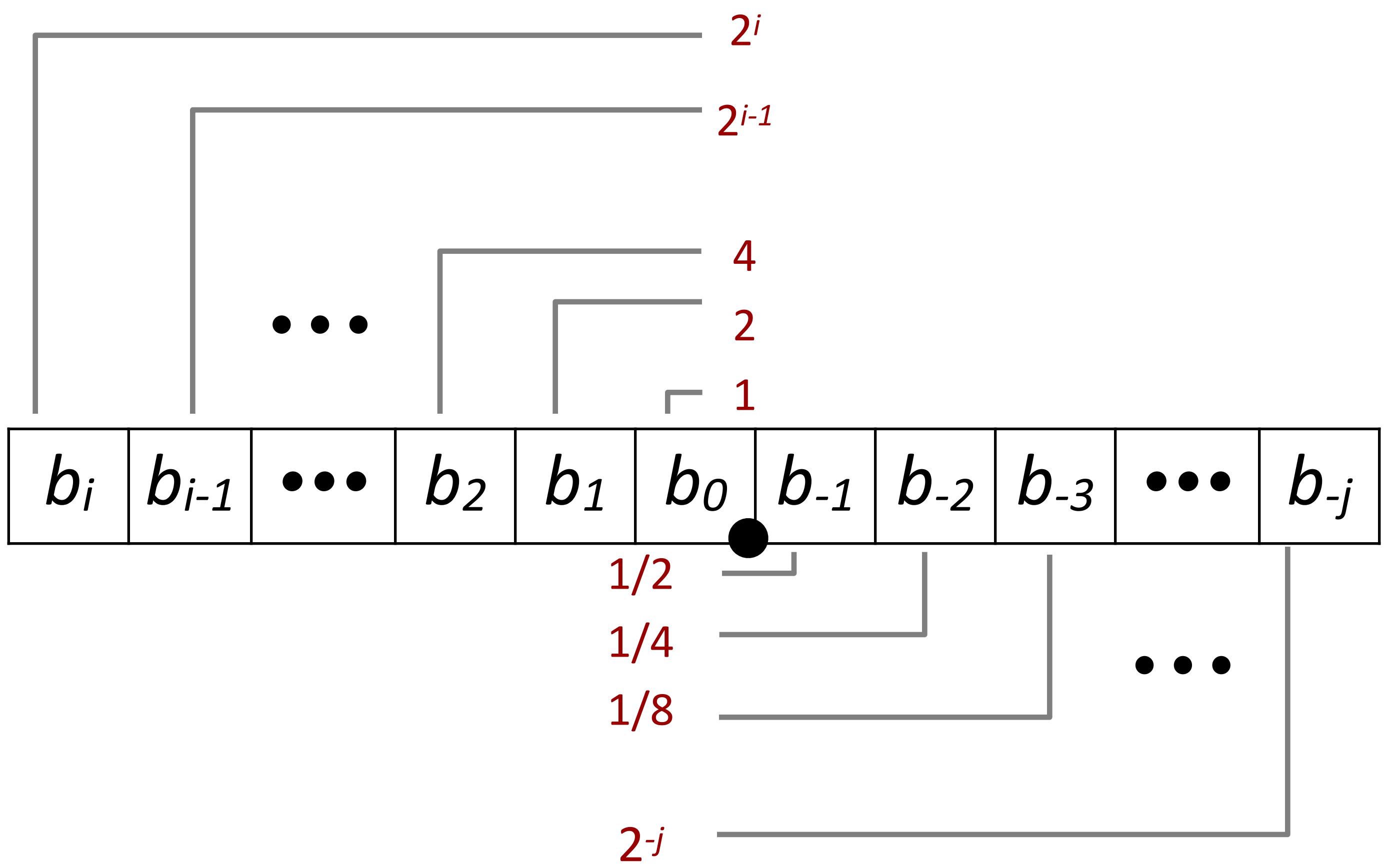
$$= 2.25_{10}$$

Fractional Binary Numbers



Fixed-Point Representation

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- Binary point stays fixed

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Decimal	Binary
0	0000.
1	0001.
2	0010.
3	0011.
4	0100.
5	0101.
6	0110.
7	0111.
8	1000.
9	1001.
10	1010.
11	1011.
12	1100.
13	1101.
14	1110.
15	1111.

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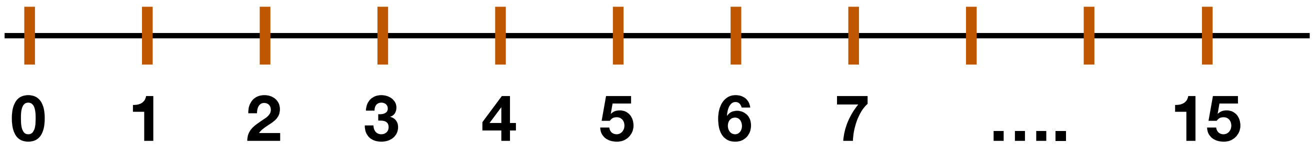

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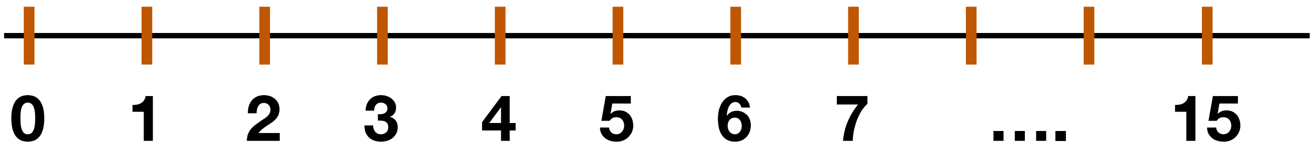

Decimal	Binary
0	00.00
0.25	00.01
0.5	00.10
0.75	00.11
1	01.00
1.25	01.01
1.5	01.10
1.75	01.11
2	10.00
2.25	10.01
2.5	10.10
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3	11.00
3.25	11.01
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  - The interval in this example is 0.25_{10}

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Fixed-Point Representation

- Binary point stays fixed
- Fixed interval between representable numbers
  - The interval in this example is 0.25_{10}

- Still need to remember the binary point, but just once for all numbers, which is implicit given the data type
- Usual arithmetics still work
  - No need to align (already aligned)

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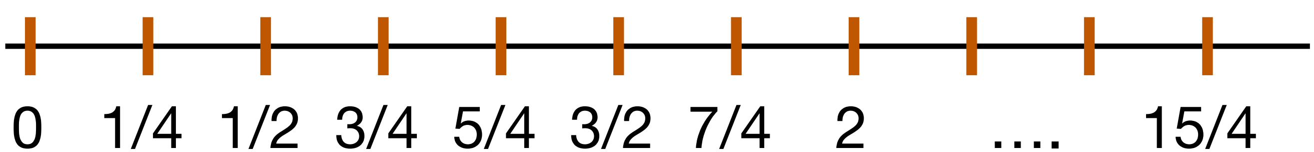

$b_3b_2.b_1b_0$

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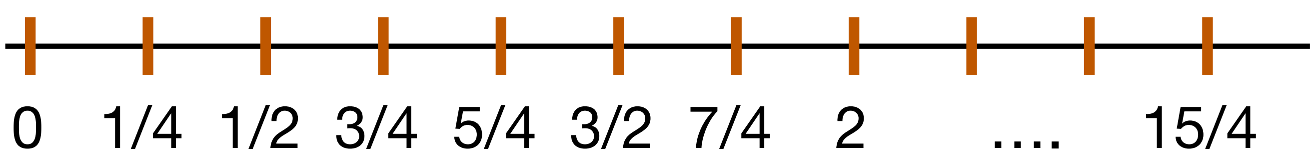

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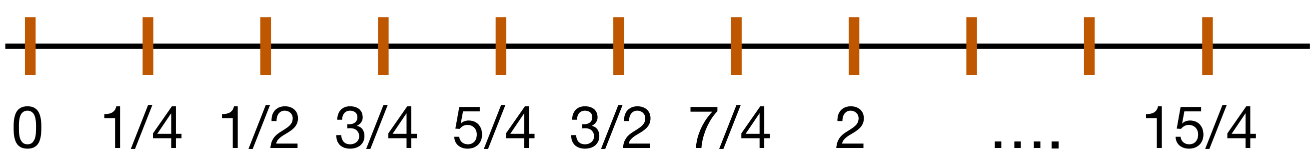
Decimal Value	Binary Representation
1/3	0.0101010101[01]…
1/5	0.001100110011[0011]…
1/10	0.0001100110011[0011]…



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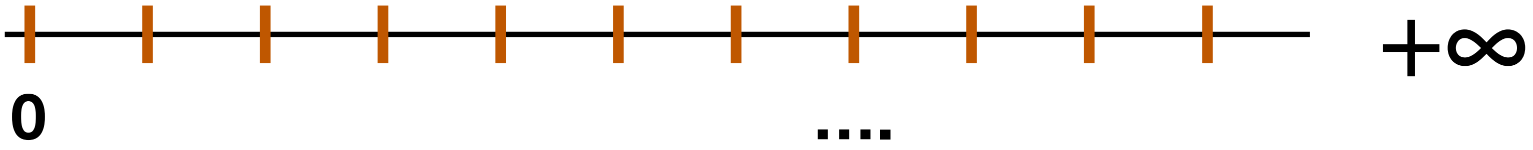

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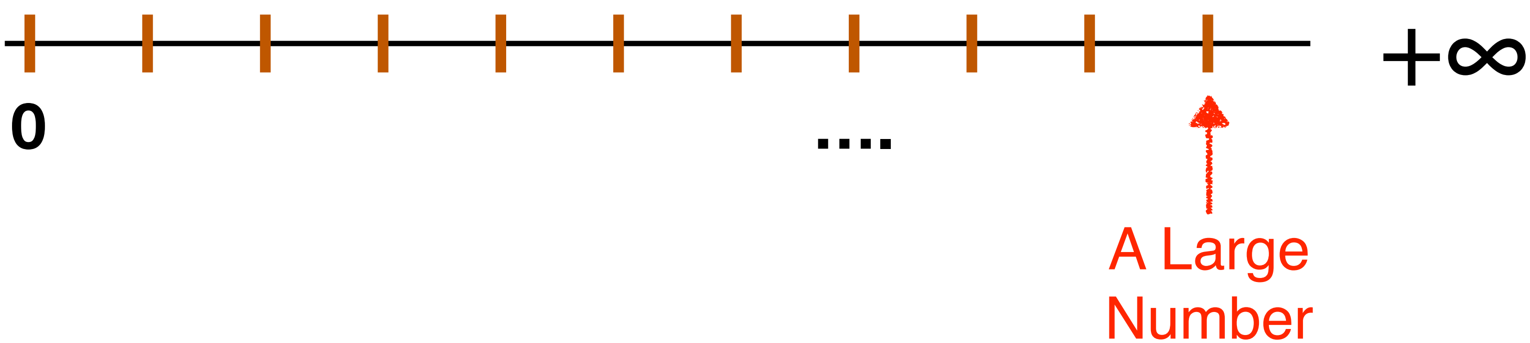

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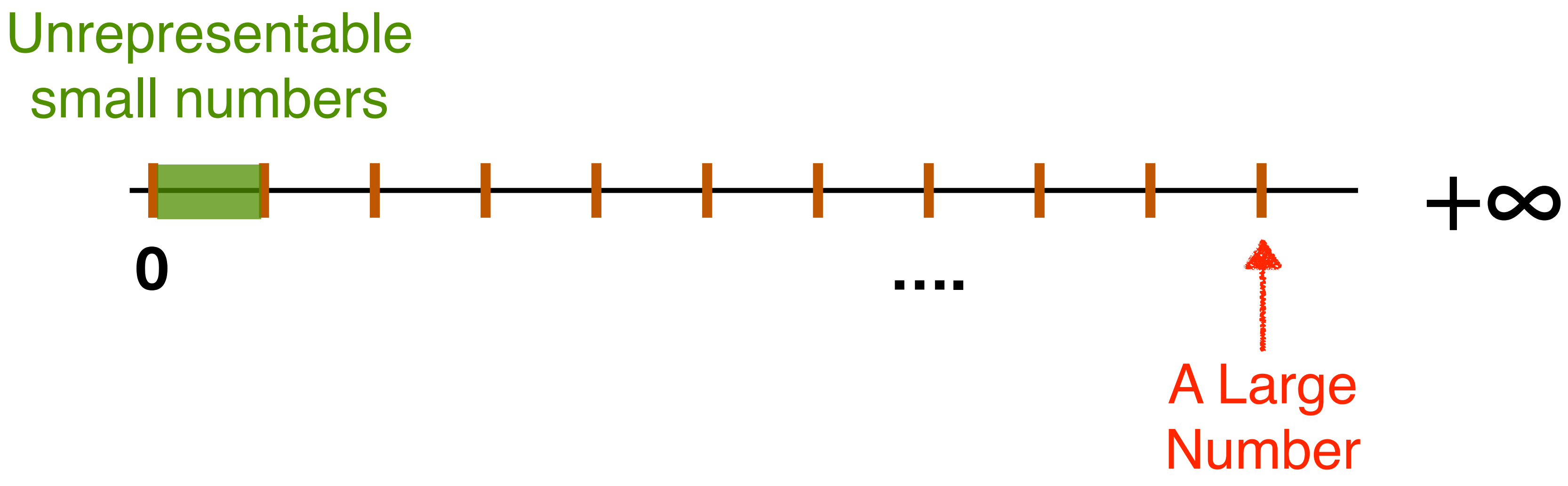

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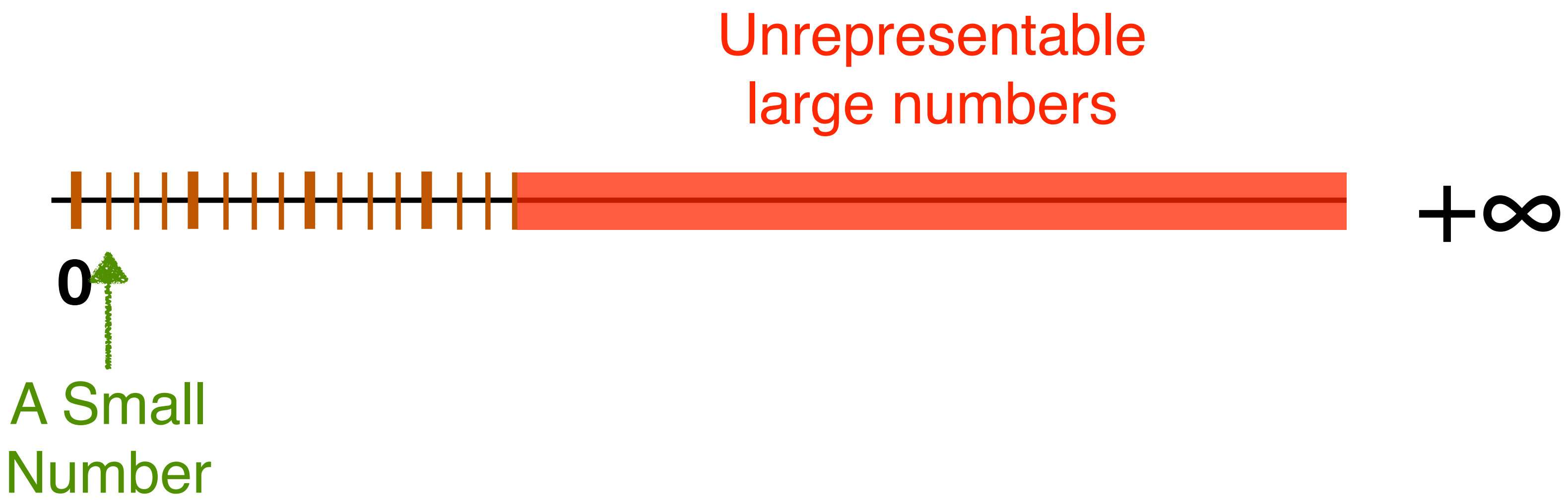

Today: Floating Point

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- Floating point in C
- Summary

Primer: (Normalized) Scientific Notation

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Decimal Value	**Scientific Notation**
2	2×10^{0}
-4,321.768	-4.321768×10^{3}
0.000 000 007 51	7.51×10^{-9}

Primer: (Normalized) Scientific Notation

- In decimal: $M \times 10^E$
  - *E* is an integer
  - Normalized form: $1 <= |M| < 10$

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Exponent

Significand Base

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1110110110110	$(-1)^0$ 1.110110110110 x 2^{12}
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- In binary: $(-1)^s \, M \, 2^E$
- Normalized form:
  - $1 <= M < 2$
  - $M = 1.b_0 b_1 b_2 b_3 \ldots$

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Sign

Exponent

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    Fraction

Sign Exponent

$$(-1)^s M \times 2^E$$

Significand Base

- If I tell you that there is a number where:
  - Fraction = 0101
  - s = 1
  - E = 10
  - You could reconstruct the number as $(-1)^1 1.0101 \times 2^{10}$

Primer: Floating Point Representation

- In binary: $(-1)^s\, M\, 2^E$
- Normalized form:
    - $1 <= M < 2$
    - $M = 1.\underbrace{b_0 b_1 b_2 b_3 \ldots}_{\text{Fraction}}$

Sign Exponent

$$(-1)^s\, M \times 2^E$$

Significand Base

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- Encoding

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Sign Exponent

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Significand Base

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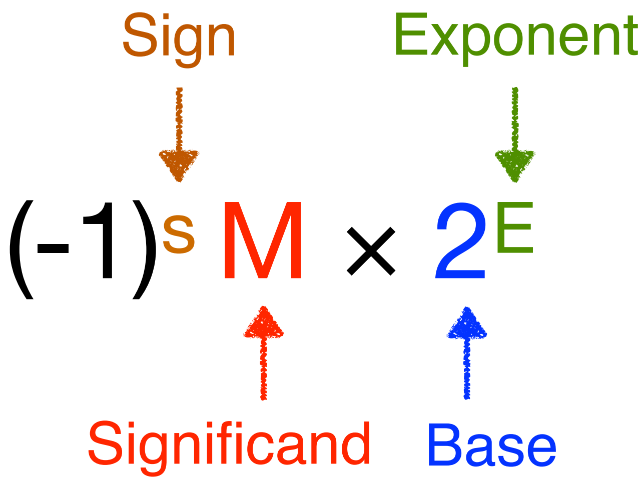
    Fraction
- Encoding
  - MSB s is sign bit s



Primer: Floating Point Representation

- In binary: $(-1)^s\, M\, 2^E$
- Normalized form:
  - $1 <= M < 2$
  - $M = 1.b_0b_1b_2b_3\ldots$

    Fraction
- Encoding
  - MSB s is sign bit s
  - *exp* field encodes Exponent (but not exactly the same, more later)

Sign Exponent

$$(-1)^s\, M \times 2^E$$

Significand Base

Primer: Floating Point Representation

- In binary: $(-1)^s\ M\ 2^E$
- Normalized form:
  - $1 <= M < 2$
  - $M = 1.\underbrace{b_0 b_1 b_2 b_3 \ldots}_{\text{Fraction}}$

$$\overset{\text{Sign}}{(-1)^{s}}\ \underset{\text{Significand}}{M} \times \underset{\text{Base}}{2}^{\overset{\text{Exponent}}{E}}$$

- Encoding
  - MSB *s* is sign bit s
  - *exp* field encodes Exponent (but not exactly the same, more later)
  - *frac* field encodes Fraction (but not exactly the same, more later)

s	exp	frac

6-bit Floating Point Example

$$v = (-1)^s \, M \, 2^E$$

s	exp	frac
1	3	2

6-bit Floating Point Example

$$v = (-1)^s\, M\, 2^E$$

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- *exp* has 3 bits, interpreted as an unsigned value

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  - If *exp* were *E*, we could represent exponents from **0 to 7**

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  - If $E = -2$, *exp* is 1 (001_2)

6-bit Floating Point Example

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  - How about negative exponent?
  - Subtract a bias term: *E = exp - bias* (i.e., exp = E + bias)
  - bias is always $2^{k-1} - 1$, where k is number of exponent bits
- Example when we use 3 bits for exp (i.e., $k = 3$):
  - bias = 3
  - If $E = -2$, *exp* is 1 (001_2)

E	exp
-3	000
-2	001
-1	010
0	011
1	100
2	101
3	110
4	111

6-bit Floating Point Example

$v = (-1)^s M 2^E$

s	exp	frac
1	3	2

- *exp* has 3 bits, interpreted as an unsigned value
  - If *exp* were *E*, we could represent exponents from **0 to 7**
  - How about negative exponent?
  - Subtract a bias term: *E = exp - bias* (i.e., exp = E + bias)
  - bias is always $2^{k-1} - 1$, where k is number of exponent bits
- Example when we use 3 bits for exp (i.e., *k* = 3):
  - bias = 3
  - If *E* = -2, *exp* is 1 (001_2)
  - Reserve 000 and 111 for other purposes (more on this later)

E	exp
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

$$v = (-1)^s \, M \, 2^E$$

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1	3	2

- *exp* has 3 bits, interpreted as an unsigned value
  - If *exp* were *E*, we could represent exponents from **0 to 7**
  - How about negative exponent?
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  - bias is always $2^{k-1} - 1$, where k is number of exponent bits
- Example when we use 3 bits for exp (i.e., $k = 3$):
  - bias = 3
  - If $E = -2$, *exp* is 1 (001_2)
  - Reserve 000 and 111 for other purposes (more on this later)
  - We can now represent exponents from **-2 (exp 001) to 3 (exp 110)**

E	**exp**
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

$v = (-1)^s M\ 2^E$

s	exp	frac
1	3	2

6-bit Floating Point Example

$v = (-1)^s M\, 2^E$

- *frac* has 2 bits, append them after “1.” to form M
  - *frac* = 10 implies M = 1.10

6-bit Floating Point Example

$v = (-1)^s \, M \, 2^E$

s	exp	frac
1	3	2

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- Putting it Together: An Example:

$$-10.1_2 = (-1)^1 \; 1.01 \times 2^1$$

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1	3	2

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6-bit Floating Point Example

$v = (-1)^s M 2^E$

1	exp	frac
1	3	2

- *frac* has 2 bits, append them after “1.” to form M
  - *frac* = 10 implies M = 1.10
- Putting it Together: An Example:

$$-10.1_2 = (-1)^1 \; 1.01 \times 2^1$$

E	exp
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

$$v = (-1)^s \, M \, 2^E$$

1	100	frac
1	3	2

- *frac* has 2 bits, append them after "1." to form M
  - *frac* = 10 implies M = 1.10
- Putting it Together: An Example:

$$-10.1_2 = (-1)^1 \; 1.01 \times 2^1$$

E	exp
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

$$v = (-1)^s M\, 2^E$$

1	100	frac
1	3	2

- *frac* has 2 bits, append them after “1.” to form M
  - *frac* = 10 implies M = 1.10
- Putting it Together: An Example:

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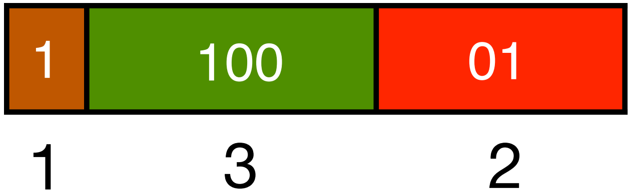
E	exp
~~-3~~	~~000~~
-2	001
-1	010
0	011
1	100
2	101
3	110
~~4~~	~~111~~

6-bit Floating Point Example

$$v = (-1)^s\, M\, 2^E$$



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E	exp
~~-3~~	~~000~~
-2	001
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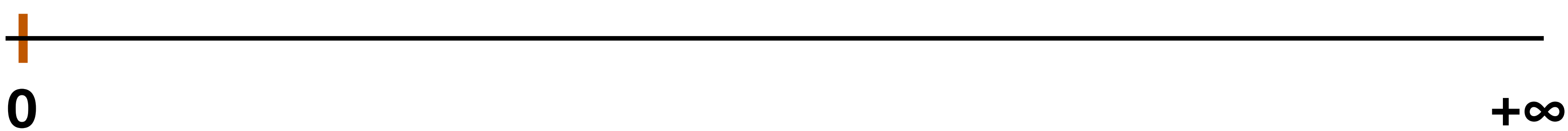
0	011
1	100
2	101
3	110
~~4~~	~~111~~

Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

s	exp	frac

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

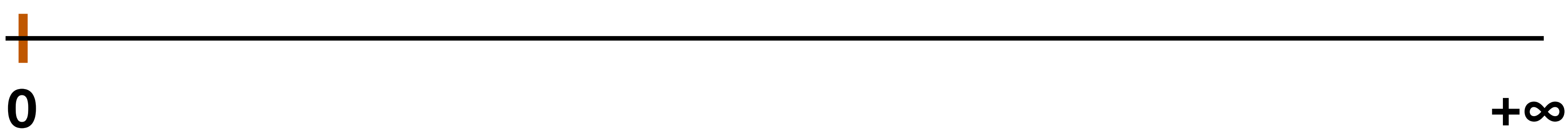


Representable Numbers (Positive Only)

$v = (-1)^s M\ 2^E$

0	exp	frac

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

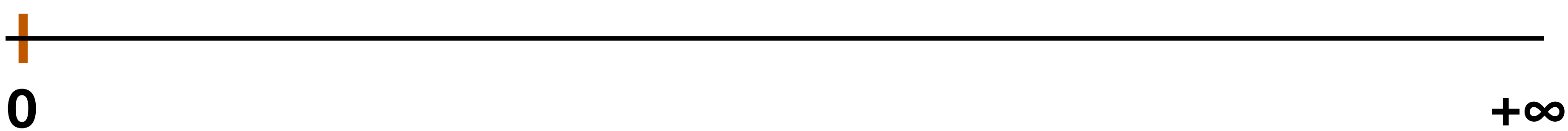


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	exp	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

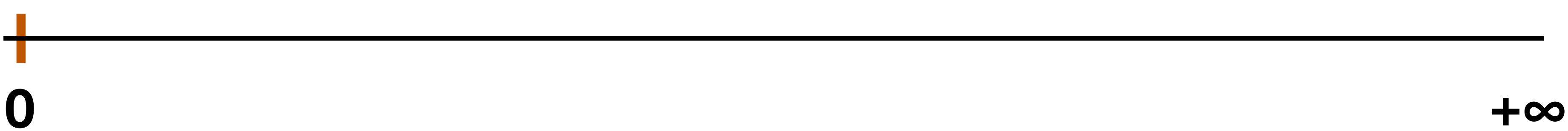


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	001	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

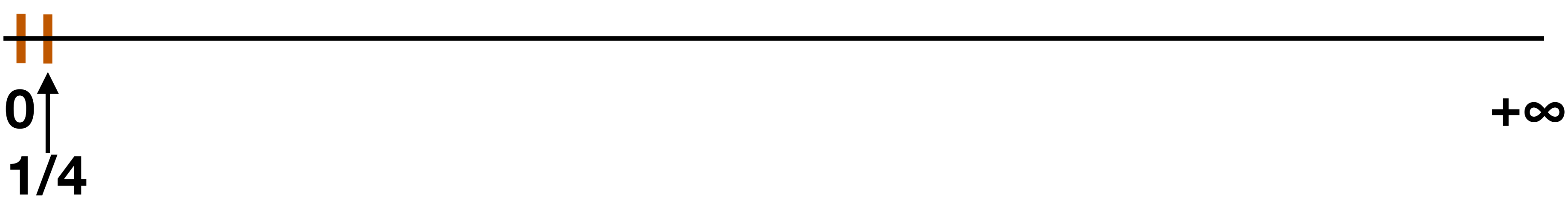


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	001	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

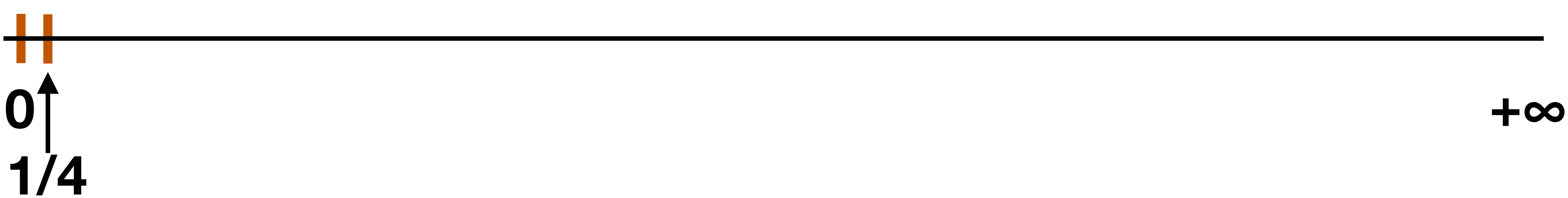


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	010	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

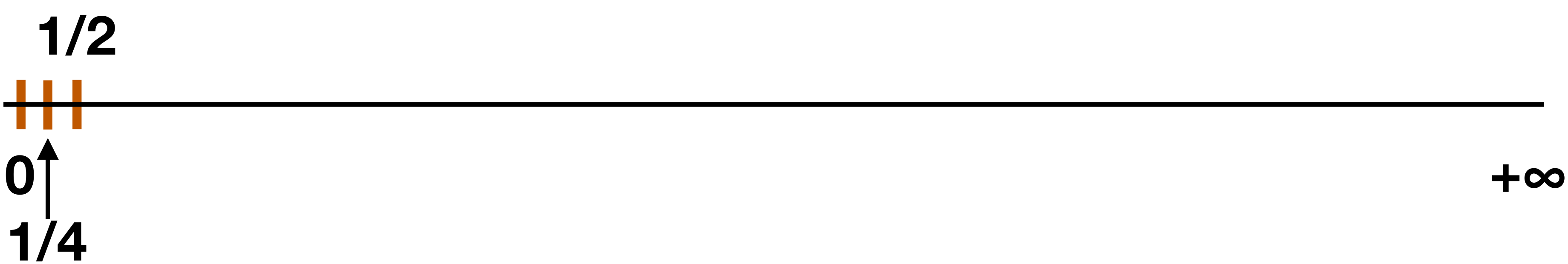


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	010	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

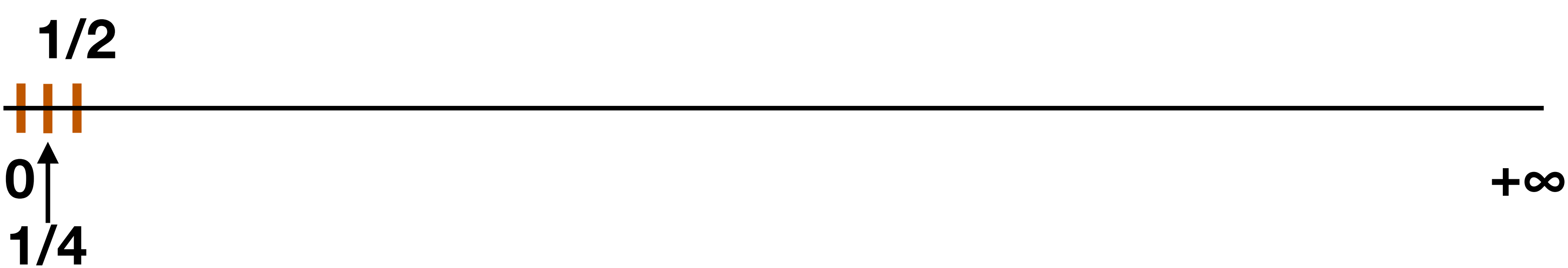


Representable Numbers (Positive Only)

$v = (-1)^s M\, 2^E$

0	011	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

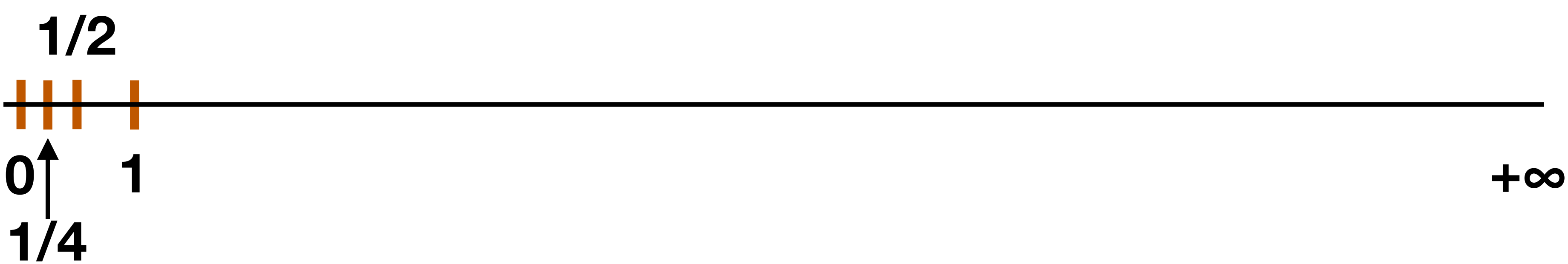


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	011	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~



Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	100	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
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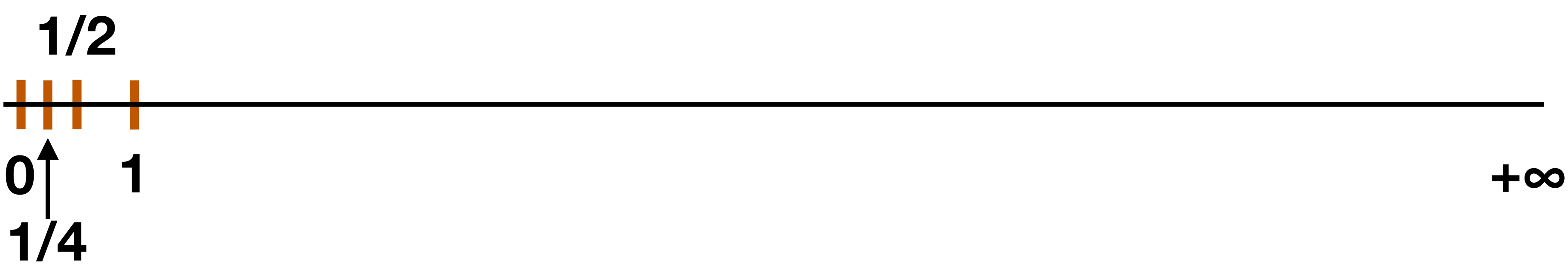
0	011	~~4~~	~~111~~



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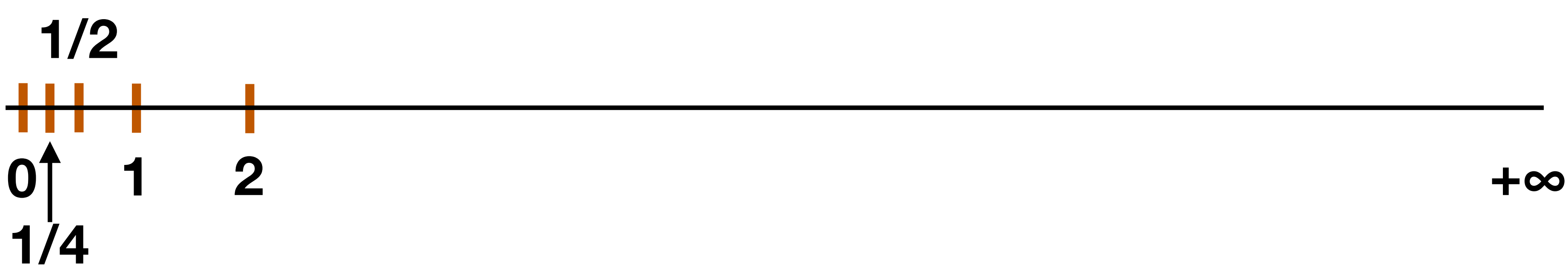

Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	101	00

E	exp	E	exp
~~-3~~	~~000~~	1	100
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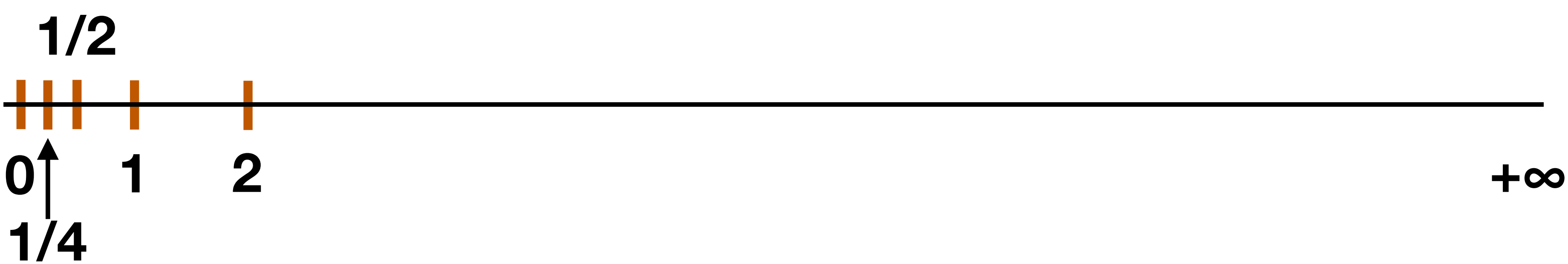
0	011	~~4~~	~~111~~



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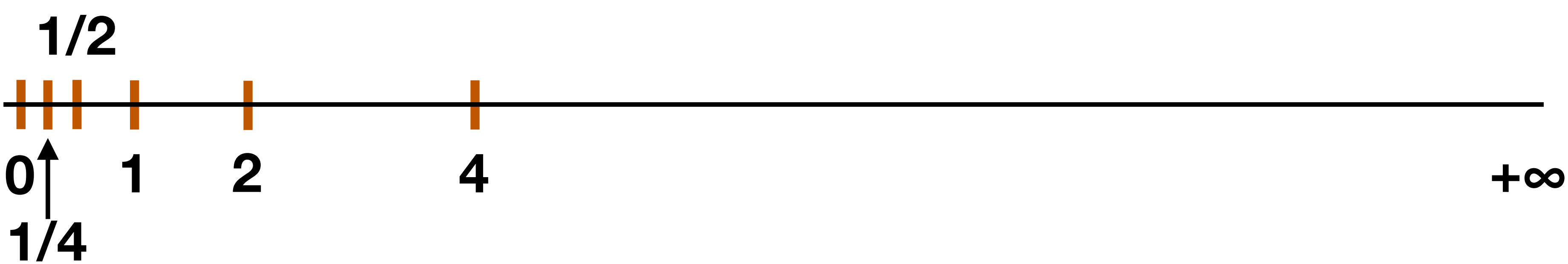

Representable Numbers (Positive Only)

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0	110	00

E	exp	E	exp
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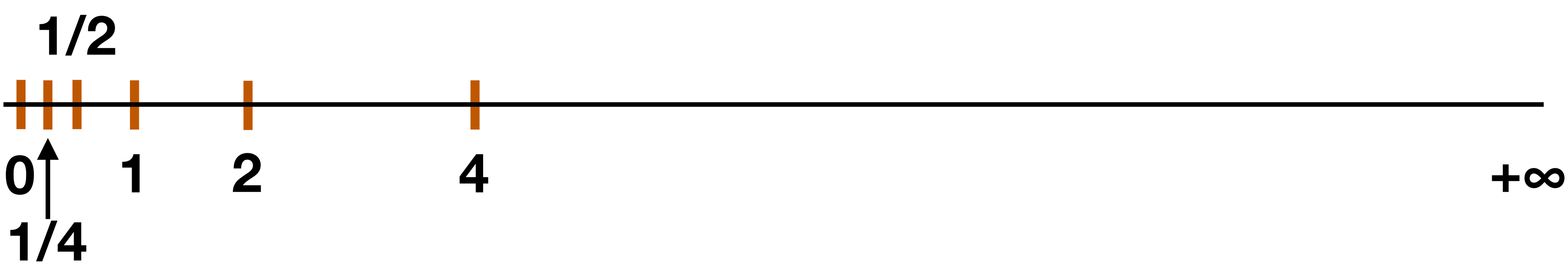
0	011	~~4~~	~~111~~



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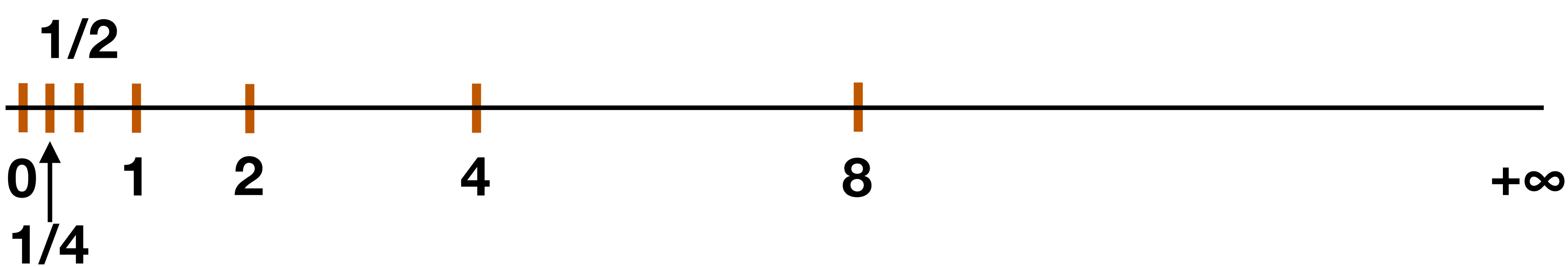

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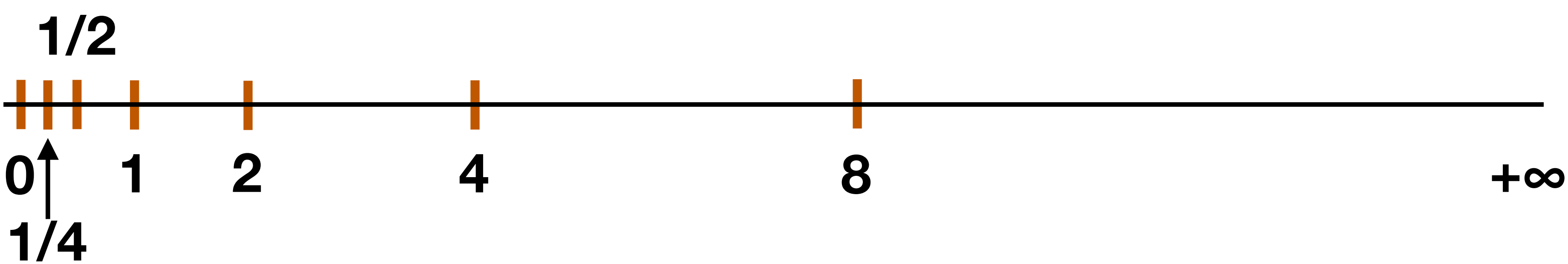

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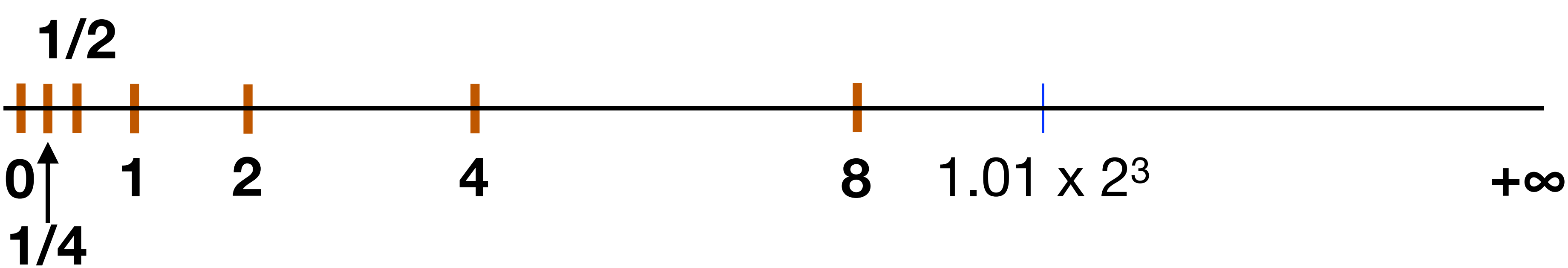

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0	110	01

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-2	001	2	101
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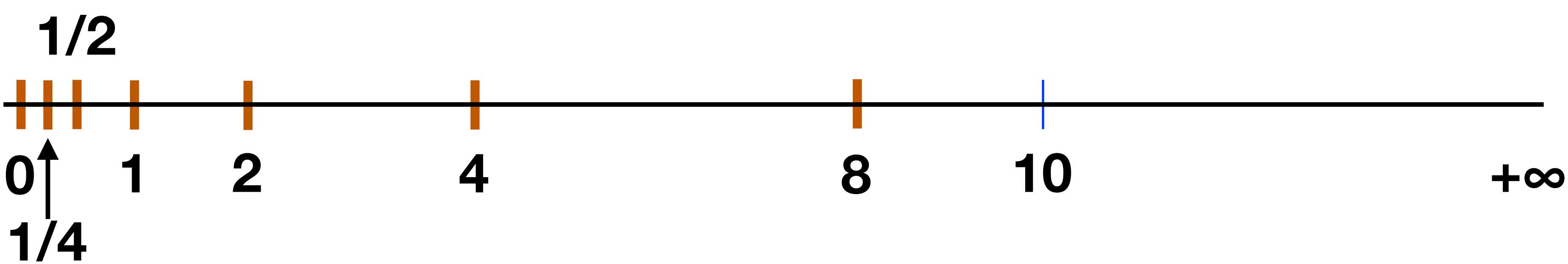
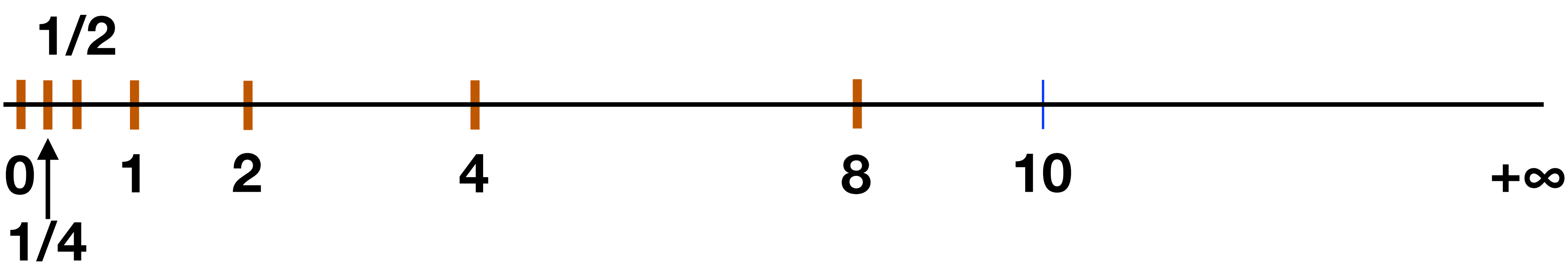


Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	110	10

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~



Representable Numbers (Positive Only)

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0	110	10

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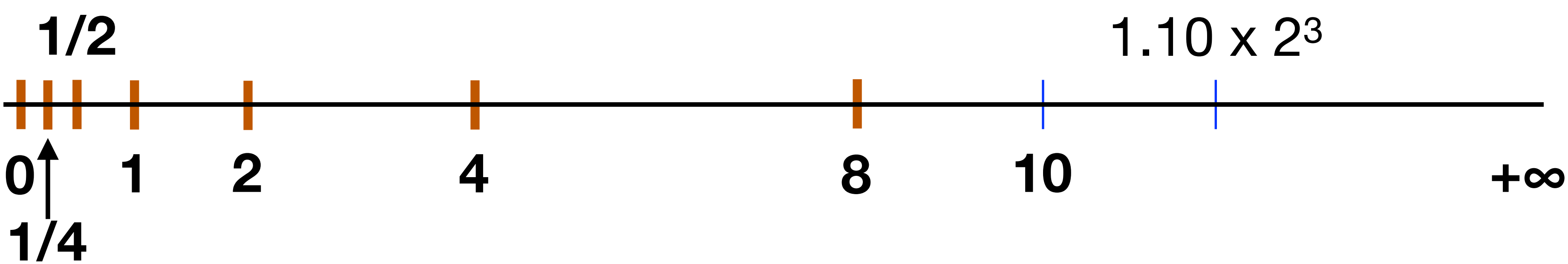
0	011	~~4~~	~~111~~

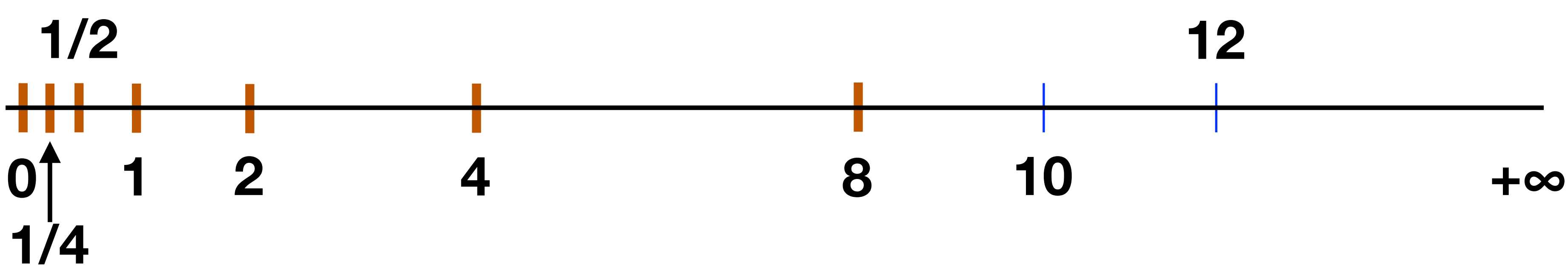


Representable Numbers (Positive Only)

$v = (-1)^s M\ 2^E$

0	110	10

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

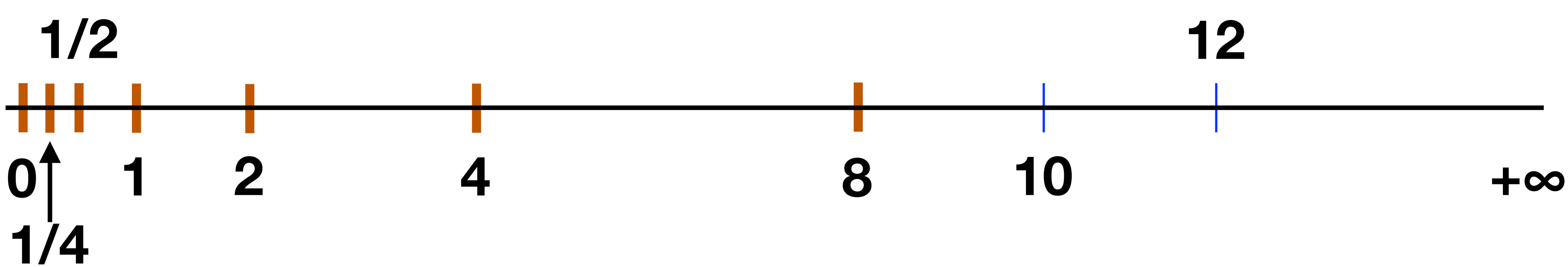


Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	110	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

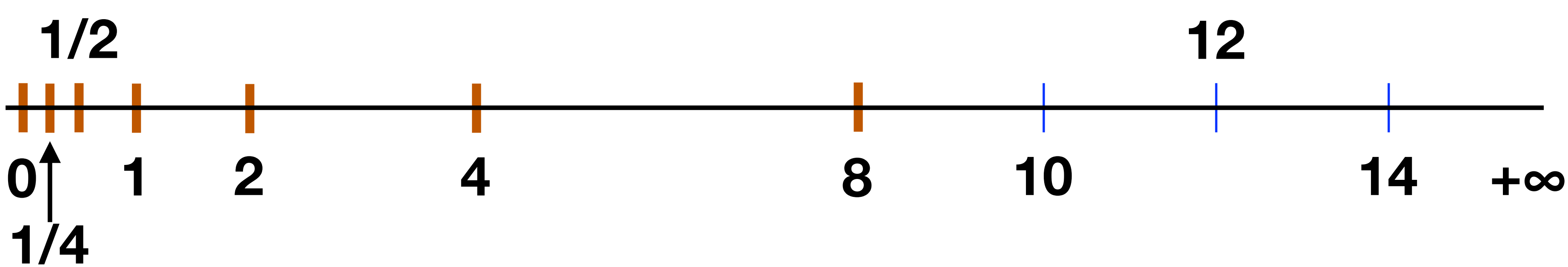


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	110	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

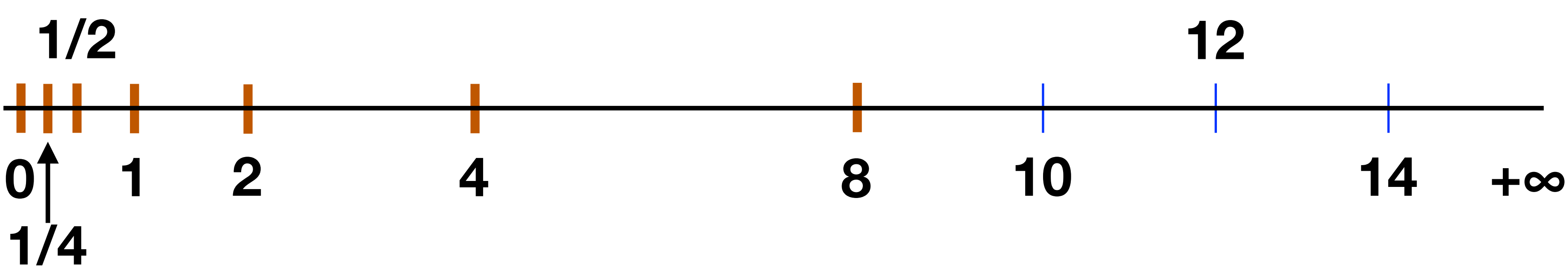


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	101	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

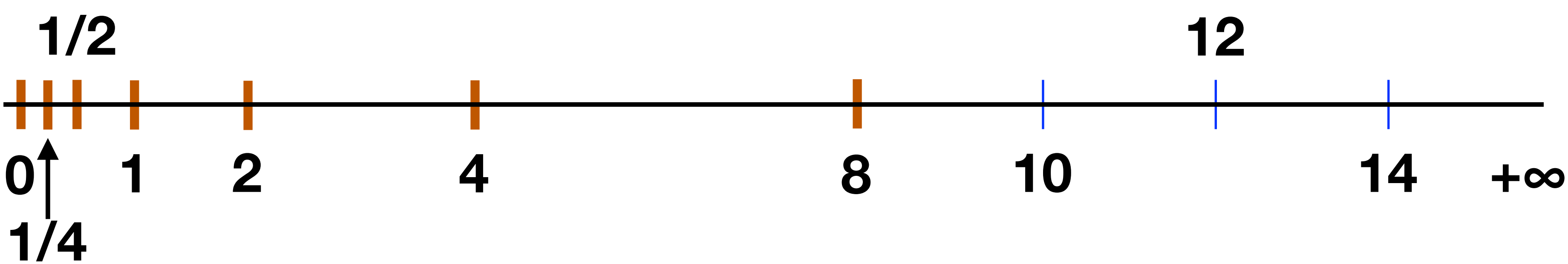


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	101	01

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

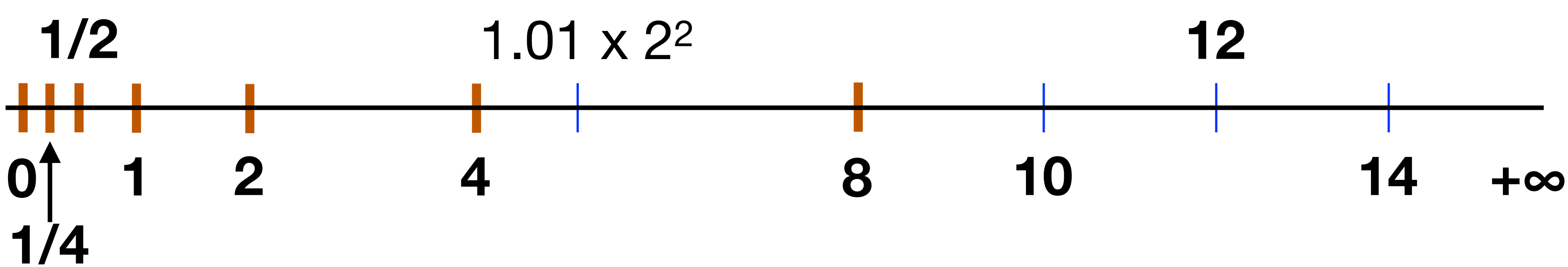


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	101	01

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

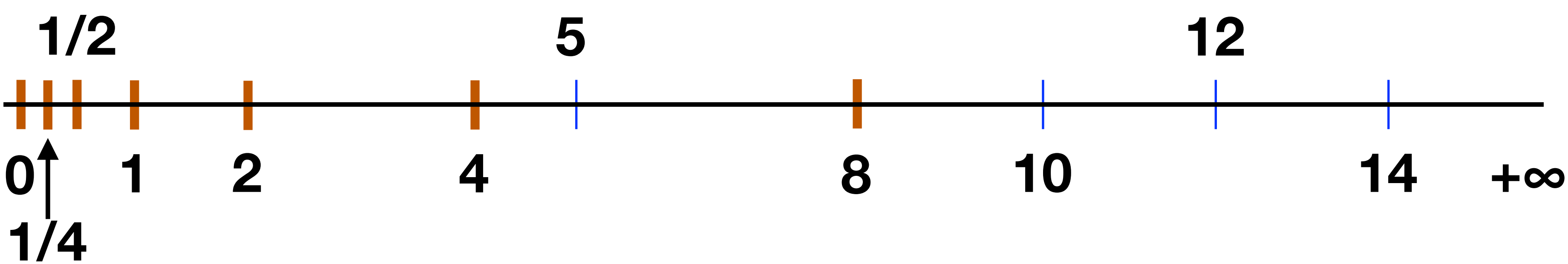


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	101	01

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

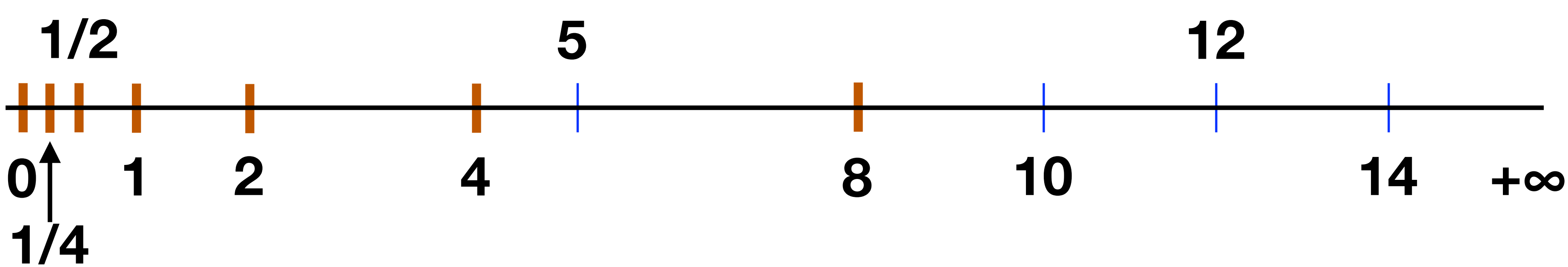


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	101	10

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

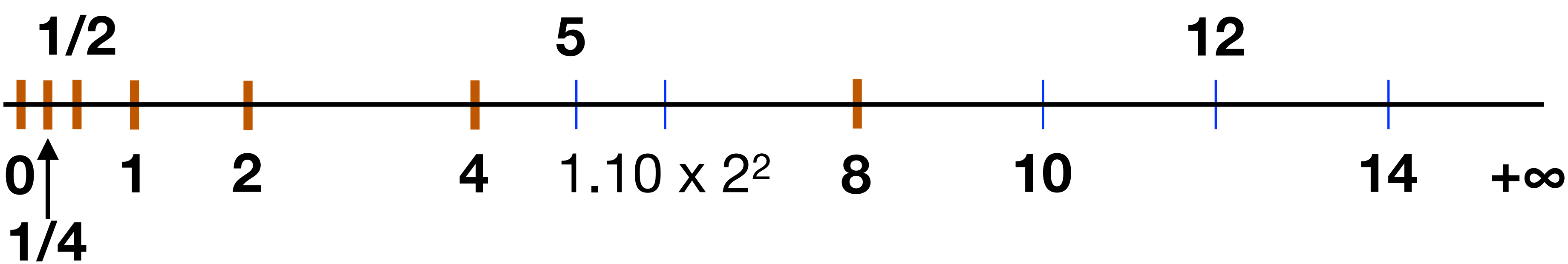


Representable Numbers (Positive Only)

$$v = (-1)^s M 2^E$$

0	101	10

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

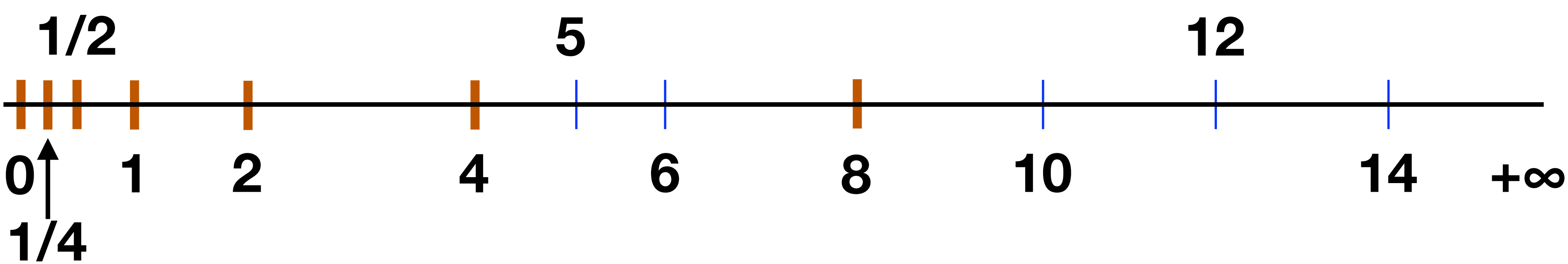


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	101	10

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

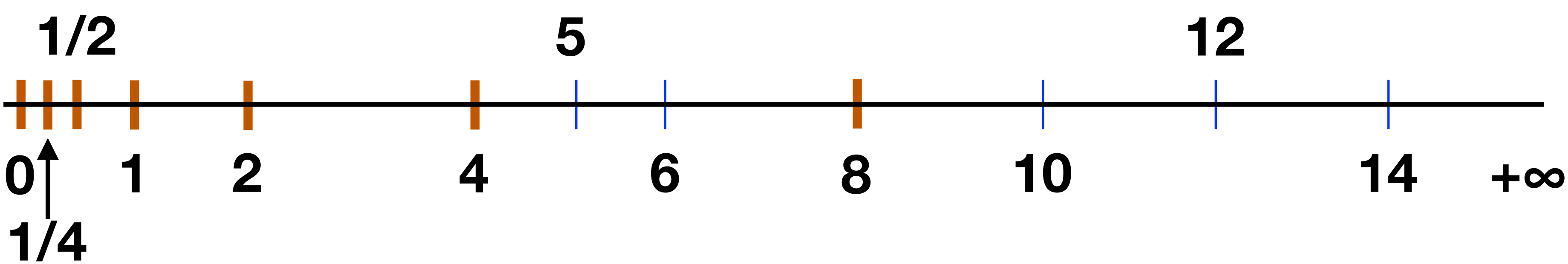


Representable Numbers (Positive Only)

$v = (-1)^s M\, 2^E$

0	101	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

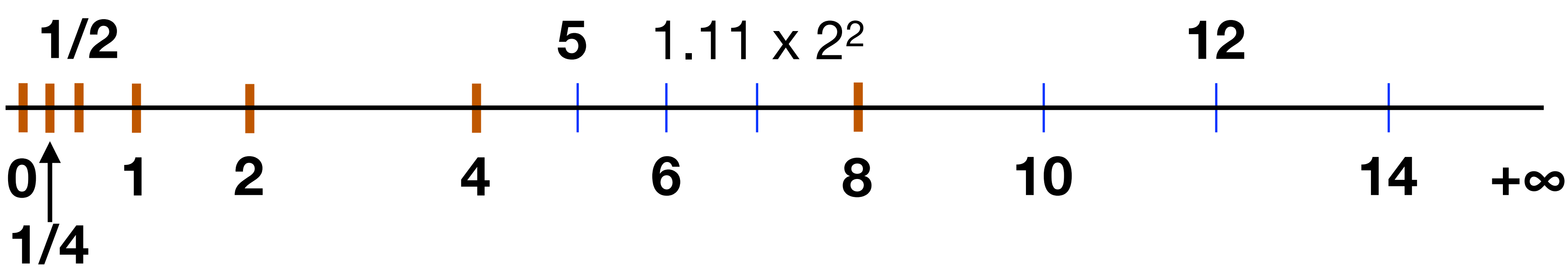


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	101	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

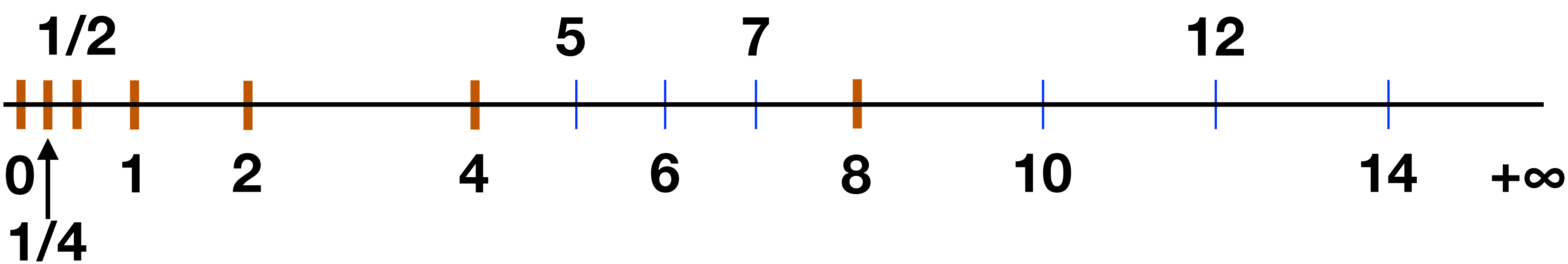


Representable Numbers (Positive Only)

$$v = (-1)^s M 2^E$$

0	101	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

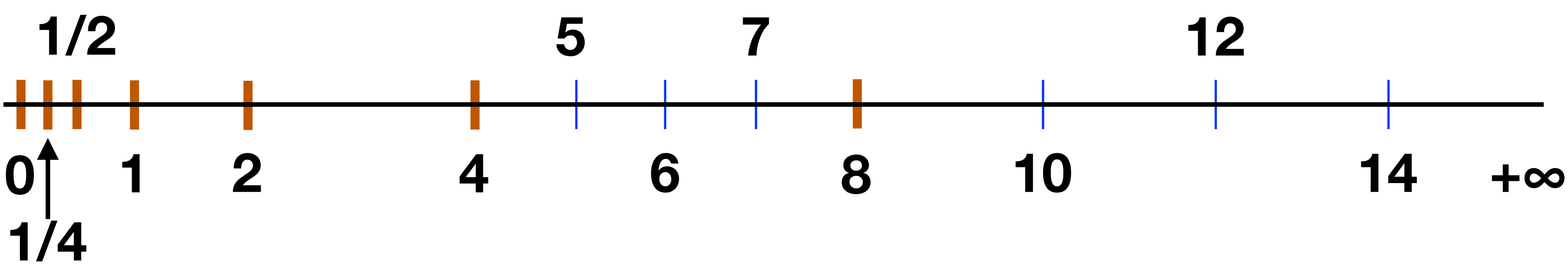


Representable Numbers (Positive Only)

$v = (-1)^s M\ 2^E$

0	100	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

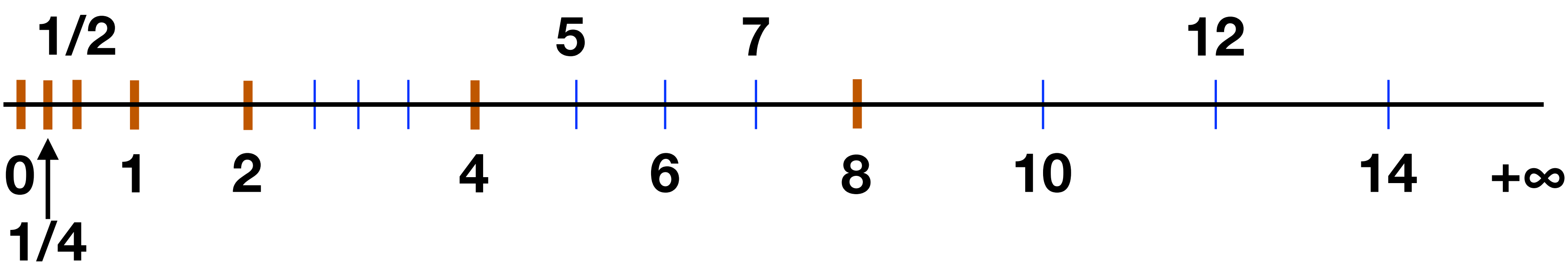


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	100	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

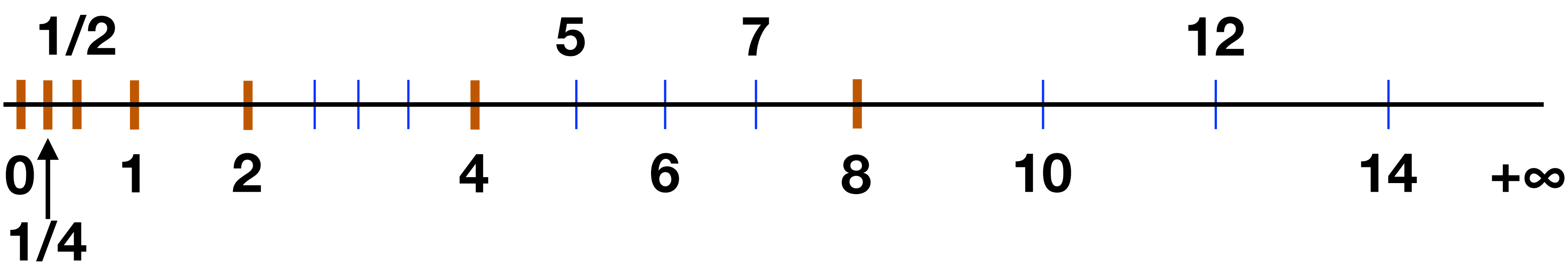


Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	011	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

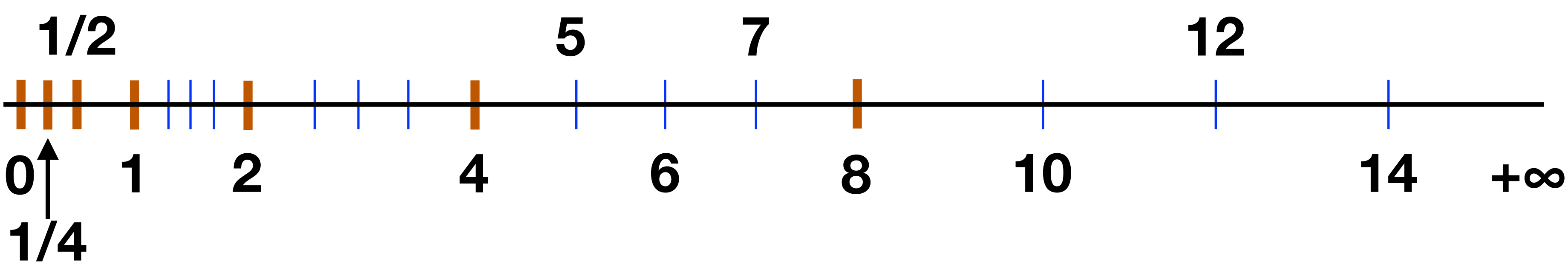


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	011	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

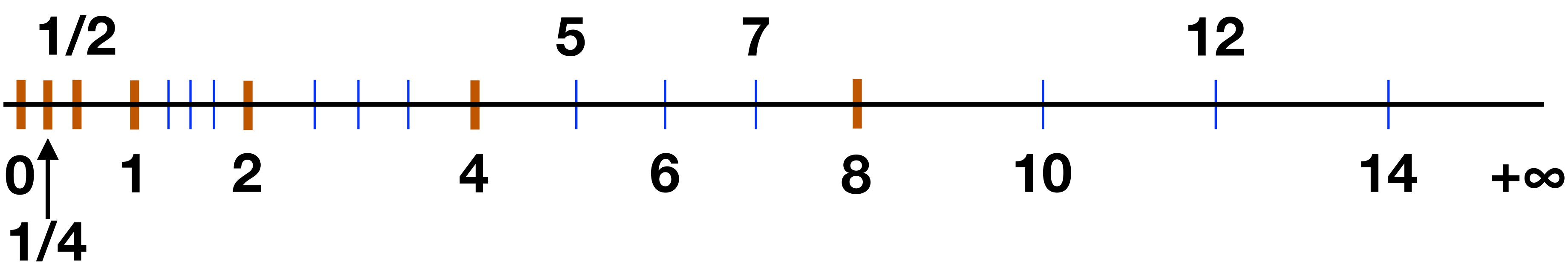


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	010	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

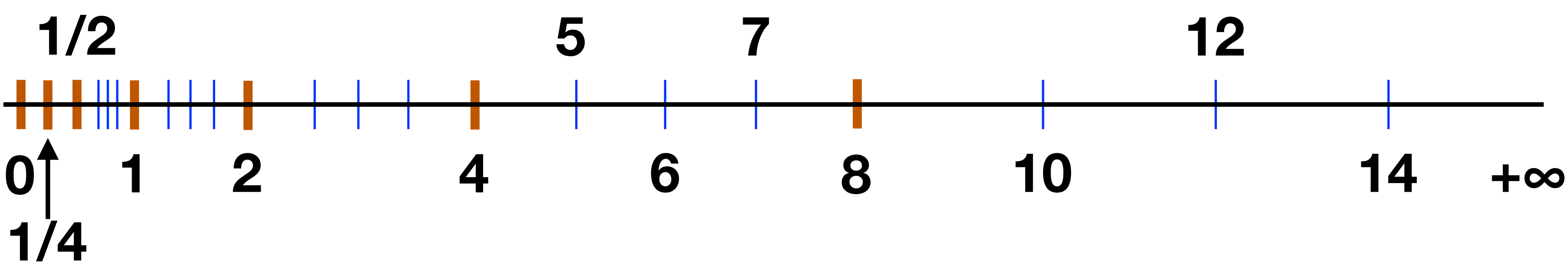


Representable Numbers (Positive Only)

$v = (-1)^s\, M\, 2^E$

0	010	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

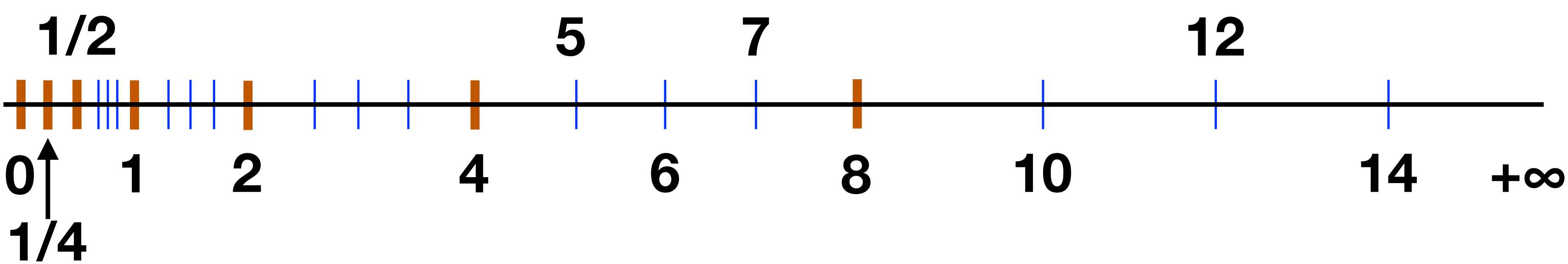


Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	001	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

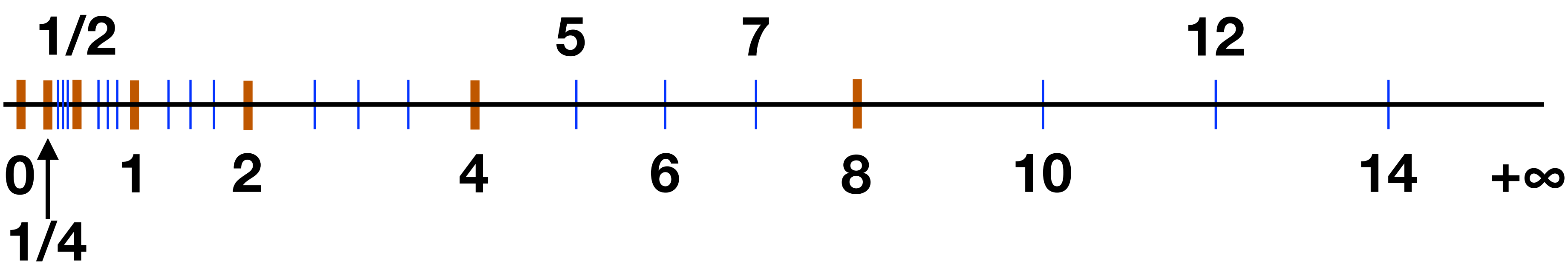


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

0	001	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~



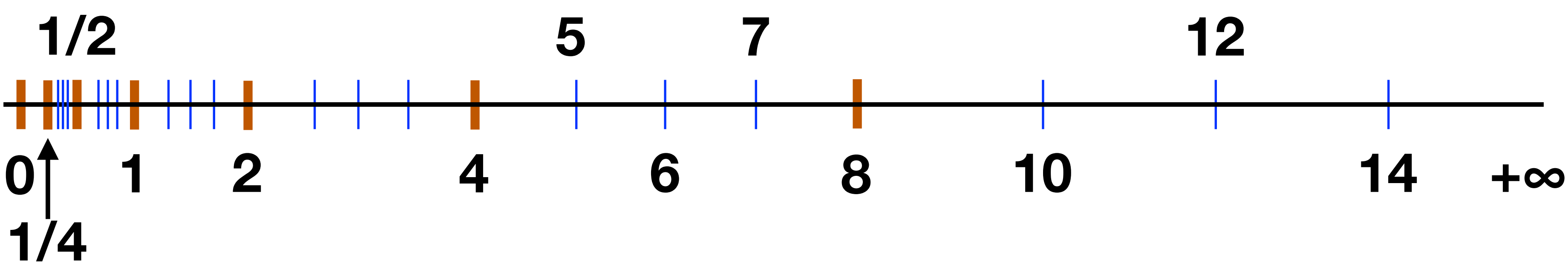
Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$

0	001	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Uneven interval (c.f., fixed interval in fixed-point)
  - More dense toward 0, sparser toward infinite
  - Allow encoding small and large numbers at the same time



Representable Numbers (Positive Only)

$v = (-1)^s M\, 2^E$

0	001	11

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

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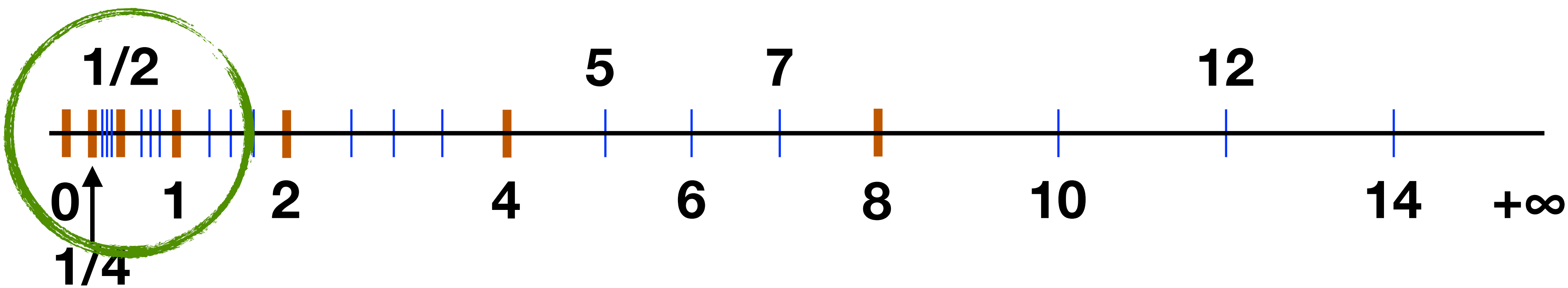
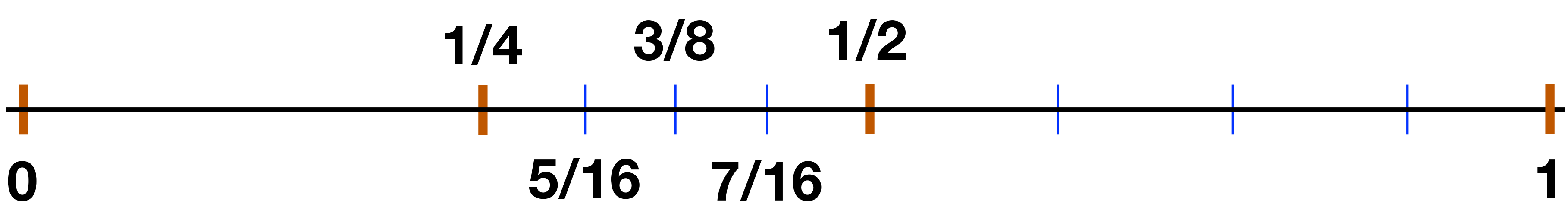


Representable Numbers (Positive Only)

$v = (-1)^s M\, 2^E$

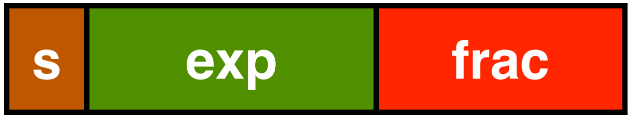
s	exp	frac

E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

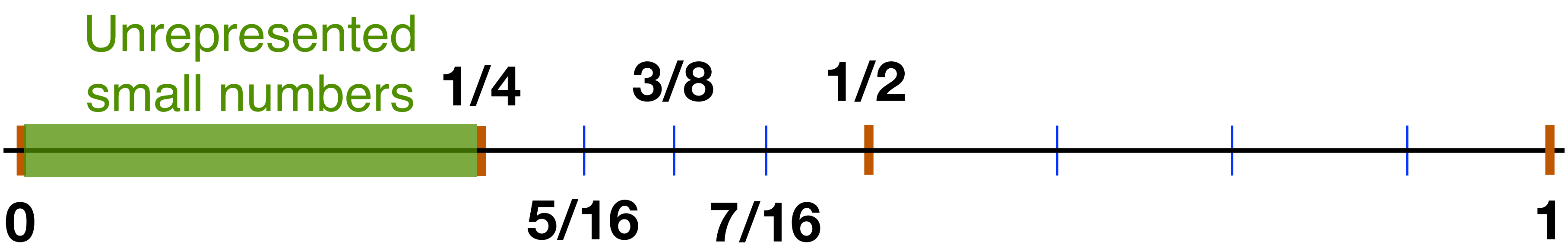


Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$



E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~



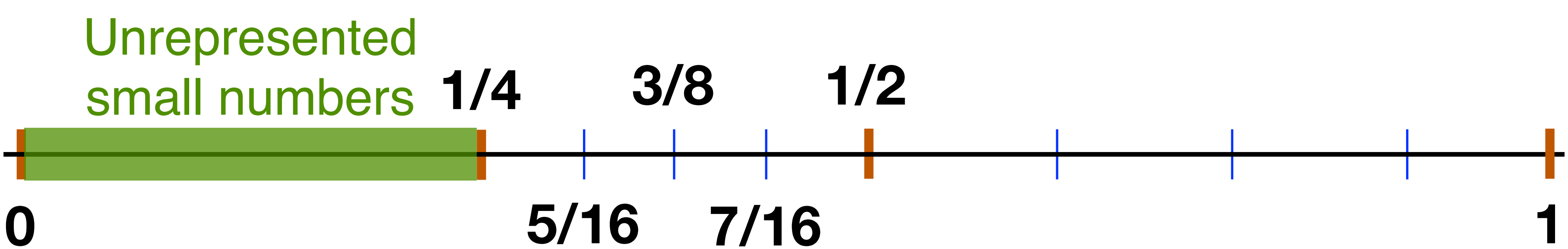
Representable Numbers (Positive Only)

$v = (-1)^s M \, 2^E$

s	exp	frac

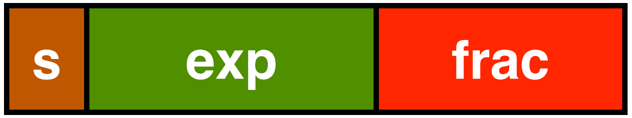
E	exp	E	exp
~~-3~~	~~000~~	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Always round to 0 is inelegant



Representable Numbers (Positive Only)

$v = (-1)^s M 2^E$



E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

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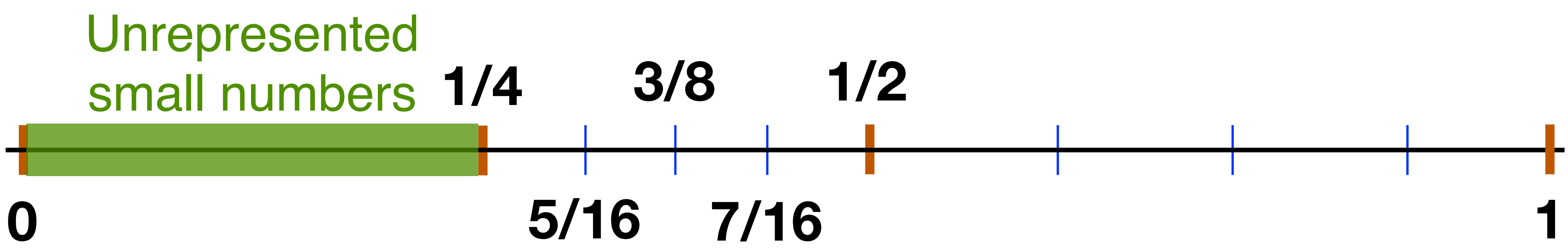

Representable Numbers (Positive Only)

$v = (-1)^s \, M \, 2^E$

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E	exp	E	exp
-3	000	1	100
-2	001	2	101
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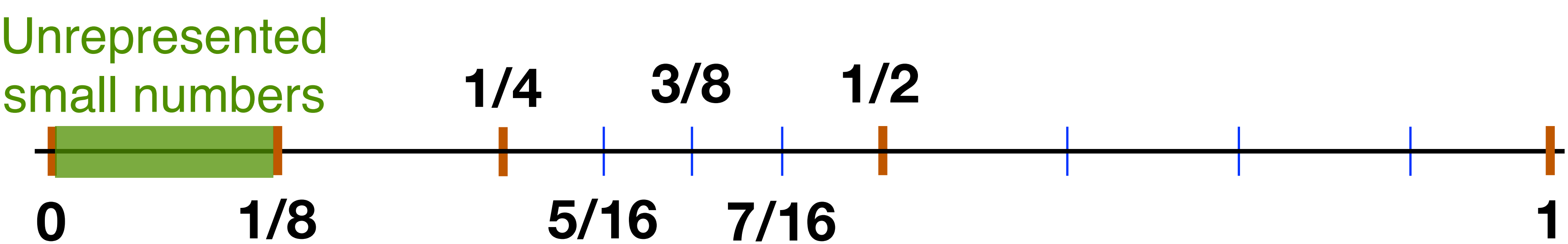

Representable Numbers (Positive Only)

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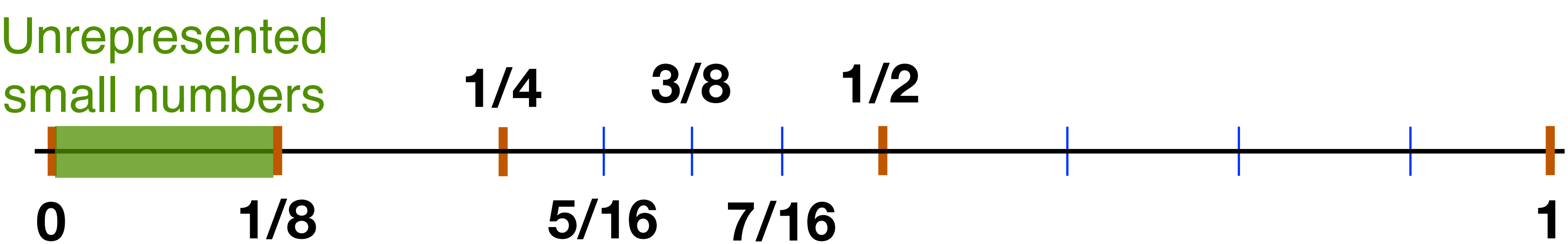
s	exp	frac

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Always round to 0 is inelegant
- Using 000 for *exp* doesn't solve it either



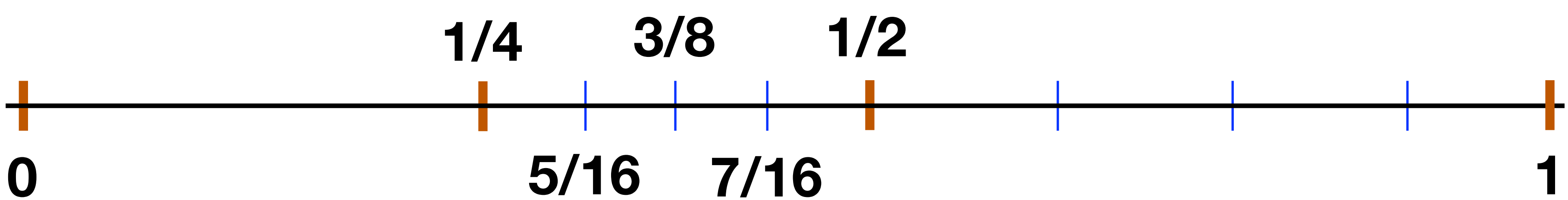
Subnormal (De-normalized) Numbers

$$v = (-1)^s \, M \, 2^E$$

s	exp	frac

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Idea: Evenly divide between 0 and 1/4 rather than exponentially decreasing **when *exp* = 0** (subnormal/denormalized numbers)



Subnormal (De-normalized) Numbers

$$v = (-1)^s \, M \, 2^E$$

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-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

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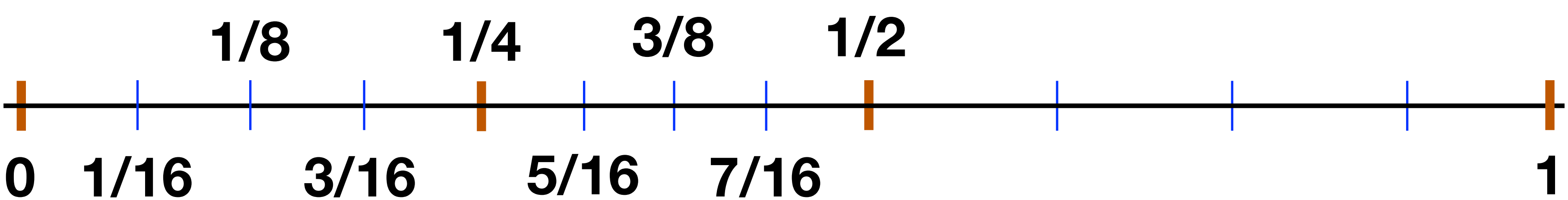


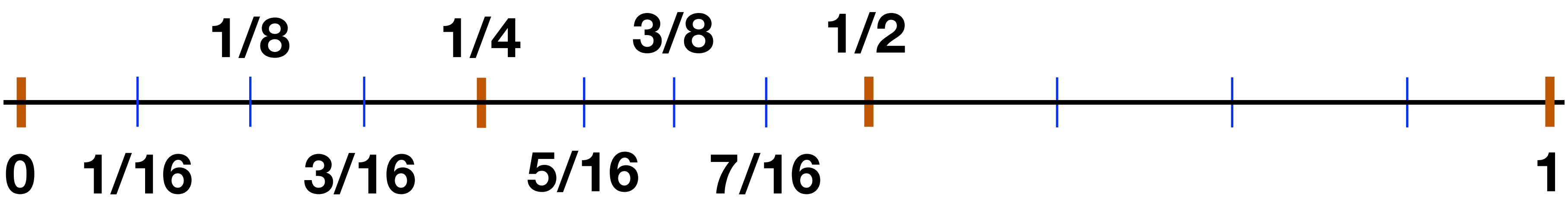
Subnormal (De-normalized) Numbers

$v = (-1)^s \, M \, 2^E$

s	exp	frac

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Idea: Evenly divide between 0 and 1/4 rather than exponentially decreasing **when *exp* = 0** (subnormal/denormalized numbers)
- $E = (exp + 1) - \text{bias}$ (instead of *exp* - *bias*)
- $M = 0.frac$ (instead of 1.frac)



Subnormal (De-normalized) Numbers

$v = (-1)^s M 2^E$

s	exp	frac

E	exp	E	exp
-3	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

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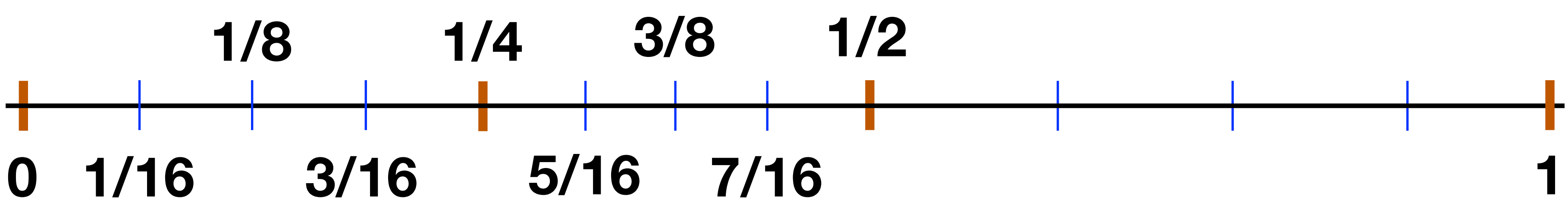


0	000	01

$= (-1)^0\ 0.01 \times 2^{(0+1-3)} = 1/16$

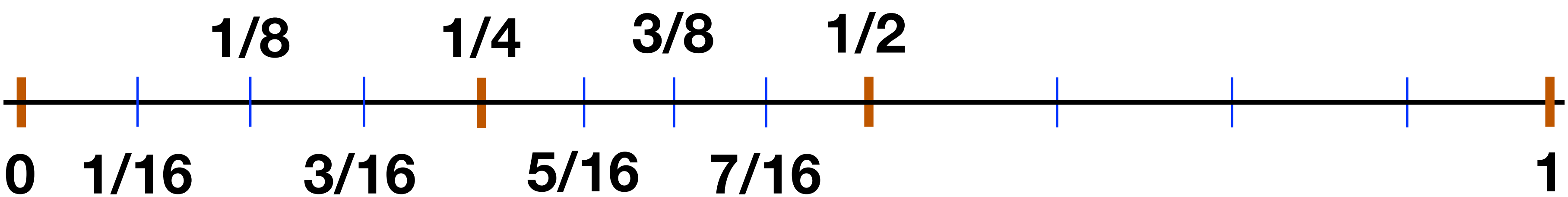
Subnormal (De-normalized) Numbers

$v = (-1)^s M 2^E$

s	exp	frac

E	exp	E	exp
$\boxed{-3}$	$\boxed{000}$	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- Idea: Evenly divide between 0 and 1/4 rather than exponentially decreasing **when *exp* = 0** (subnormal/denormalized numbers)
- $E = (exp + 1) - bias$ (instead of *exp* - *bias*)
- $M = 0.frac$ (instead of 1.frac)
- Subnormal numbers allow graceful underflow

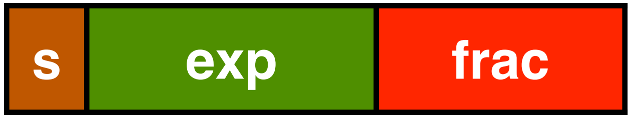


0	000	01

$= (-1)^0\ 0.01 \times 2^{(0+1-3)} = 1/16$

Special Values

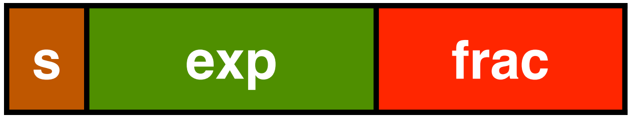
$v = (-1)^s \, M \, 2^E$



E	exp	E	exp
-2	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

Special Values

$v = (-1)^s M 2^E$



E	exp	E	exp
-2	000	1	100
-2	001	2	101
-1	010	3	110
0	011	~~4~~	~~111~~

- There are many special values in scientific computing
    - +/- ∞, Not-a-Numbers (NaNs) (e.g., 0 / 0, 0 / ∞, ∞ / ∞, sqrt(–1), ∞ - ∞, ∞ x 0, etc.)

Special Values

$v = (-1)^s M 2^E$

s	exp	frac

E	exp	E	exp
-2	000	1	100
-2	001	2	101
-1	010	3	110
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- exp = 111 is reserved to represent these numbers

Special Values

$v = (-1)^s \, M \, 2^E$

s	exp	frac

E	exp	E	exp
-2	000	1	100
-2	001	2	101
-1	010	3	110
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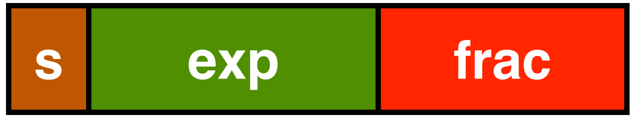
Special Values

$$v = (-1)^s \, M \, 2^E$$



E	exp	E	exp
-2	000	1	100
-2	001	2	101
-1	010	3	110
0	011		111

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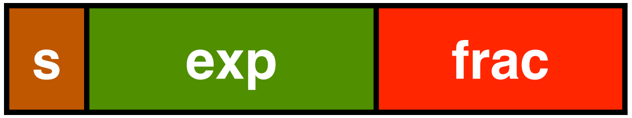
  - +/- ∞, Not-a-Numbers (NaNs) (e.g., 0 / 0, 0 / ∞, ∞ / ∞, sqrt(–1), ∞ - ∞, ∞ x 0, etc.)
- exp = 111 is reserved to represent these numbers
- exp = 111, frac = 00
  - +/- ∞ (depending on the *s* bit). Overflow results.
  - Arithmetic on ∞ is exact: 1.0/0.0 = –1.0/–0.0 = + ∞, 1.0/–0.0 = -∞

Special Values

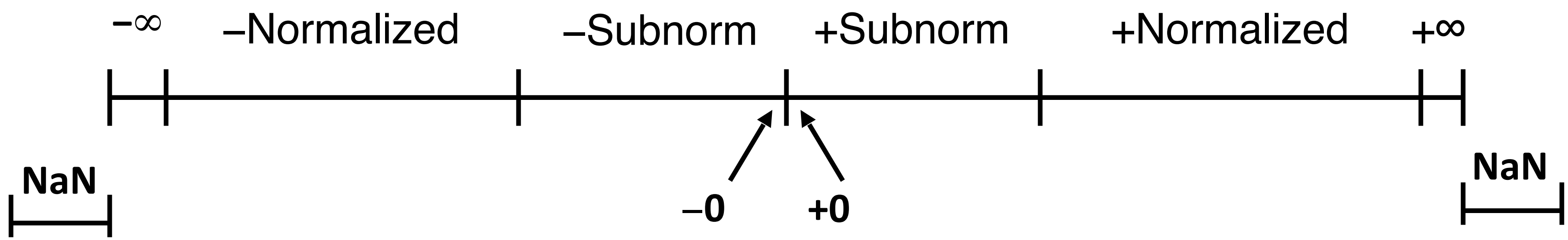
$v = (-1)^s M 2^E$



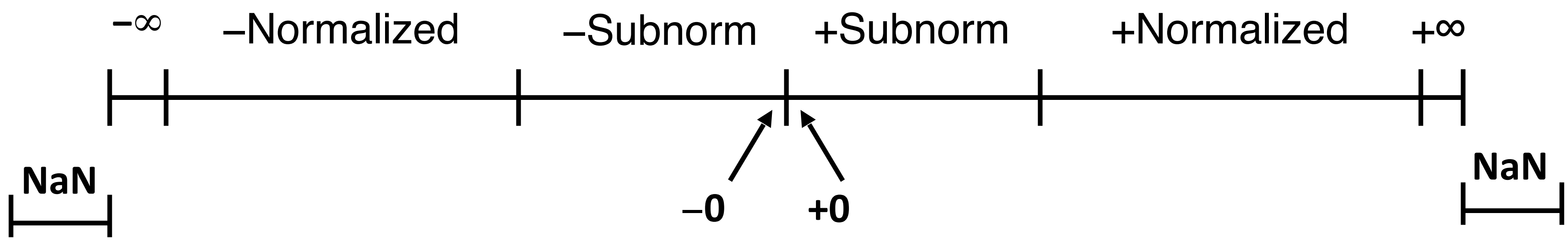
E	exp	E	exp
-2	000	1	100
-2	001	2	101
-1	010	3	110
0	011		111

- There are many special values in scientific computing
  - +/- ∞, Not-a-Numbers (NaNs) (e.g., 0 / 0, 0 / ∞, ∞ / ∞, sqrt(–1), ∞ - ∞, ∞ x 0, etc.)
- exp = 111 is reserved to represent these numbers
- exp = 111, frac = 00
  - +/- ∞ (depending on the *s* bit). Overflow results.
  - Arithmetic on ∞ is exact: 1.0/0.0 = –1.0/–0.0 = + ∞, 1.0/–0.0 = -∞
- exp = 111, frac != 00
  - Represent NaNs

Visualization: Floating Point Encodings

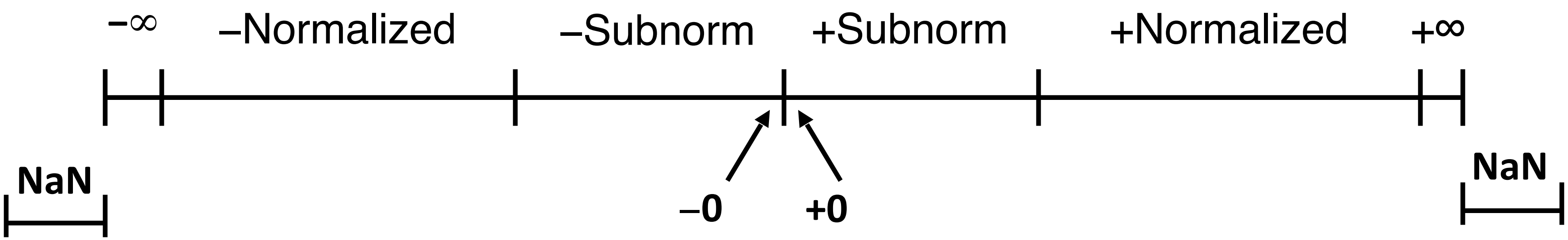


Visualization: Floating Point Encodings

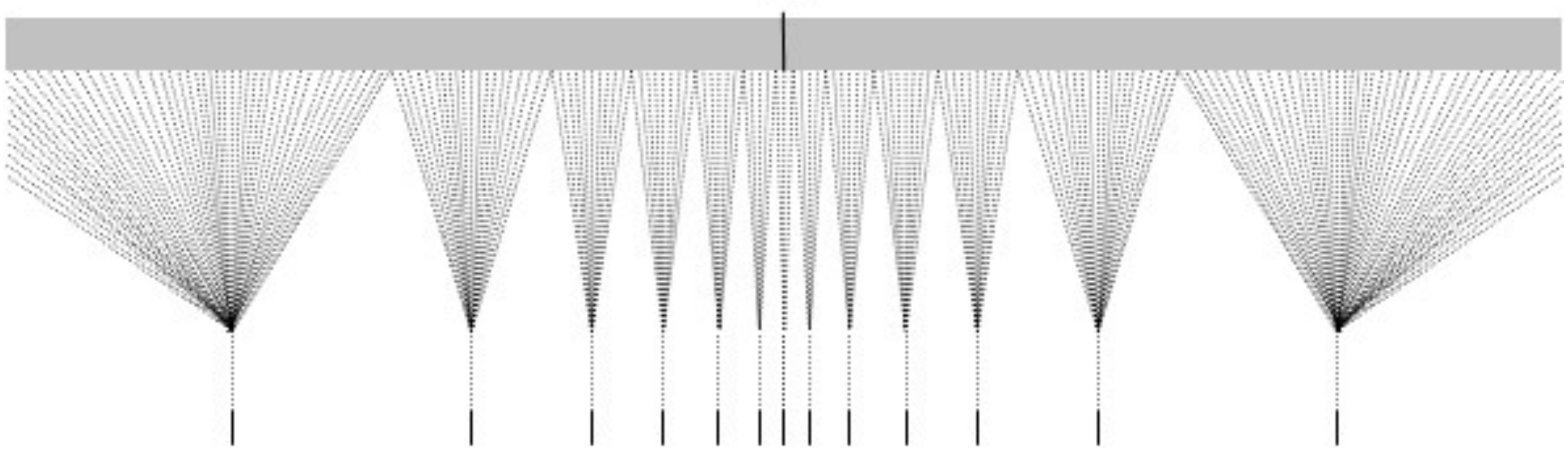


Infinite Amount of Real Numbers

Visualization: Floating Point Encodings

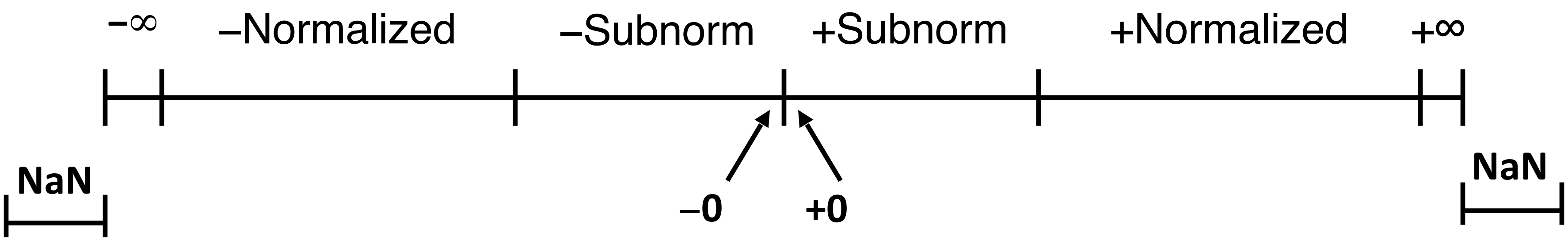


Infinite Amount of Real Numbers



Finite Amount of Floating Point Numbers

Visualization: Floating Point Encodings



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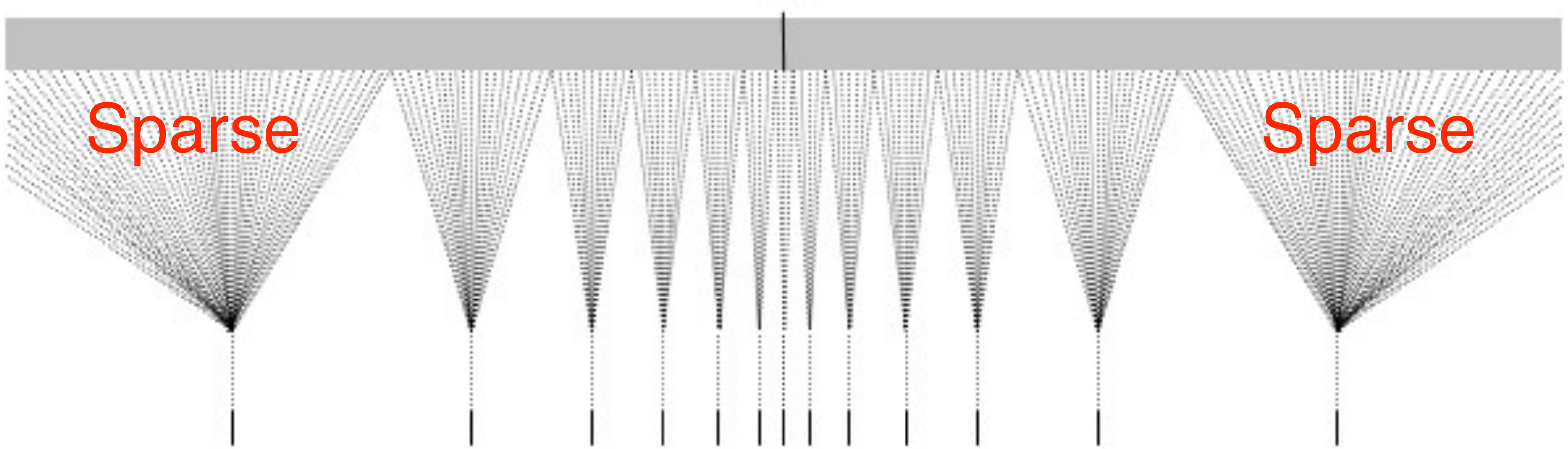

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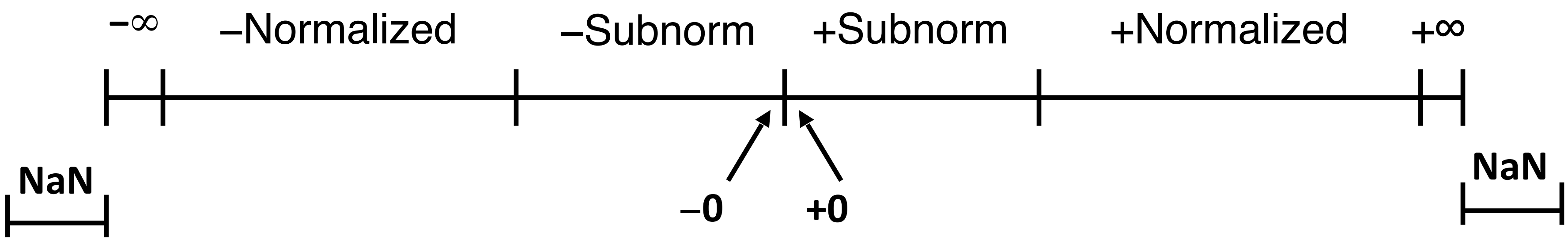

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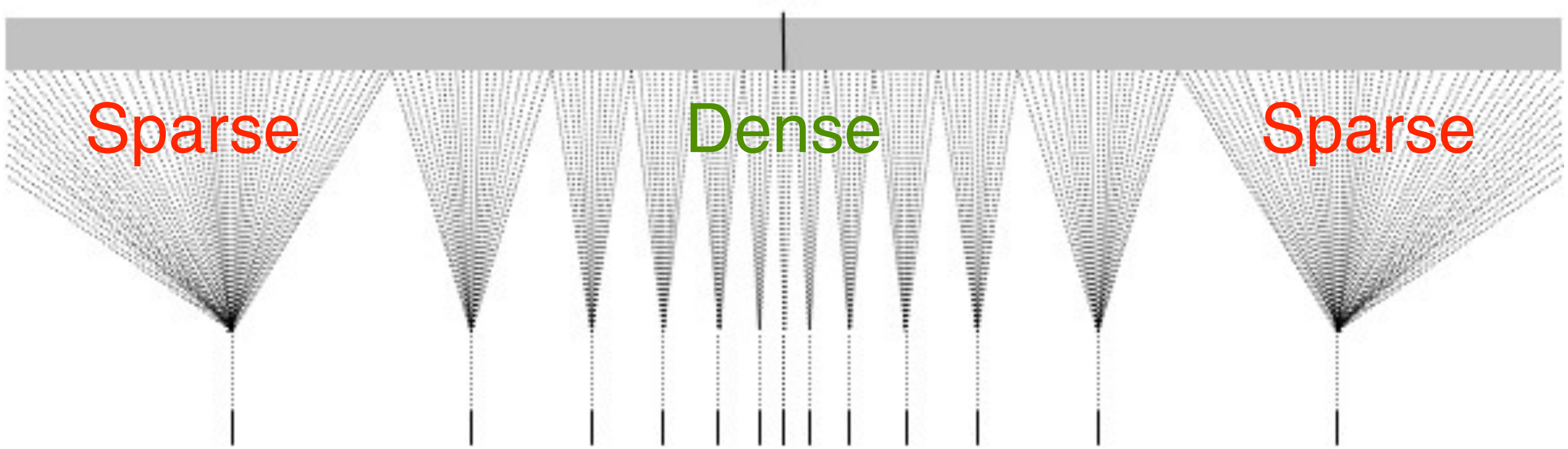

Finite Amount of Floating Point Numbers

Today: Floating Point

- Background: Fractional binary numbers and fixed-point
- Floating point representation
- IEEE 754 standard
- Rounding, addition, multiplication
- Floating point in C
- Summary

IEEE 754 Floating Point Standard

- Single precision: 32 bits

- Double precision: 64 bits

IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs (and even GPUs and other processors)

- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Single Precision (32-bit) Example

$v = (-1)^s \, M \, 2^E$

$\text{bias} = 2^{(8-1)}-1 = 127$

s	exp	frac
1	8-bit	23-bit

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Floating Point Computations

- The problem: Computing on floating point numbers might produce a result that can't be precisely represented
- Basic idea
  - We perform the operation & produce the infinitely precise result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into frac

Rounding Modes (Decimal)

- Common ones:
  - Towards zero (chop)
  - Round down ($-\infty$)
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Rounding Mode	**1.40**	**1.60**	**1.50**	**2.50**	**-1.50**
Towards zero	1	1	1	2	-1
Round down (-∞)	1	1	1	2	-2
Round up (+∞)	2	2	2	3	-1
Nearest even (default)	1	2	2	2	-2

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1.000110	1.001	1.001 is the nearest (up)
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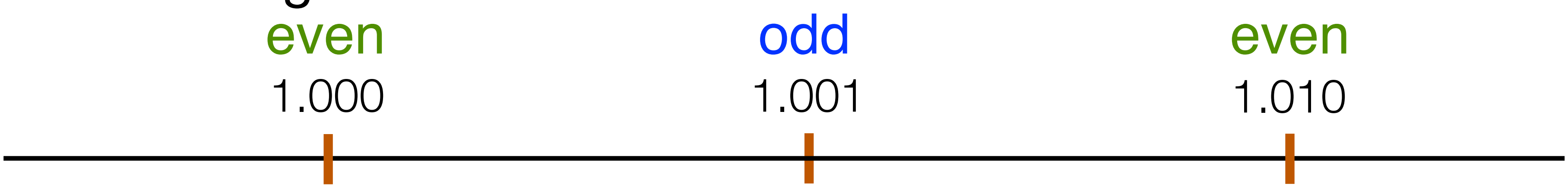

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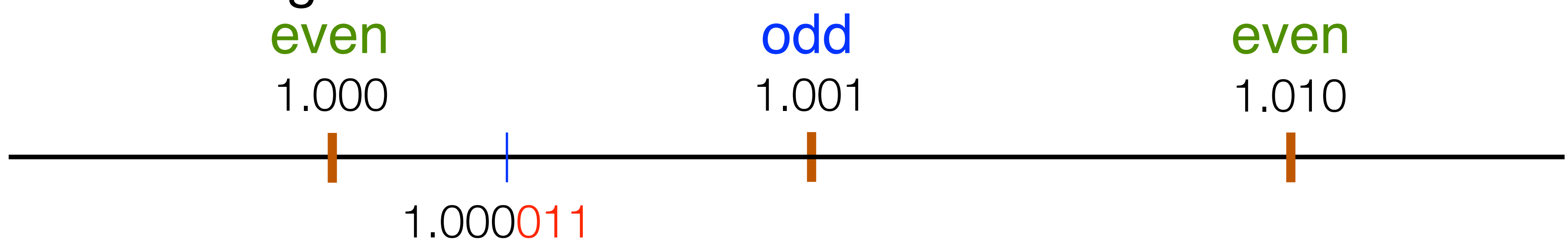

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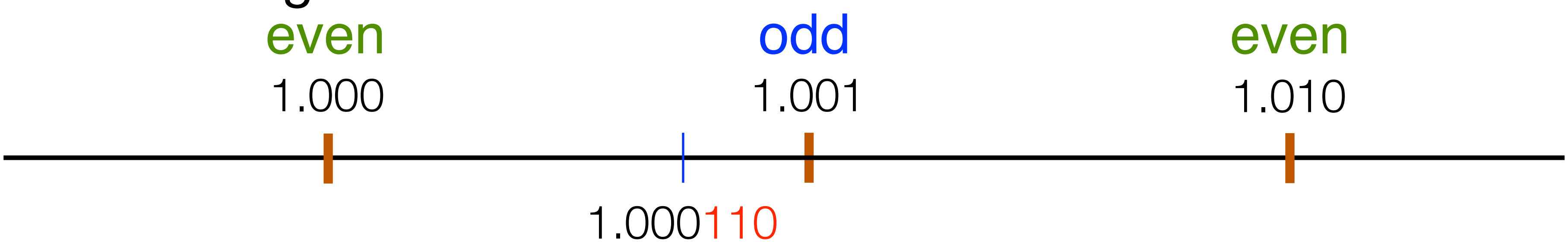

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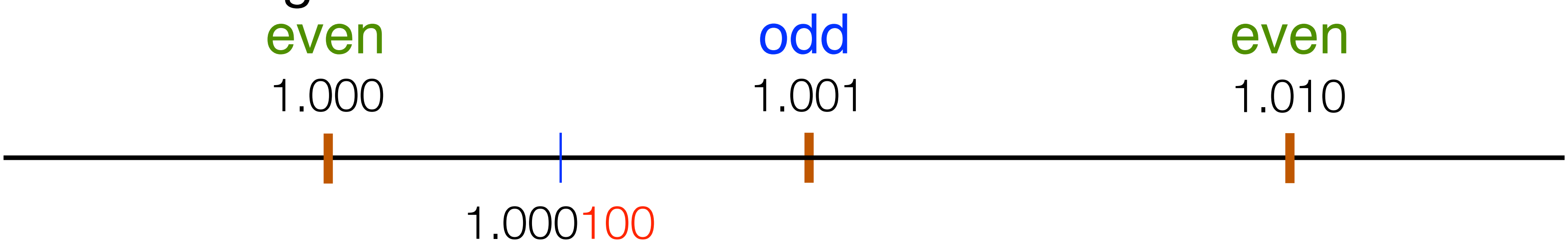

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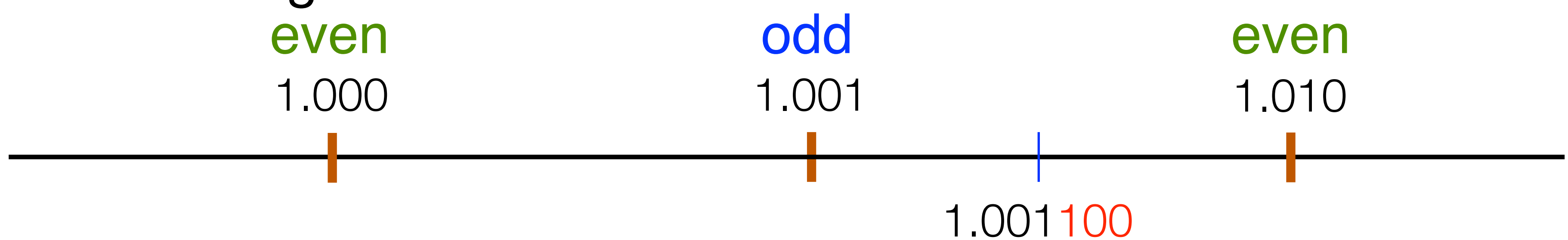

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  - If $M \geq 2$, shift M right, increment E
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  - Overflow if E out of range
  - Round M to fit *frac* precision

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- Implementation
  - Biggest chore is multiplying significands

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