CSC 252: Computer Organization Spring 2023: Lecture 3

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Announcement

- Programming Assignment 1 is out
  - Details: https://www.cs.rochester.edu/courses/252/spring2023/labs/assignment1.html
  - Due on Jan. 27, 11:59 PM
  - You have 3 slip days

15	16	17	18 Today	19	20	21
22	23	24	25	26	27 Due	28

Announcement

- Programming assignment 1 is in C language. Seek help from TAs.
- TAs are best positioned to answer your questions about programming assignments!!!
- Programming assignments do NOT repeat the lecture materials. They ask you to synthesize what you have learned from the lectures and work out something new.

Encoding Negative Numbers

- Solution 2: Two's Complement

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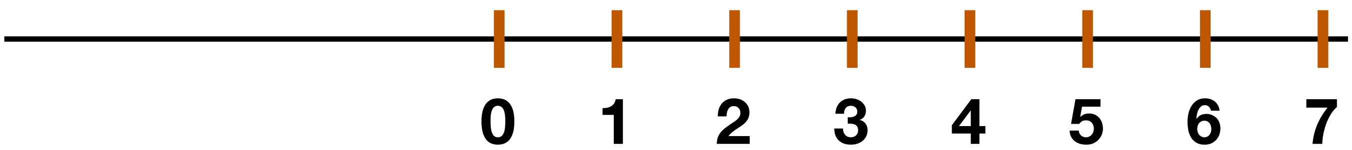
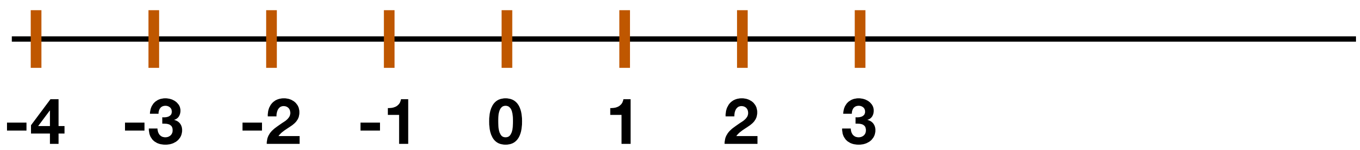


Unsigned	Binary
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Encoding Negative Numbers

- Solution 2: Two's Complement



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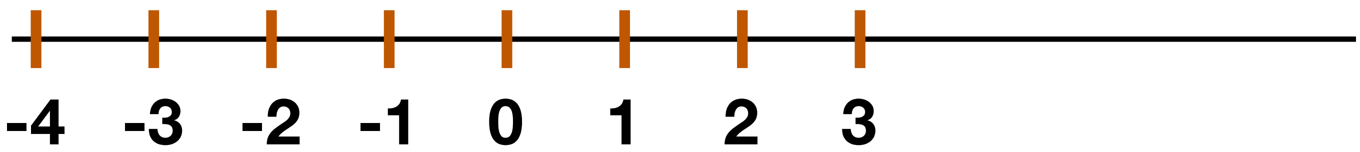

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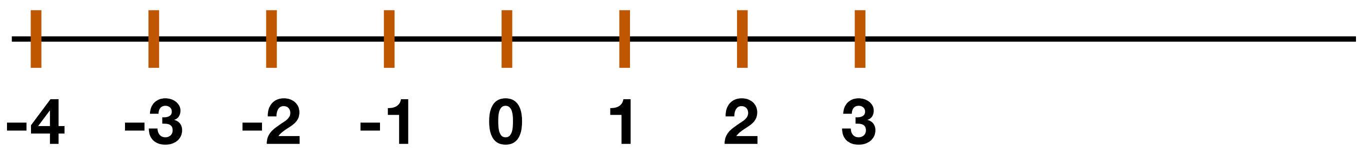

Signed Weight	Unsigned Weight	Bit Position
2^0	2^0	0
2^1	2^1	1
-2^2	2^2	2

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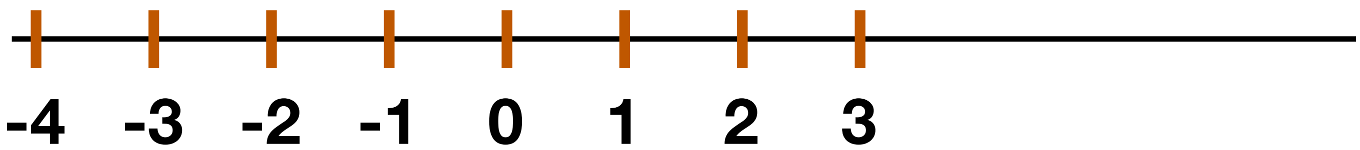

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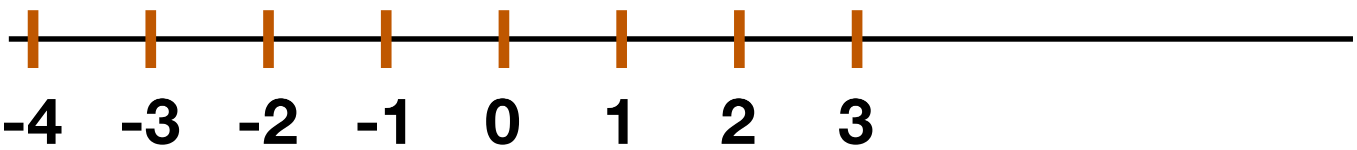

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$$101_2 = 1*2^0 + 0*2^1 - 1*2^2 = -3_{10}$$

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0	0	000
1	1	001
2	2	010
3	3	011
-4	4	100
-3	5	101
-2	6	110
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Two-Complement Encoding Example

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Two-Complement Implications

- Only 1 zero
- Usual arithmetic still works
- There is a bit that represents the sign!
- Most widely used in today's machines

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0	000
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```
    010
+)  101
-------
    111
```

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-1	111

```
    010
+)  101
-------
    111
```

```
      2
+)   -3
-------
     -1
```

Numeric Ranges

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- Unsigned Values
  - *UMin* = 0

    000…0
  - *UMax* = $2^w - 1$

    111…1

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  - *TMin* = -2^{w-1}

    100…0
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    011…1

Values for *W* = 16

	Decimal	Hex	Binary
UMax	**65535**	**FF FF**	11111111 11111111
TMax	**32767**	7F FF	01111111 11111111
TMin	**-32768**	80 00	10000000 00000000
-1	**-1**	**FF FF**	11111111 11111111
0	**0**	00 00	00000000 00000000

Numeric Ranges

- Unsigned Values
  - *UMin* = 0
    000…0
  - *UMax* = $2^w - 1$
    111…1
- Two's Complement Values
  - *TMin* = -2^{w-1}
    100…0
  - *TMax* = $2^{w-1} - 1$
    011…1
- Other Values
  - Minus 1
    111…1

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-1	**-1**	**FF FF**	**11111111 11111111**
0	**0**	**00 00**	**00000000 00000000**

Data Representations in C (in Bytes)

- By default variables are signed
- Unless explicitly declared as unsigned (e.g., `unsigned int`)
- Signed variables use two-complement encoding

C Data Type	32-bit	64-bit
`char`	1	1
`short`	2	2
`int`	4	4
`long`	4	8

Data Representations in C (in Bytes)

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

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- C Language
  - `#include <limits.h>`
  - Declares constants, e.g.,
    - `ULONG_MAX`
    - `LONG_MAX`
    - `LONG_MIN`
  - Values platform specific

Can We Represent Fractions in Binary?

- What does 10.01_2 mean?
- C.f., Decimal
  - $12.45 = 1*10^1 + 2*10^0 + 4*10^{-1} + 5*10^{-2}$
- $10.01_2 = 1*2^1 + 0*2^0 + 0*2^{-1} + 1*2^{-2} = 2.25_{10}$

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Decimal	Binary
0	00.00
0.25	00.01
0.5	00.10
0.75	00.11
1	01.00
1.25	01.01
1.5	01.10
1.75	01.11
2	10.00
2.25	10.01
2.5	10.10
2.75	10.11
3	11.00
3.25	11.01
3.5	11.10
3.75	11.11

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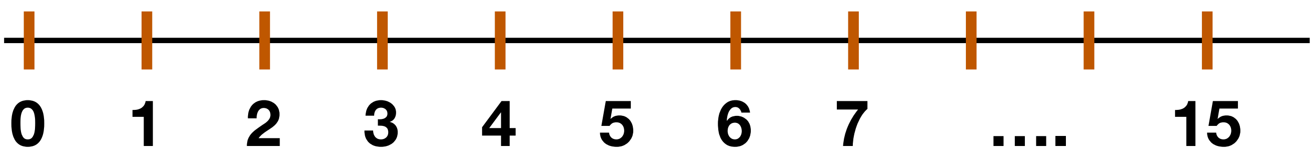

Decimal	Binary
0	00.00
0.25	00.01
0.5	00.10
0.75	00.11
1	01.00
1.25	01.01
1.5	01.10
1.75	01.11
2	10.00
2.25	10.01
2.5	10.10
2.75	10.11
3	11.00
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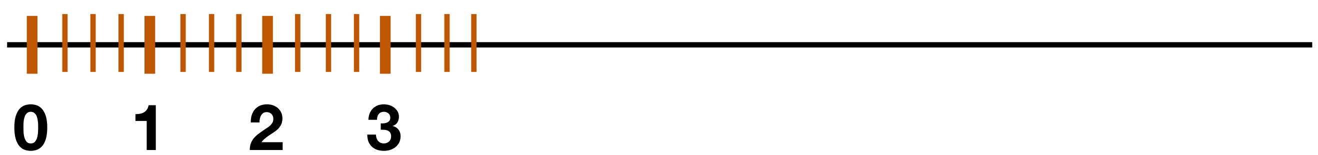

Decimal	Binary
0	00.00
0.25	00.01
0.5	00.10
0.75	00.11
1	01.00
1.25	01.01
1.5	01.10
1.75	01.11
2	10.00
2.25	10.01
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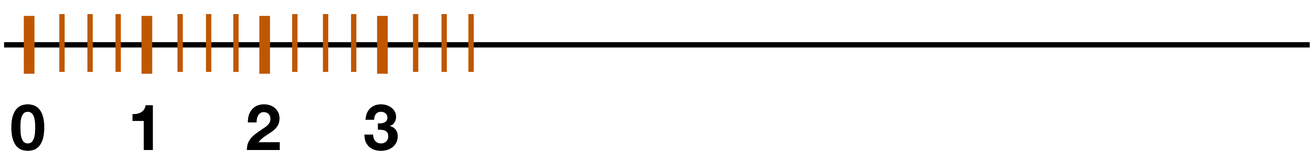


```
   01.10
+  01.01
--------
   10.11
```

```
   1.50
+  1.25
-------
   2.75
```

Decimal	Binary
0	00.00
0.25	00.01
0.5	00.10
0.75	00.11
1	01.00
1.25	01.01
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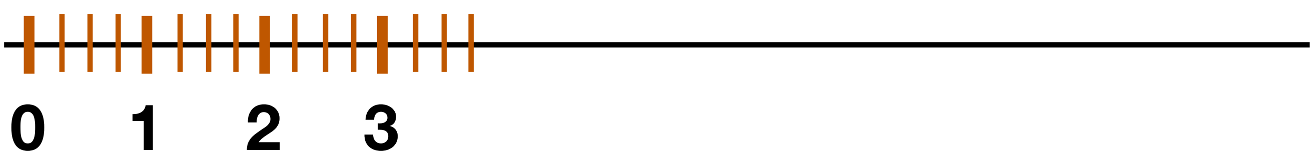

Integer Arithmetic Still Works!

```
  01.10
+ 01.01
-------
  10.11
```

```
  1.50
+ 1.25
------
  2.75
```

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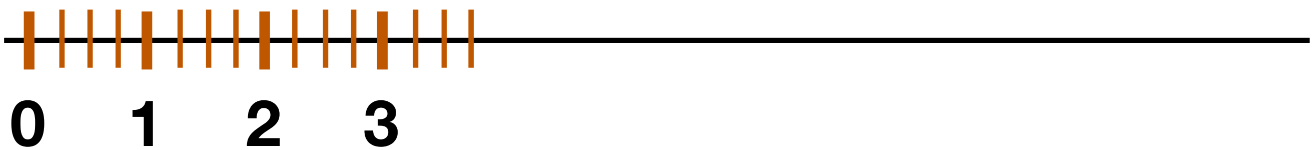
Fixed-Point Representation

- Fixed interval between two representable numbers as long as the binary point stays fixed
  - The interval is 0.25_{10} here
- Fixed-point representation of numbers
  - Integer is one special case of fixed-point

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Today: Representing Information in Binary

- Why Binary (bits)?
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting

One Bit Sequence, Two Interpretations

- A sequence of bits can be interpreted as either a signed integer or an unsigned integer

Signed	Unsigned	Binary
0	0	000
1	1	001
2	2	010
3	3	011
-4	4	100
-3	5	101
-2	6	110
-1	7	111

Signed vs. Unsigned Conversion in C

- What happens when we convert between signed and unsigned numbers?
- Casting (In C terminology)
  - Explicit casting between signed & unsigned

```
int tx, ty = -4;
unsigned ux = 7, uy;
tx = (int) ux; // U2T
uy = (unsigned) ty; // T2U
```

  - Implicit casting
    - e.g., assignments, function calls

```
tx = ux;
uy = ty;
```

Mapping Between Signed & Unsigned

- Mappings between unsigned and two's complement numbers: Keep bit representations and reinterpret

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Mapping Signed ↔ Unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15

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T2U

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Signed $\rightarrow$ **T2U** $\rightarrow$ Unsigned

Signed $\leftarrow$ **U2T** $\leftarrow$ Unsigned

Mapping Signed ↔ Unsigned

Bits	Signed		Unsigned
0000	0		0
0001	1		1
0010	2		2
0011	3	=	3
0100	4		4
0101	5		5
0110	6		6
0111	7		7
1000	-8		8
1001	-7		9
1010	-6		10
1011	-5		11
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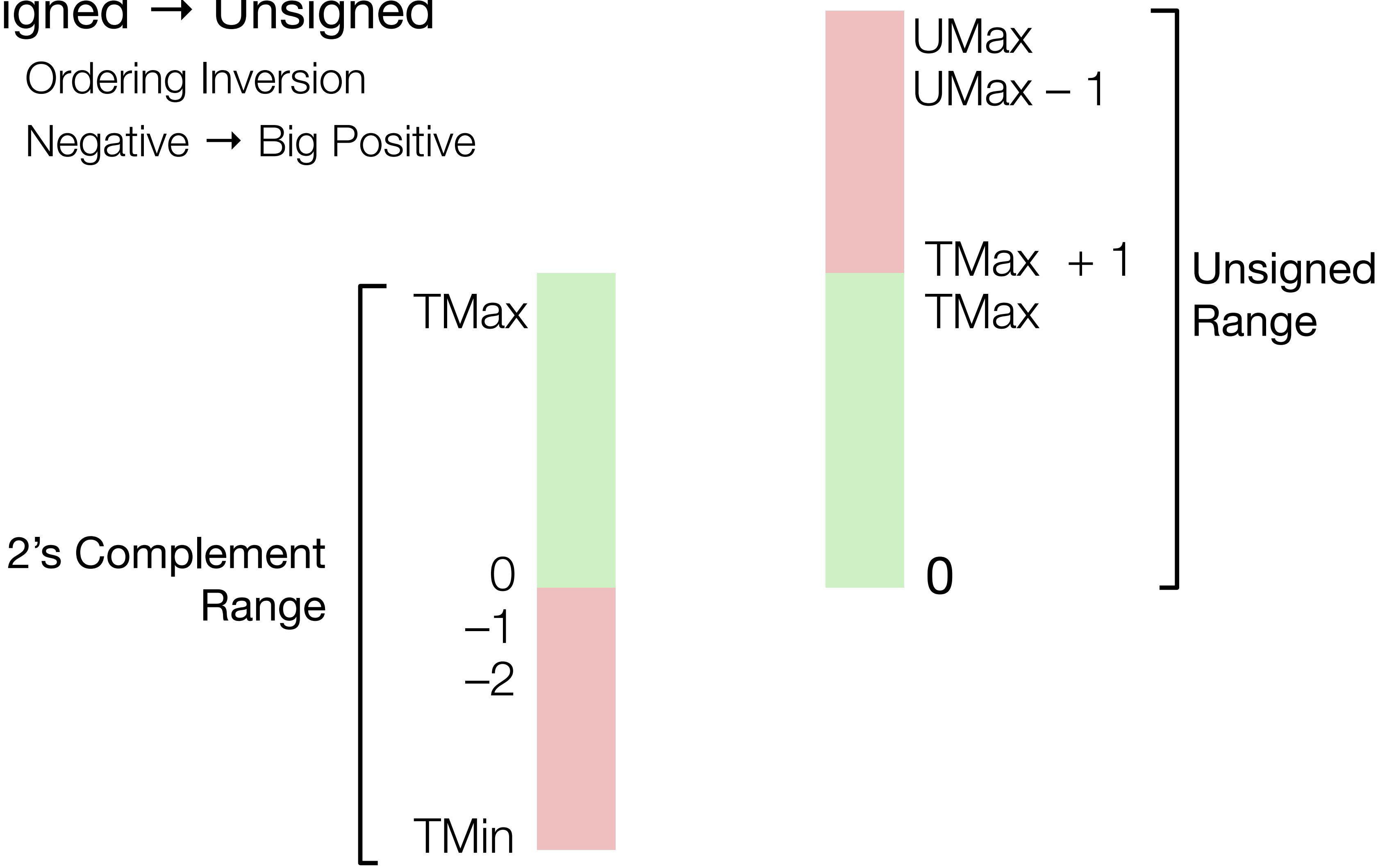
=

+/- 16

Conversion Visualized

- Signed → Unsigned
  - Ordering Inversion
  - Negative → Big Positive



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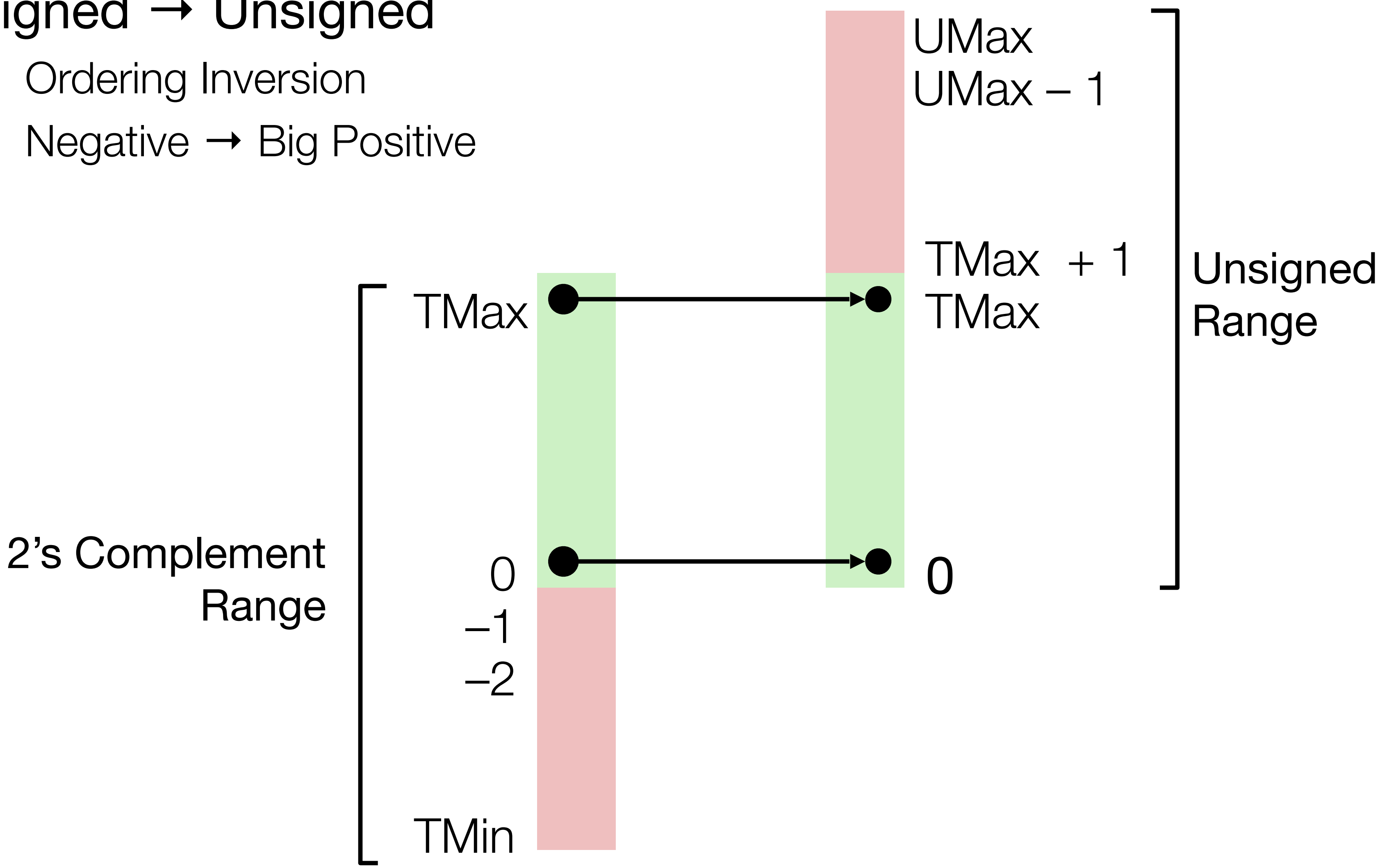

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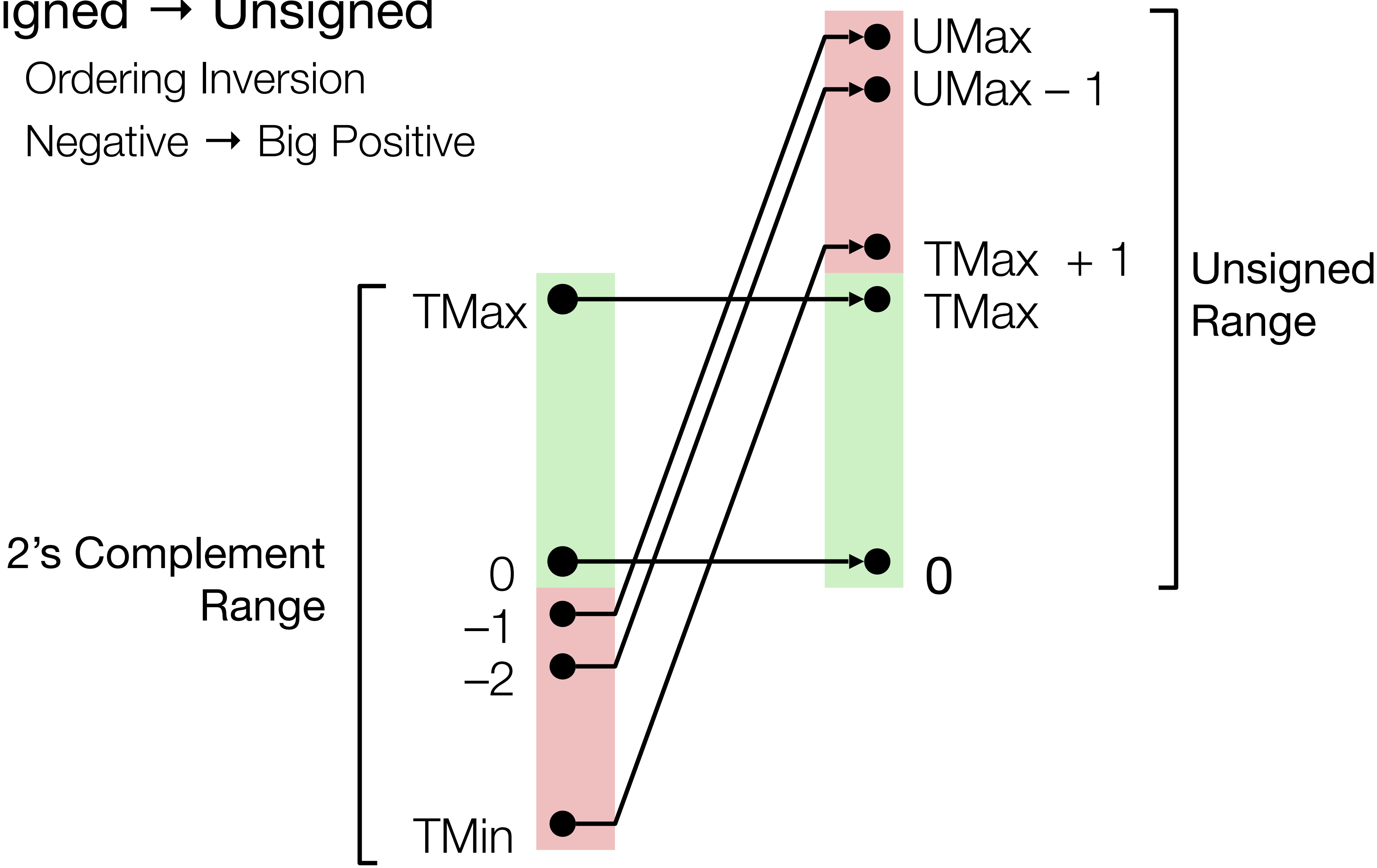

Today: Representing Information in Binary

- Why Binary (bits)?
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
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  - Addition, negation, multiplication, shifting

The Problem

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

C Data Type	# of Bytes
char	1
short	2
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- Converting from smaller to larger integer data type
- Should we preserve the value?
- Can we preserve the value?
- How?

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- Should we preserve the value?
- Can we preserve the value?
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	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

Signed Extension

- Task:
  - Given w-bit signed integer x
  - Convert it to $(w+k)$-bit integer with same value

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- Rule:
  - Make k copies of sign bit:
  - $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$

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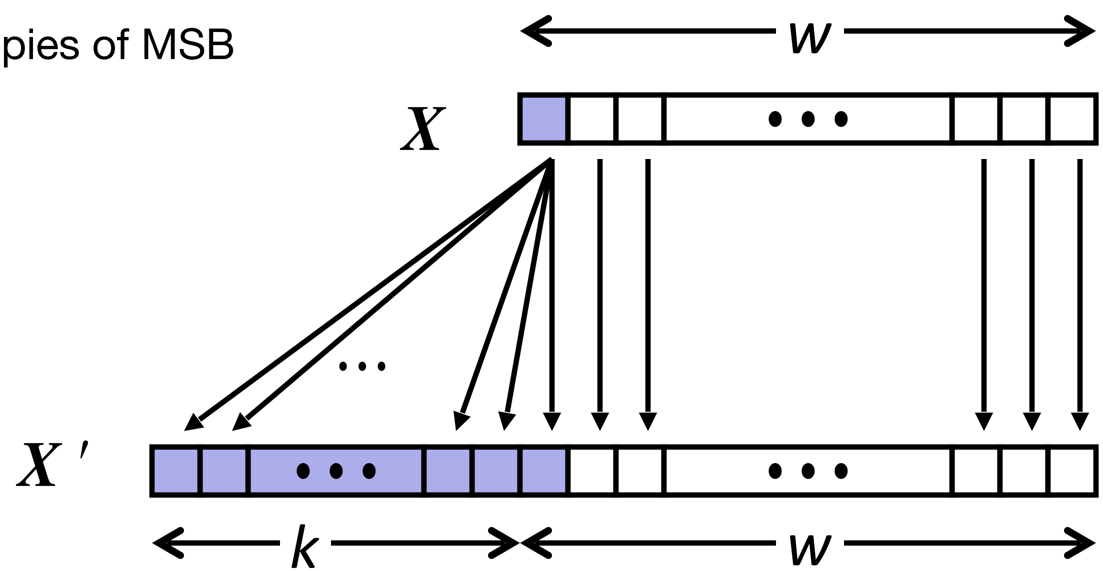


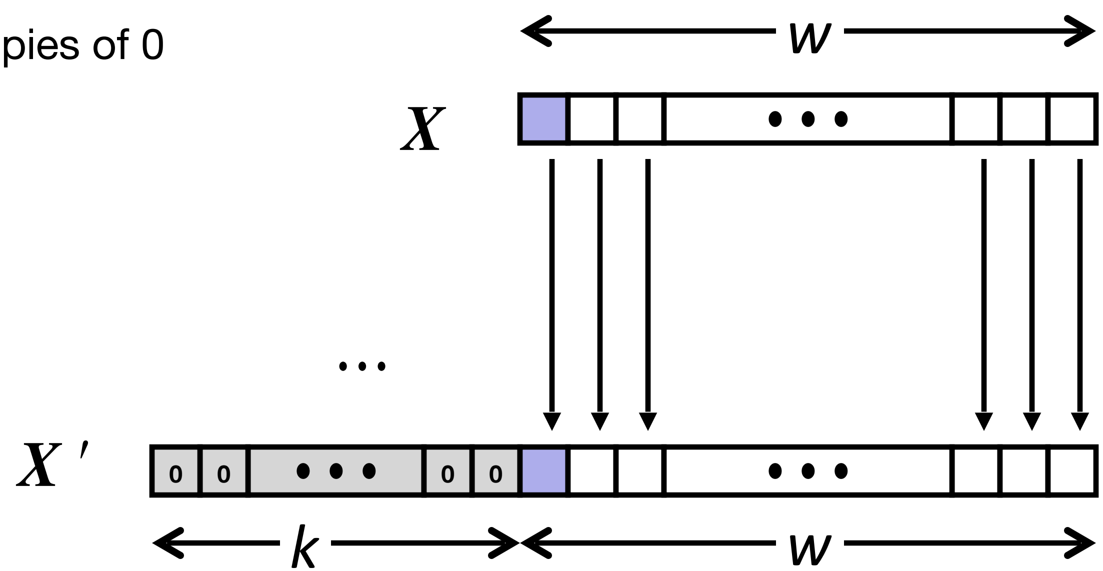
Another Problem

```
unsigned short x = 47981;
unsigned int  ux = x;
```

	Decimal	Hex	Binary
x	47981	BB 6D	10111011 01101101
ux	47981	00 00 BB 6D	00000000 00000000 10111011 01101101

Unsigned (Zero) Extension

- Task:
  - Given w-bit unsigned integer x
  - Convert it to $(w+k)$-bit integer with same value
- Rule:
  - Simply pad zeros:
  - $X' = \underbrace{0, \ldots, 0}_{k \text{ copies of } 0}, x_{w-1}, x_{w-2}, \ldots, x_0$



Yet Another Problem

```
int    x =  53191;
short sx = (short) x;
```

	Decimal	Hex	Binary
x	53191	00 00 CF C7	00000000 00000000 11001111 11000111
sx	-12345	CF C7	11001111 11000111

Yet Another Problem

```
int    x =  53191;
short sx = (short) x;
```

	Decimal	Hex	Binary
x	53191	00 00 CF C7	00000000 00000000 11001111 11000111
sx	-12345	CF C7	11001111 11000111

- Truncating (e.g., int to short)
  - C's implementation: leading bits are truncated, results reinterpreted
  - So can't always preserve the numerical value

Today: Representing Information in Binary

- Why Binary (bits)?
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting

Unsigned Addition

Unsigned	Binary
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Unsigned Addition

- Similar to Decimal Addition

Unsigned	**Binary**
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Unsigned Addition

- Similar to Decimal Addition
- Suppose we have a new data type that is 3-bit wide (c.f., **`short`** has 16 bits)

Unsigned	Binary
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Normal Case

```
    010
+)  101
-------
    111
```

```
      2
+)    5
-------
      7
```

Unsigned Addition

- Similar to Decimal Addition
- Suppose we have a new data type that is 3-bit wide (c.f., **`short`** has 16 bits)
- Might overflow: result can't be represented within the size of the data type

Unsigned	Binary
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Normal Case

```
    010          2
+)  101     +)   5
-------     ------
    111          7
```

Overflow Case

```
    110          6
+)  101     +)   5
-------     ------
   1011         11
```

Unsigned Addition

- Similar to Decimal Addition
- Suppose we have a new data type that is 3-bit wide (c.f., **`short`** has 16 bits)
- Might overflow: result can't be represented within the size of the data type

Unsigned	Binary
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Normal Case

```
    010          2
+)  101     +)   5
-------     ------
    111          7
```

Overflow Case

```
    110          6
+)  101     +)   5
-------     ------
   1011         11    <- True Sum
```

Unsigned Addition

- Similar to Decimal Addition
- Suppose we have a new data type that is 3-bit wide (c.f., **`short`** has 16 bits)
- Might overflow: result can't be represented within the size of the data type

Unsigned	Binary
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Normal Case

```
    010          2
+)  101      +)  5
--------     -------
    111          7
```

Overflow Case

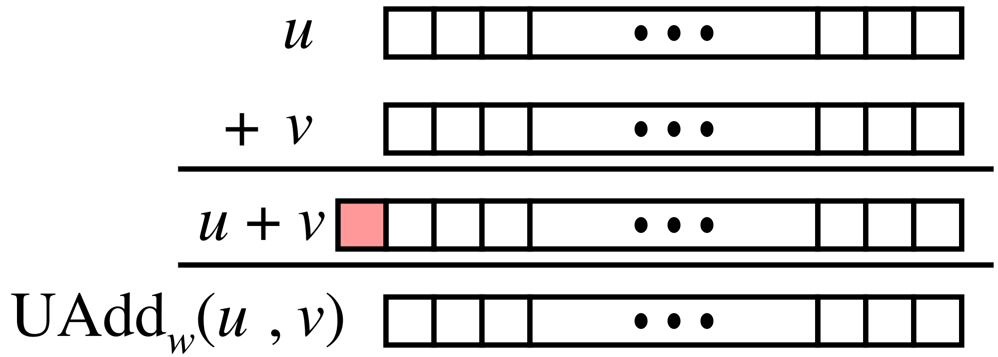
```
    110          6
+)  101      +)  5
--------     -------
   1011         11     ← True Sum
    011          3     ← Sum with same bits
```

Unsigned Addition in C

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits

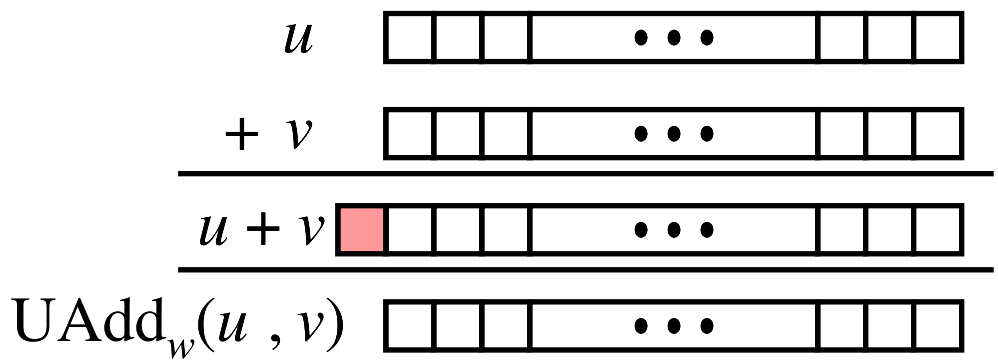


Unsigned Addition in C

Operands: *w* bits

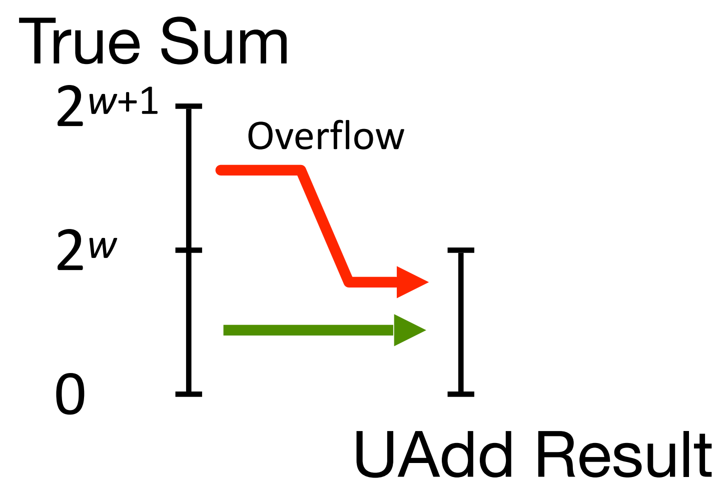
True Sum: *w*+1 bits

Discard Carry: *w* bits



- Standard Addition Function
  - Ignores carry output
- Implements Modular Arithmetic

  $s = \mathrm{UAdd}_w(u\ ,\ v) = u + v \mod 2^w$



Two's Complement Addition

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Two's Complement Addition

- Has identical bit-level behavior as unsigned addition (a big advantage over sign-magnitude)

Signed	**Binary**
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Two's Complement Addition

- Has identical bit-level behavior as unsigned addition (a big advantage over sign-magnitude)

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Normal Case

```
    010
+)  101
-------
    111
```

```
      2
+)   -3
-------
     -1
```

Two's Complement Addition

- Has identical bit-level behavior as unsigned addition (a big advantage over sign-magnitude)
- Overflow can also occur

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Normal Case

```
    010          2
+)  101     +)  -3
-------     ------
    111         -1
```

Overflow Case

```
    110         -2
+)  101     +)  -3
-------     ------
   1011         -5
```

Two's Complement Addition

- Has identical bit-level behavior as unsigned addition (a big advantage over sign-magnitude)
- Overflow can also occur

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Normal Case

```
    010          2
+)  101      +) -3
-------      -----
    111         -1
```

Overflow Case

```
    110         -2
+)  101      +) -3
-------      -----
   1011         -5
    011          3
```

Two's Complement Addition

- Has identical bit-level behavior as unsigned addition (a big advantage over sign-magnitude)
- Overflow can also occur

Min →

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Normal Case

```
    010          2
+)  101     +)  -3
-------     -------
    111         -1
```

Overflow Case

```
    110         -2
+)  101     +)  -3
-------     -------
   1011         -5
    011          3
```

Negative Overflow

Two's Complement Addition

- Has identical bit-level behavior as unsigned addition (a big advantage over sign-magnitude)
- Overflow can also occur

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Min → -4

Normal Case

```
    010          2
+)  101      +) -3
-------      ------
    111         -1
```

Overflow Case

```
    110         -2          011          3
+)  101      +) -3      +)  001      +)  1
-------      ------     -------      -----
   1011         -5         0100          4
    011          3
```

Negative Overflow

Two's Complement Addition

- Has identical bit-level behavior as unsigned addition (a big advantage over sign-magnitude)
- Overflow can also occur

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Min → -4

Normal Case

```
    010          2
+)  101     +)  -3
-------     ------
    111         -1
```

Overflow Case

```
    110         -2          011          3
+)  101     +)  -3      +)  001     +)   1
-------     ------      -------     ------
   1011         -5         0100          4
    011          3          100         -4
```

Negative Overflow

Two's Complement Addition

- Has identical bit-level behavior as unsigned addition (a big advantage over sign-magnitude)
- Overflow can also occur

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Max → 3

Min → -4

Normal Case

```
    010          2
+)  101      +) -3
-------      -----
    111         -1
```

Overflow Case

```
    110         -2         011          3
+)  101     +)  -3     +)  001     +)   1
-------     ------     -------     ------
   1011         -5        0100          4
    011          3         100         -4
```

Negative Overflow

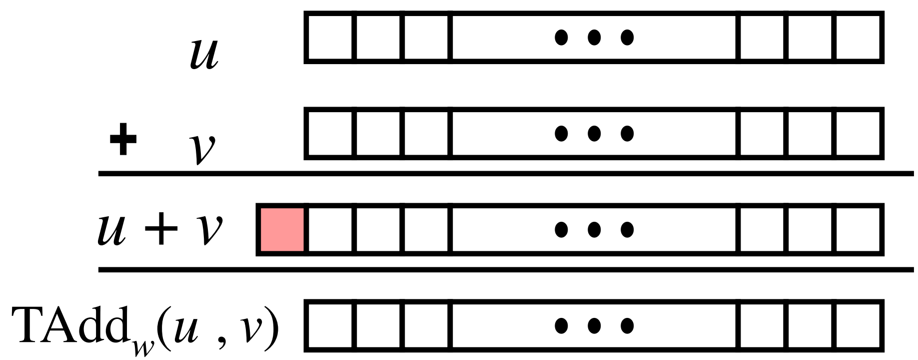
Positive Overflow

Two's Complement Addition in C

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



Is This Signed Addition an Overflow?

```
   111
+) 110
------
  1101
```

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Is This Signed Addition an Overflow?

```
   111
+) 110
------
  1101
```

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Is This Signed Addition an Overflow?

```
    111
+)  110
-------
   1101
```

Truncate

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Is This Signed Addition an Overflow?

```
    111
+)  110
-------
   1101
```

Truncate

```
     -1
+)   -2
-------
     -3
```

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

Is This Signed Addition an Overflow?

```
    111
+)  110
-------
   1101
```

Truncate

```
     -1
+)   -2
-------
     -3
```

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

- This is not an overflow by definition

Is This Signed Addition an Overflow?

```
    111
+)  110
-------
   1101
```

$\xrightarrow{\text{Truncate}}$

```
    -1
+)  -2
------
    -3
```

Signed	Binary
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

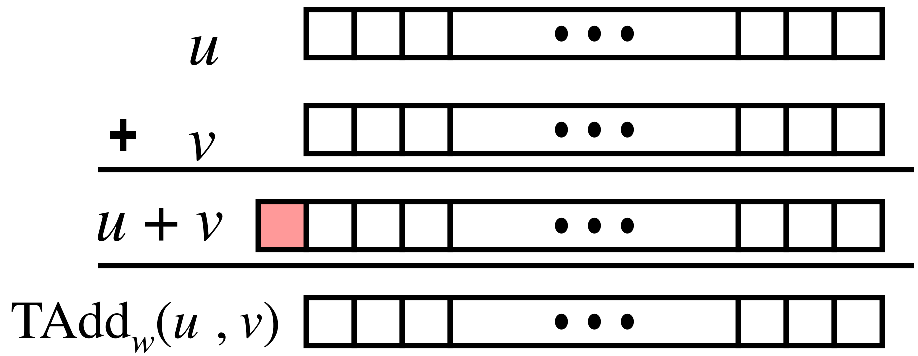
- This is not an overflow by definition
- Because the actual result can be represented using the bit width of the datatype (3 bits here)

Two’s Complement Addition in C

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits

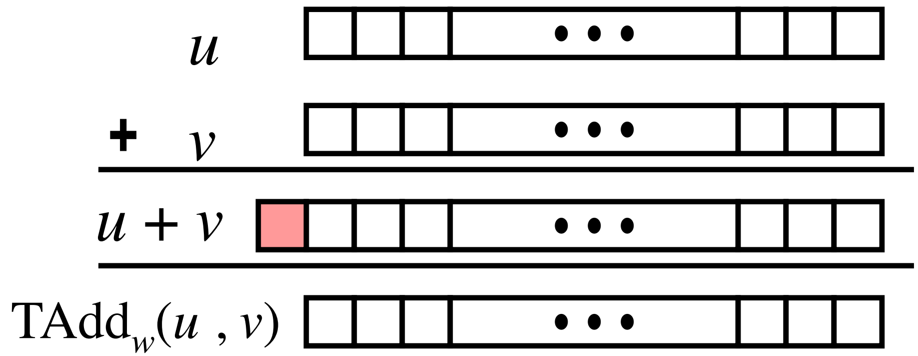


Two’s Complement Addition in C

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



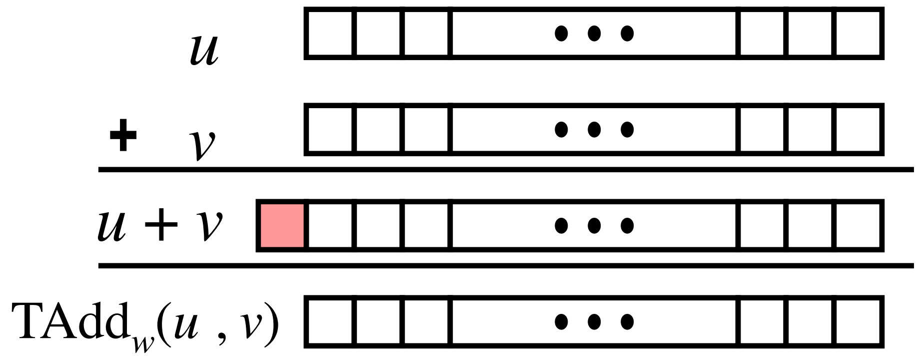
- Functionality
  - True sum requires *w*+1 bits
  - Drop off MSB
  - Treat remaining bits as 2’s comp. integer

Two's Complement Addition in C

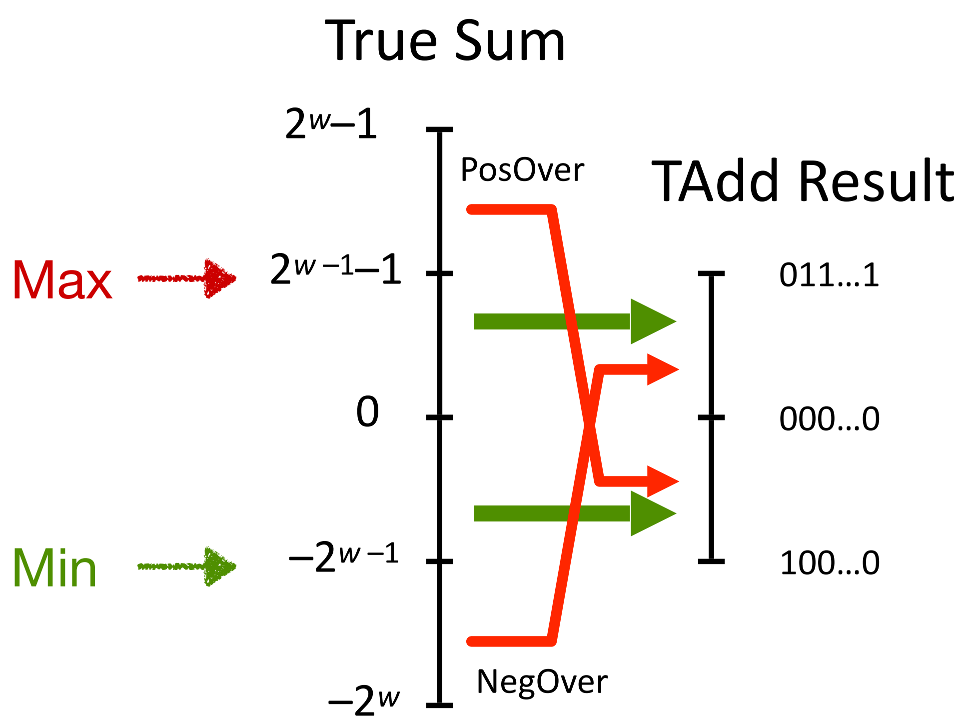
Operands: *w* bits

True Sum: *w*+1 bits

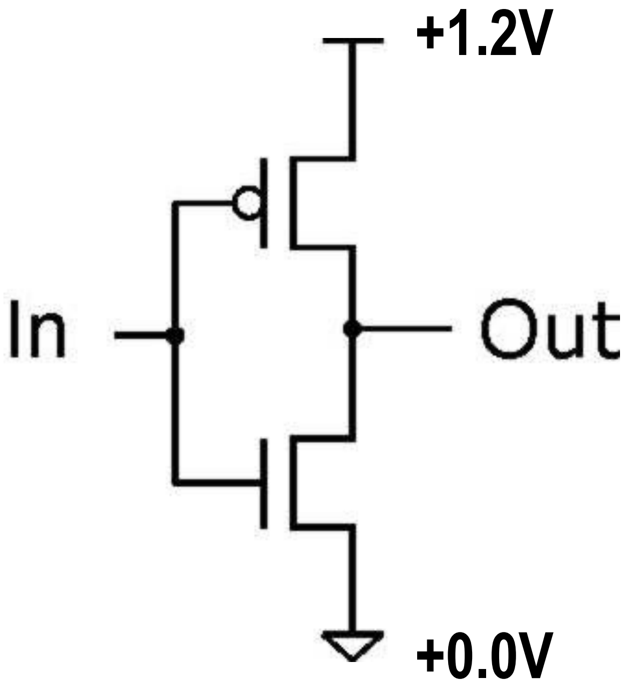
Discard Carry: *w* bits



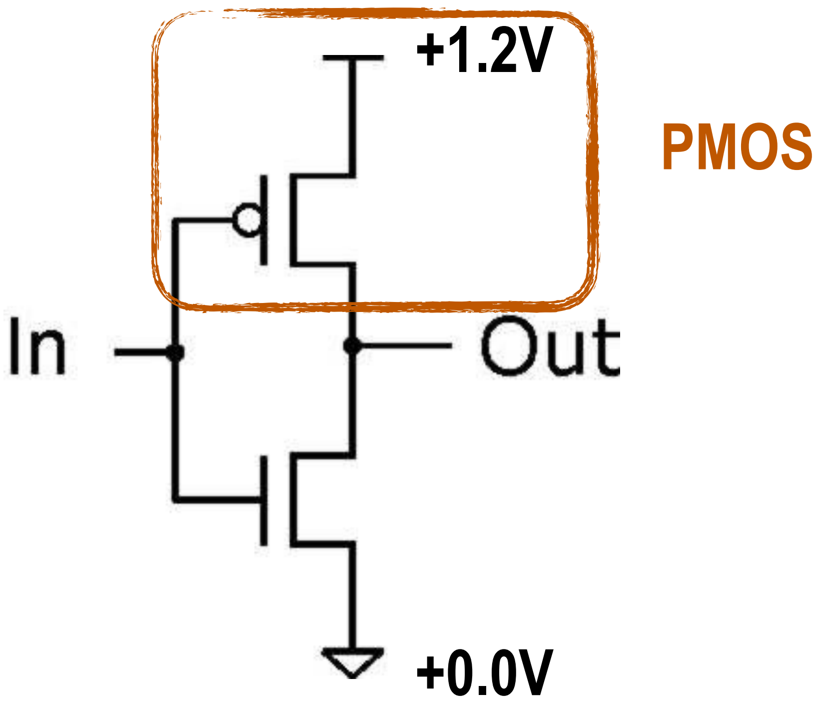
- Functionality
  - True sum requires *w*+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer



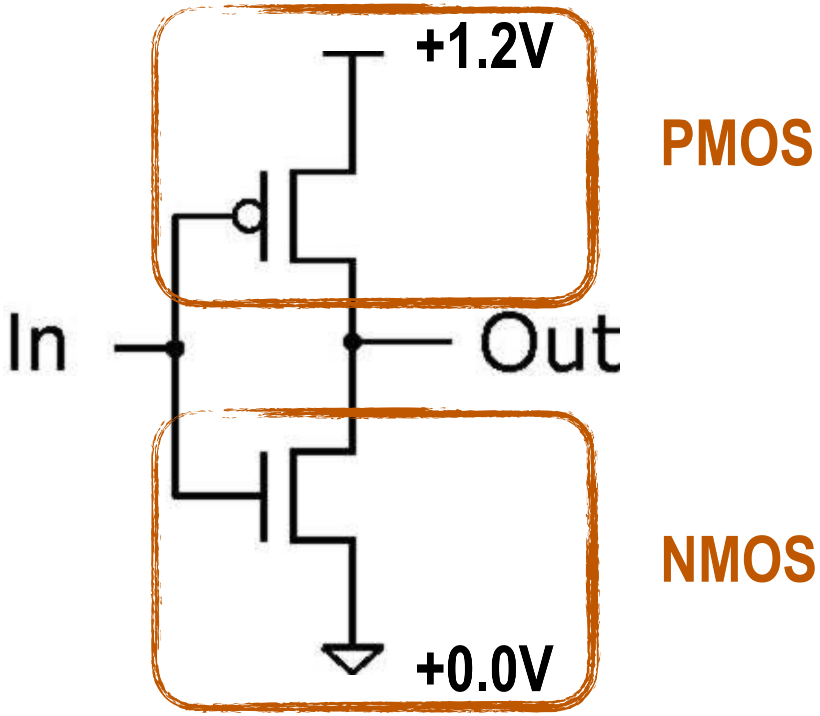
Inverter (NOT Gate)



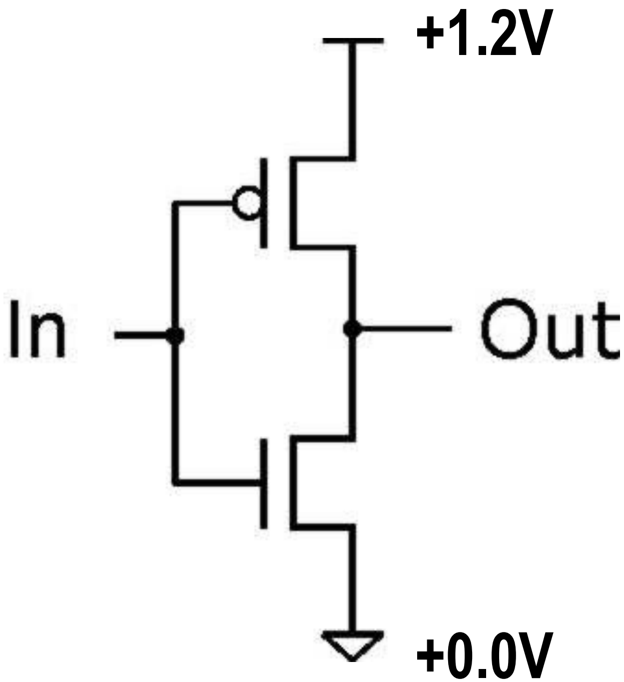
Inverter (NOT Gate)

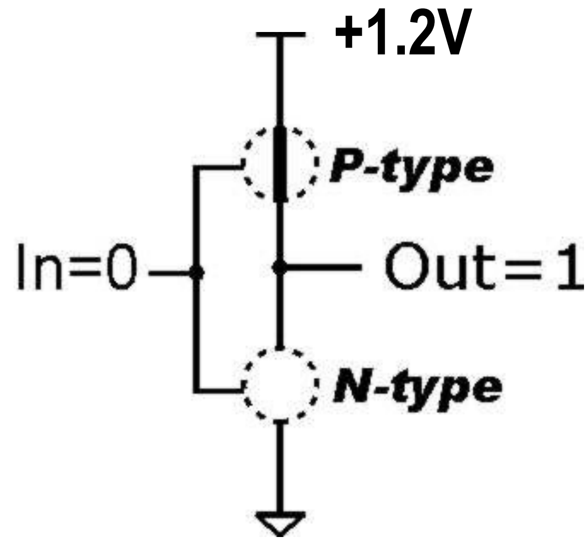


Inverter (NOT Gate)

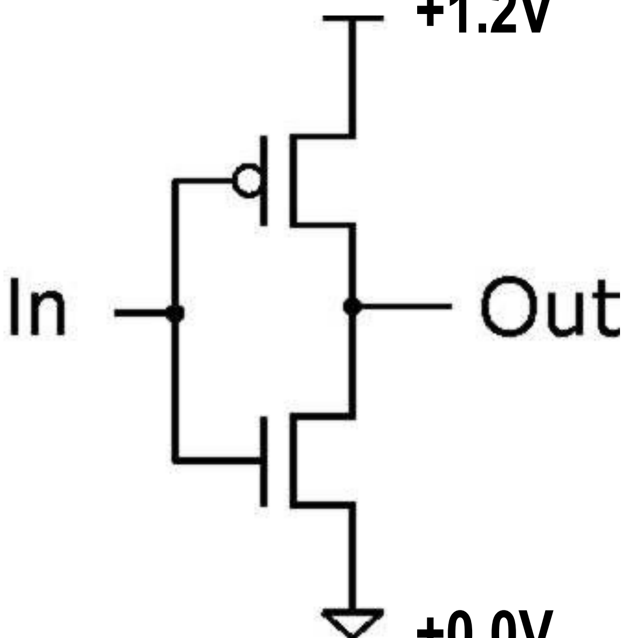


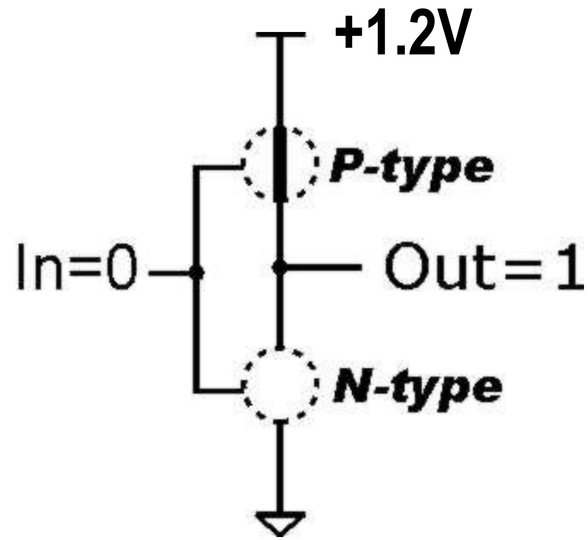
Inverter (NOT Gate)

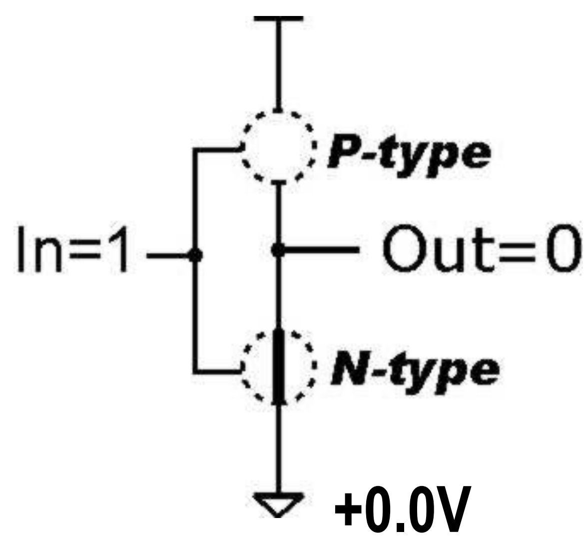




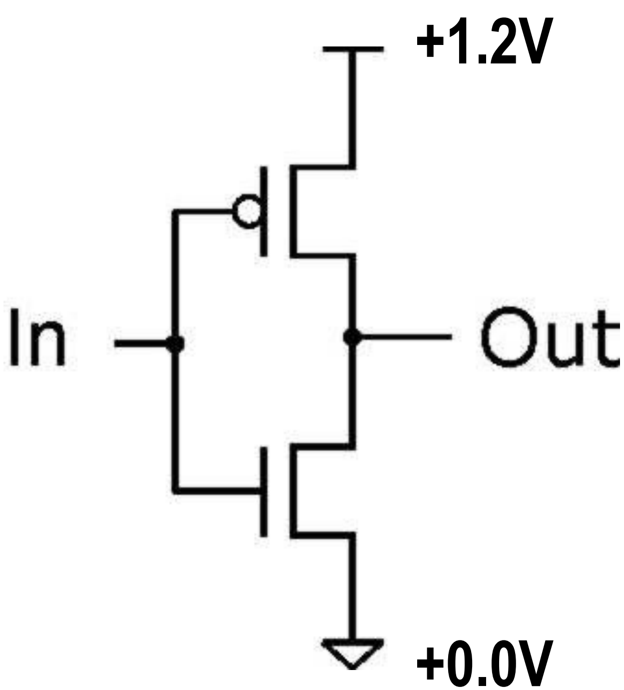
Inverter (NOT Gate)



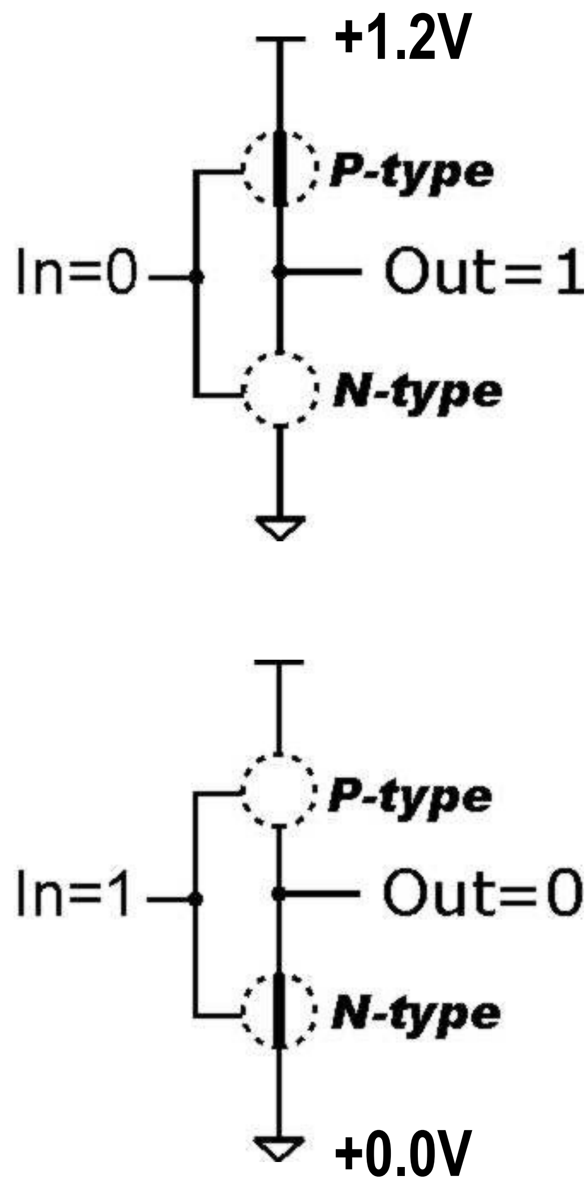




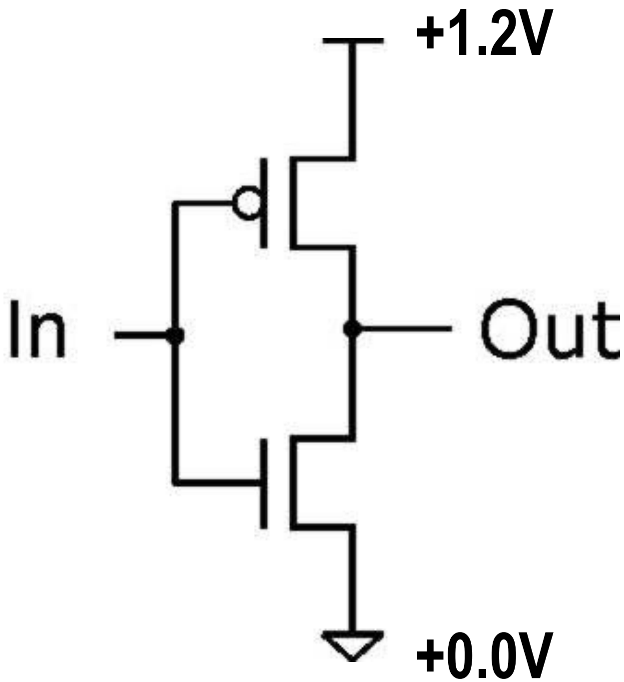
Inverter (NOT Gate)



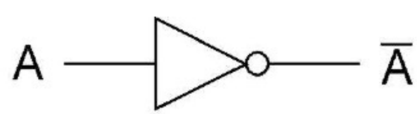
In	Out
0	1
1	0

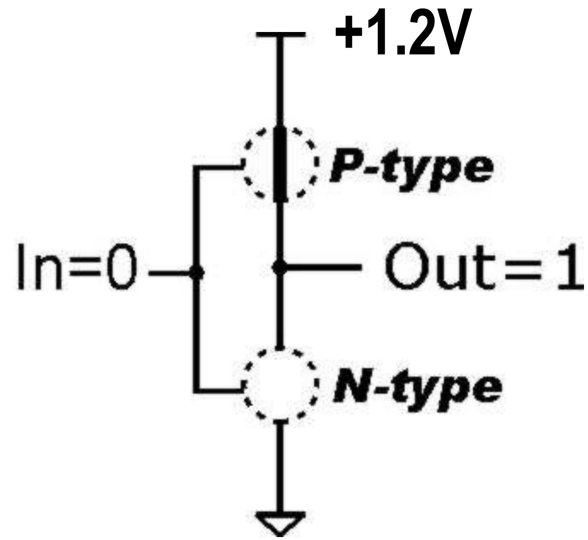


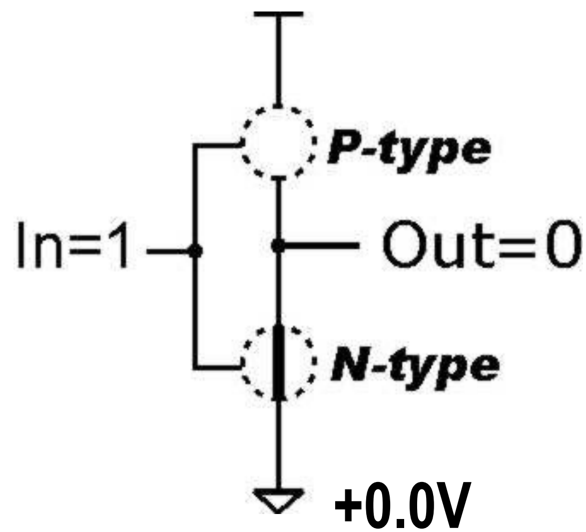
Inverter (NOT Gate)



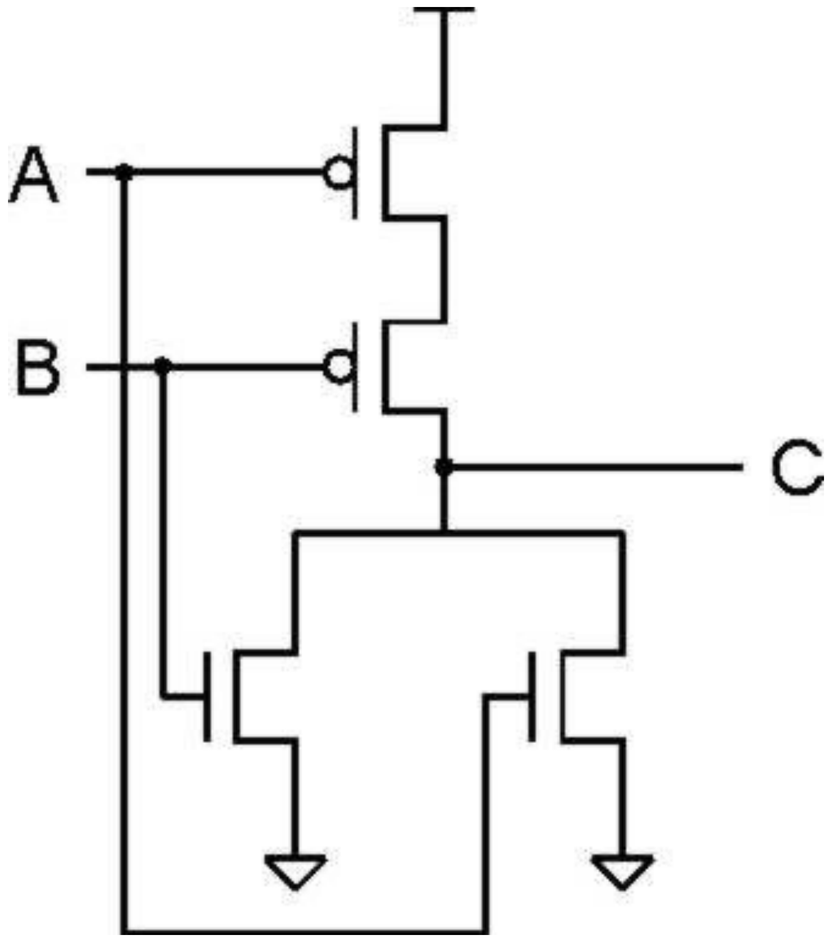
In	Out
0	1
1	0

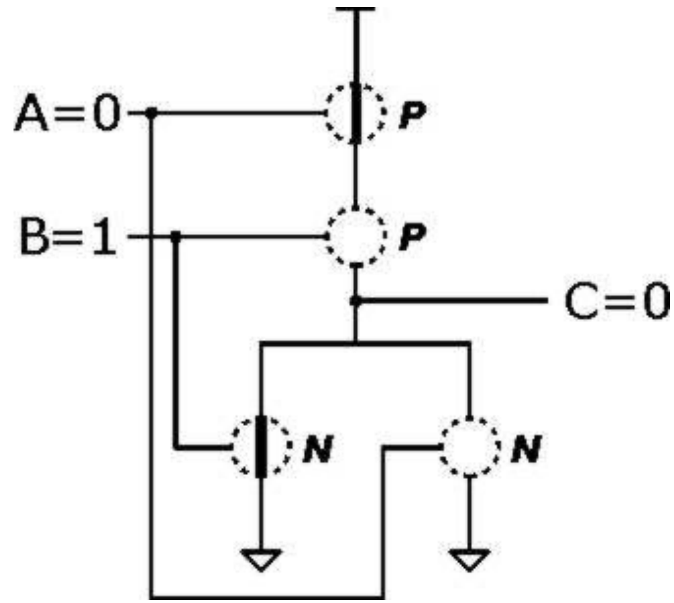






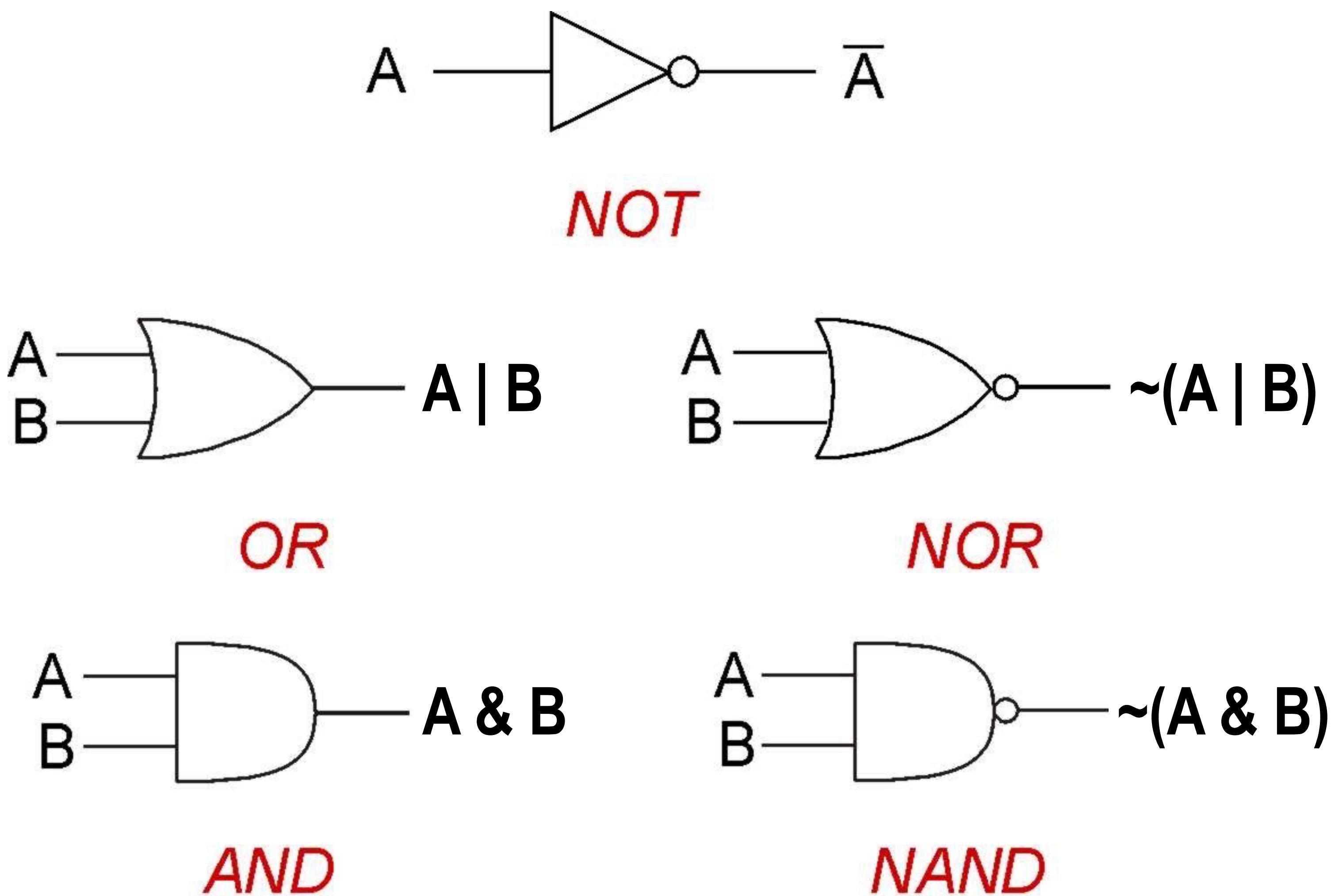
NOR Gate (NOT + OR)



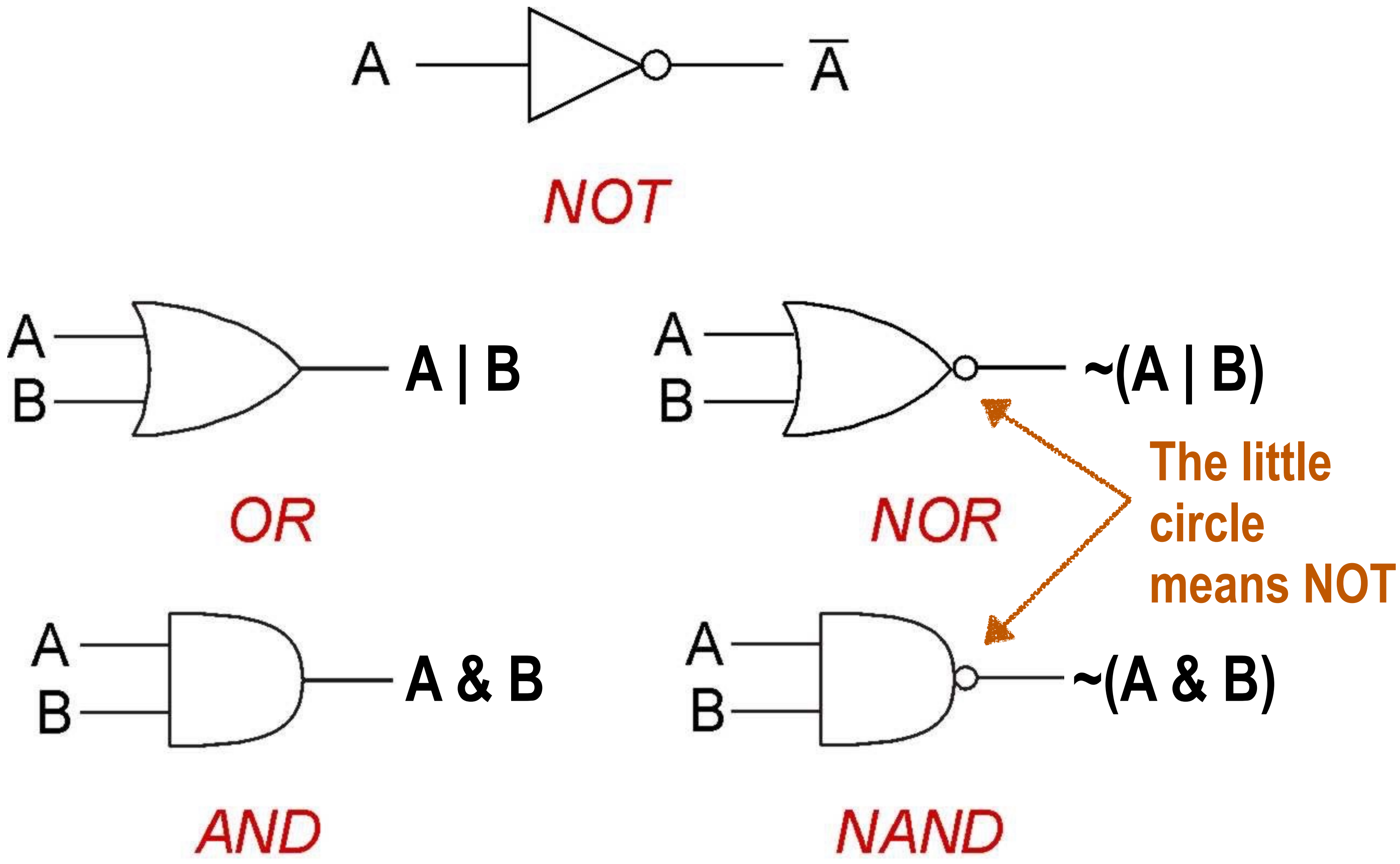


A	B	C
0	0	1
0	1	0
1	0	0
1	1	0

Basic Logic Gates



Basic Logic Gates



Full (1-bit) Adder

Add two bits and carry-in,
produce one-bit sum and carry-out.

A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Full (1-bit) Adder

Add two bits and carry-in,
produce one-bit sum and carry-out.

Truth Table

A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Full (1-bit) Adder

Truth Table

Add two bits and carry-in,
produce one-bit sum and carry-out.

$$S = (\sim A \mathbin{\&} \sim B \mathbin{\&} C_{in})$$

A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Full (1-bit) Adder

Add two bits and carry-in,
produce one-bit sum and carry-out.

$$S = (\sim A \ \& \sim B \ \& \ C_{in})$$
$$| \ (\sim A \ \& \ B \ \& \sim C_{in})$$

Truth Table

A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Full (1-bit) Adder

Truth Table

Add two bits and carry-in,
produce one-bit sum and carry-out.

$$S = (\sim A \;\&\; \sim B \;\&\; C_{in})$$
$$\quad | \; (\sim A \;\&\; B \;\&\; \sim C_{in})$$
$$\quad | \; (A \;\&\; \sim B \;\&\; \sim C_{in})$$

A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Full (1-bit) Adder

Truth Table

Add two bits and carry-in,
produce one-bit sum and carry-out.

$$S = (\sim A \ \& \ \sim B \ \& \ C_{in}) \\ \mid (\sim A \ \& \ B \ \& \ \sim C_{in}) \\ \mid (A \ \& \ \sim B \ \& \ \sim C_{in}) \\ \mid (A \ \& \ B \ \& \ C_{in})$$

A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Full (1-bit) Adder

Add two bits and carry-in,
produce one-bit sum and carry-out.

$$S = (\sim A \mathbin{\&} \sim B \mathbin{\&} C_{in}) \\ \mid (\sim A \mathbin{\&} B \mathbin{\&} \sim C_{in}) \\ \mid (A \mathbin{\&} \sim B \mathbin{\&} \sim C_{in}) \\ \mid (A \mathbin{\&} B \mathbin{\&} C_{in})$$

$$C_{ou} = (\sim A \mathbin{\&} B \mathbin{\&} C_{in}) \\ \mid (A \mathbin{\&} \sim B \mathbin{\&} C_{in}) \\ \mid (A \mathbin{\&} B \mathbin{\&} \sim C_{in}) \\ \mid (A \mathbin{\&} B \mathbin{\&} C_{in})$$

Truth Table

A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Full (1-bit) Adder

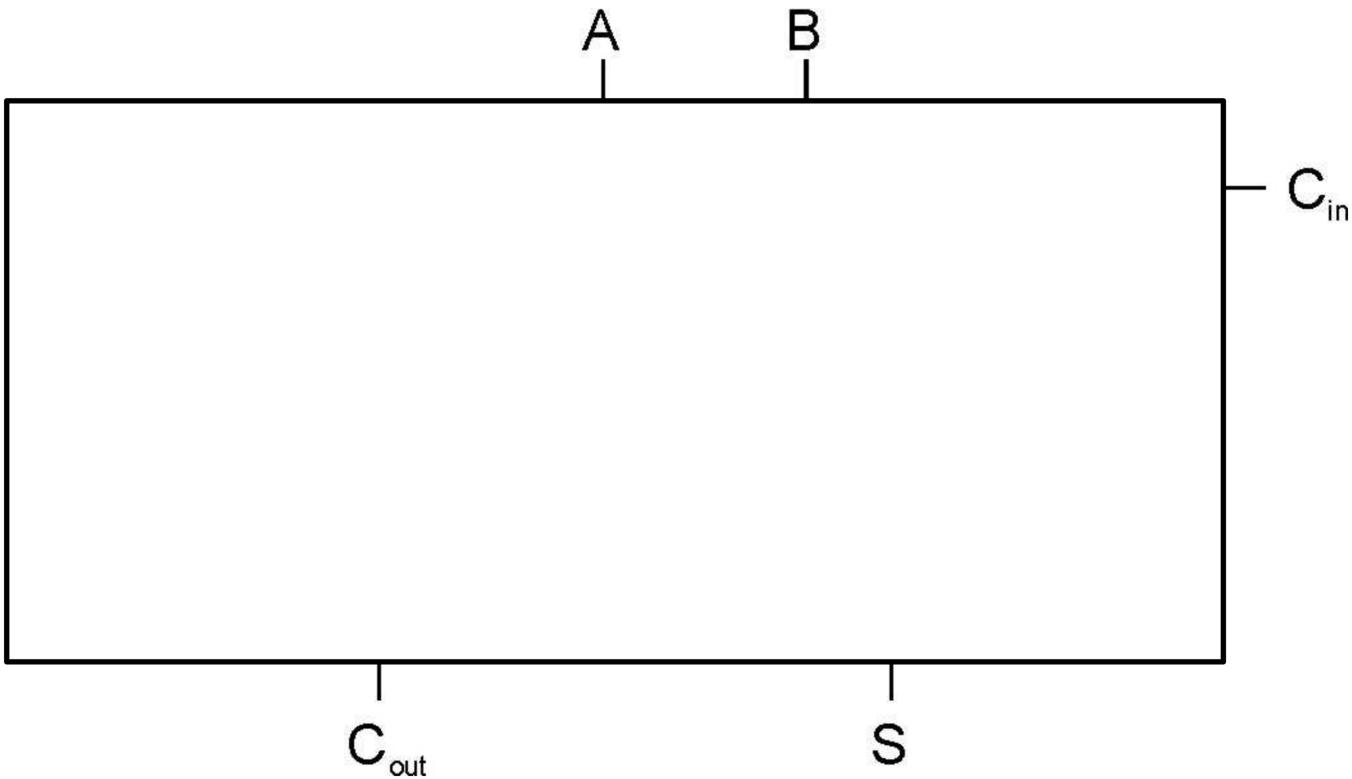
Add two bits and carry-in,
produce one-bit sum and carry-out.

$$\begin{aligned} C_{ou} = {} & (\sim A \ \& \ B \ \& \ C_{in}) \\ | \ & (A \ \& \sim B \ \& \ C_{in}) \\ | \ & (A \ \& \ B \ \& \sim C_{in}) \\ | \ & (A \ \& \ B \ \& \ C_{in}) \end{aligned}$$

Full (1-bit) Adder

$$C_{ou} = (\sim A \,\&\, B \,\&\, C_{in})$$
$$\mid (A \,\&\, \sim B \,\&\, C_{in})$$
$$\mid (A \,\&\, B \,\&\, \sim C_{in})$$
$$\mid (A \,\&\, B \,\&\, C_{in})$$

Add two bits and carry-in,
produce one-bit sum and carry-out.



Full (1-bit) Adder

$C_{ou} = (\sim A \ \&\ B \ \&\ C_{in})$
$|\ (A \ \&\ \sim B \ \&\ C_{in})$
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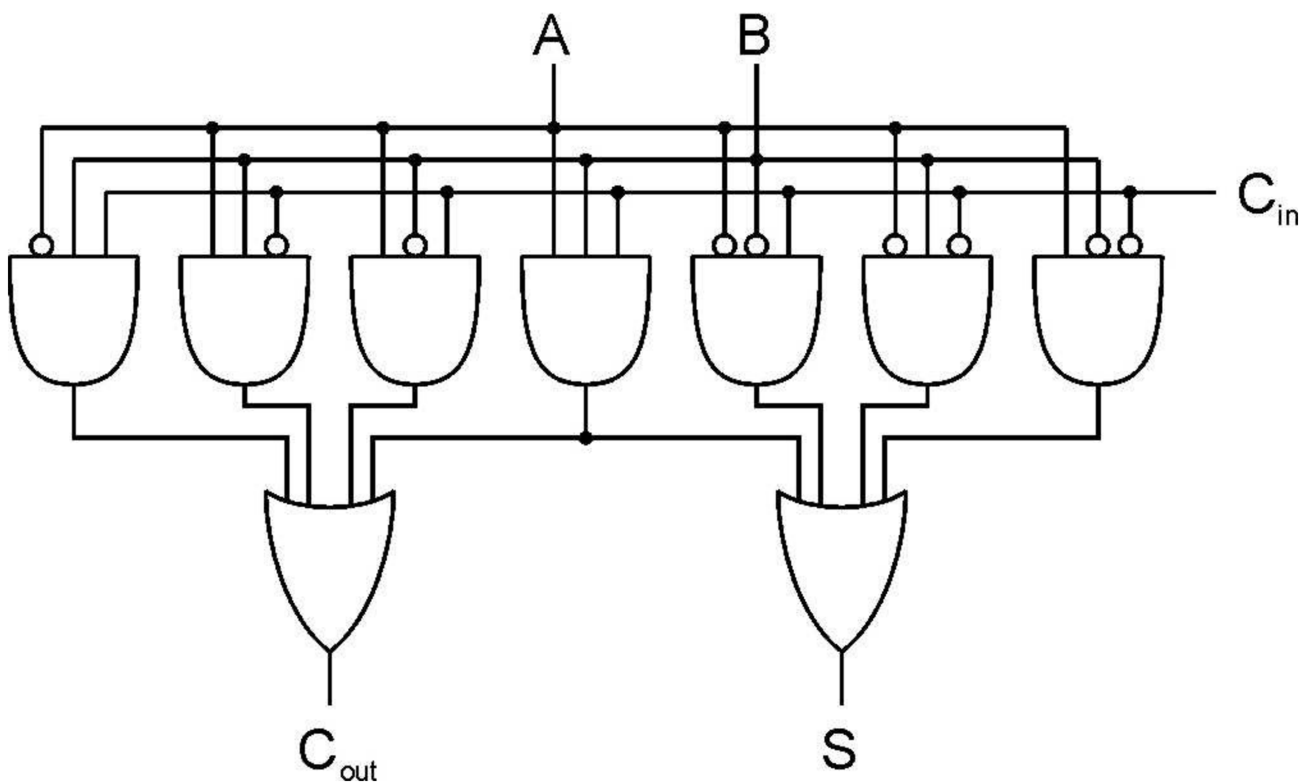

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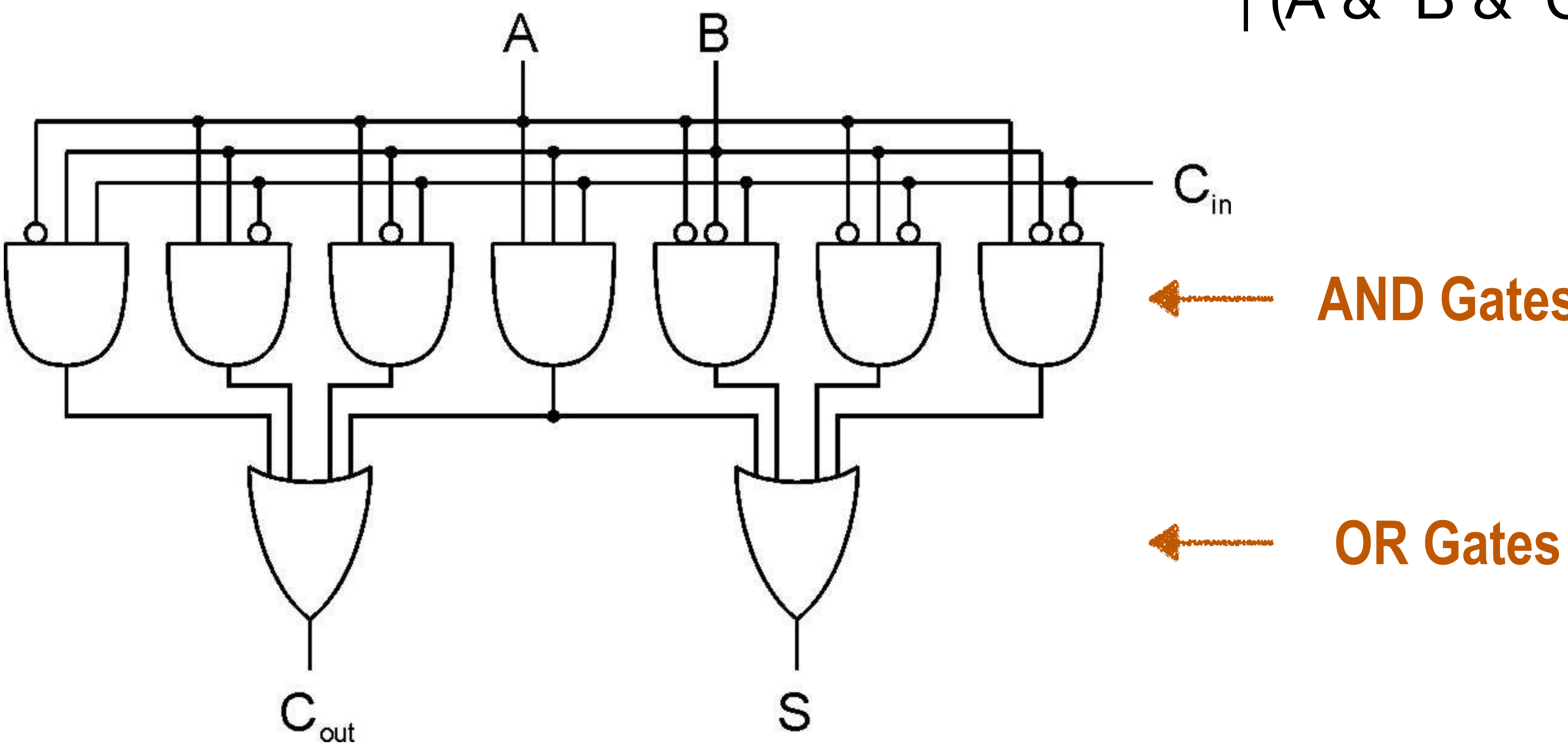

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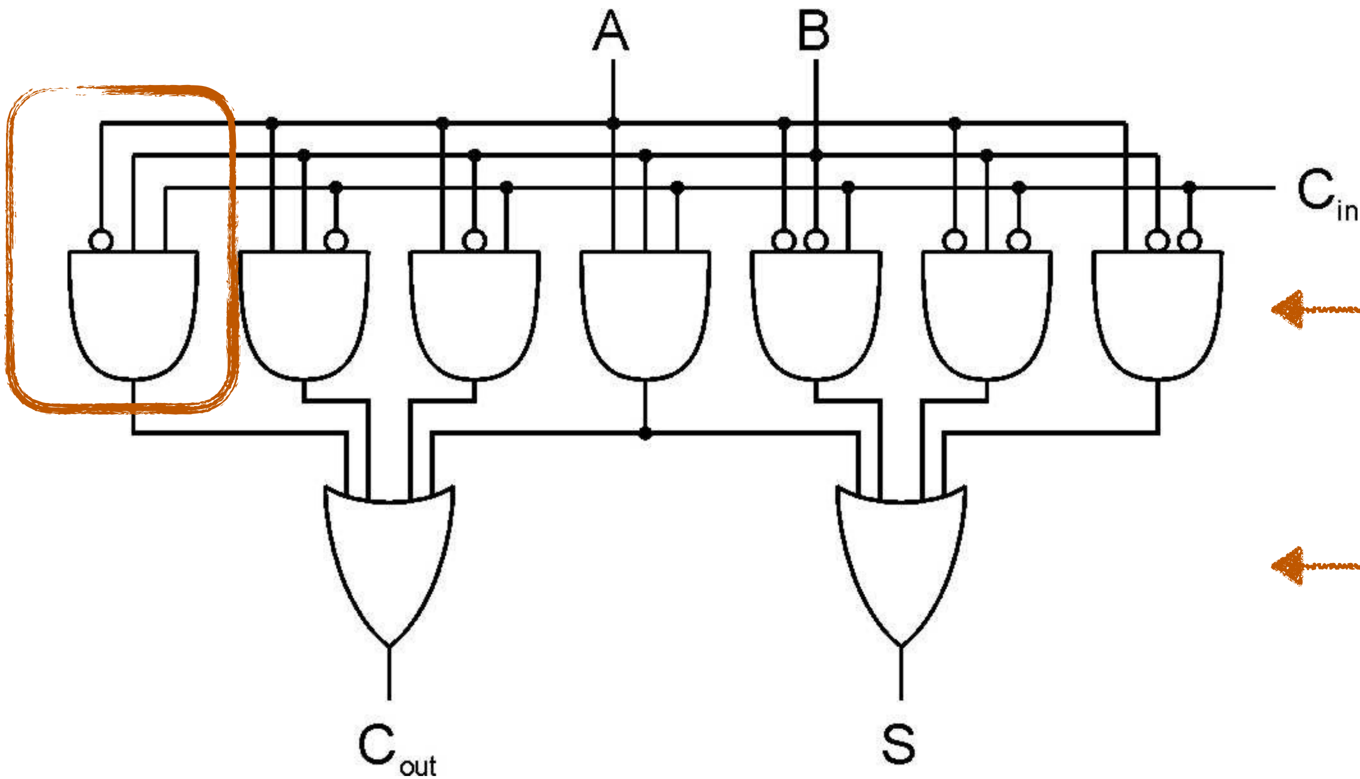

AND Gates

OR Gates

Full (1-bit) Adder

$$
\begin{aligned}
C_{ou} = {} & (\sim A \,\&\, B \,\&\, C_{in}) \\
| \; & (A \,\&\, \sim B \,\&\, C_{in}) \\
| \; & (A \,\&\, B \,\&\, \sim C_{in}) \\
| \; & (A \,\&\, B \,\&\, C_{in})
\end{aligned}
$$

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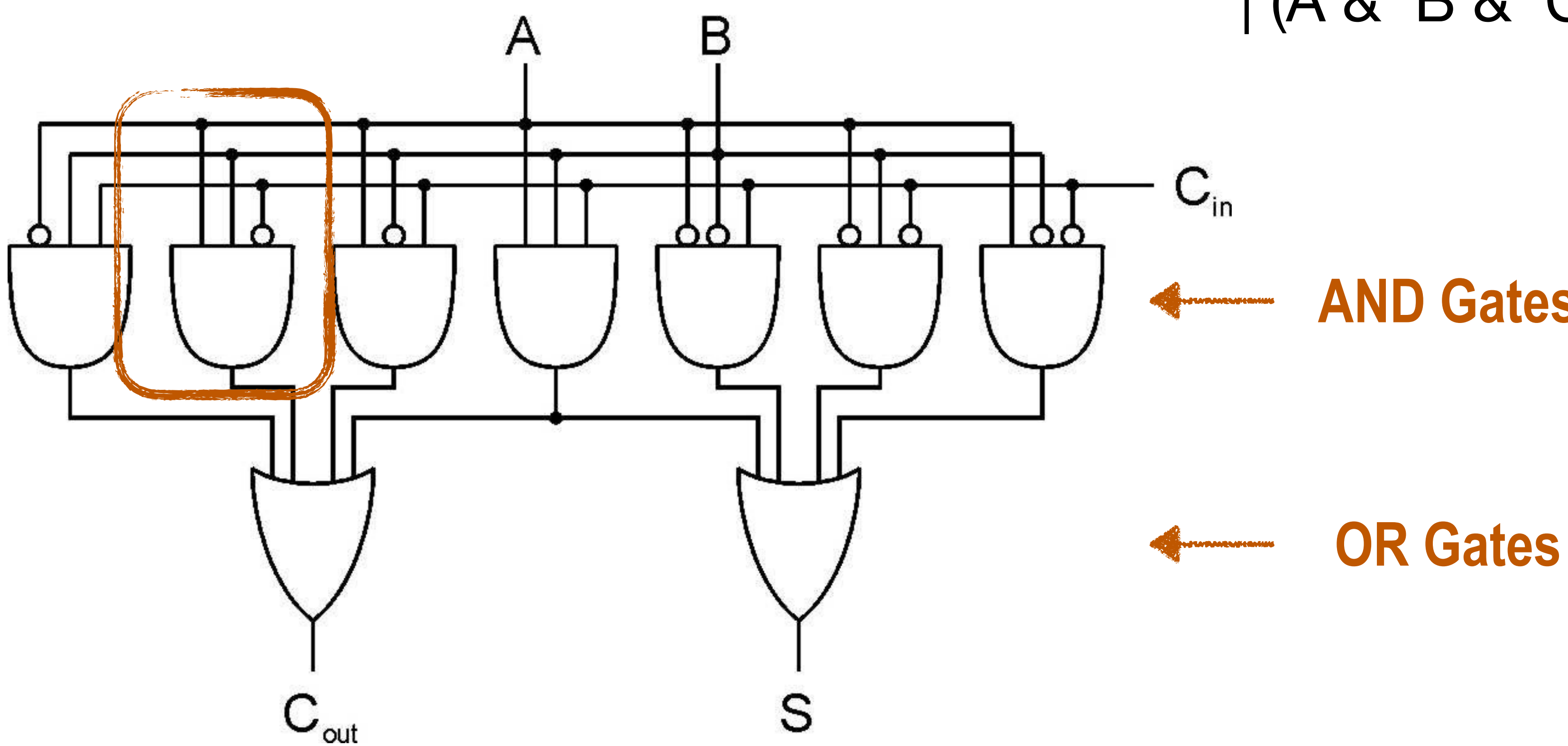

Full (1-bit) Adder

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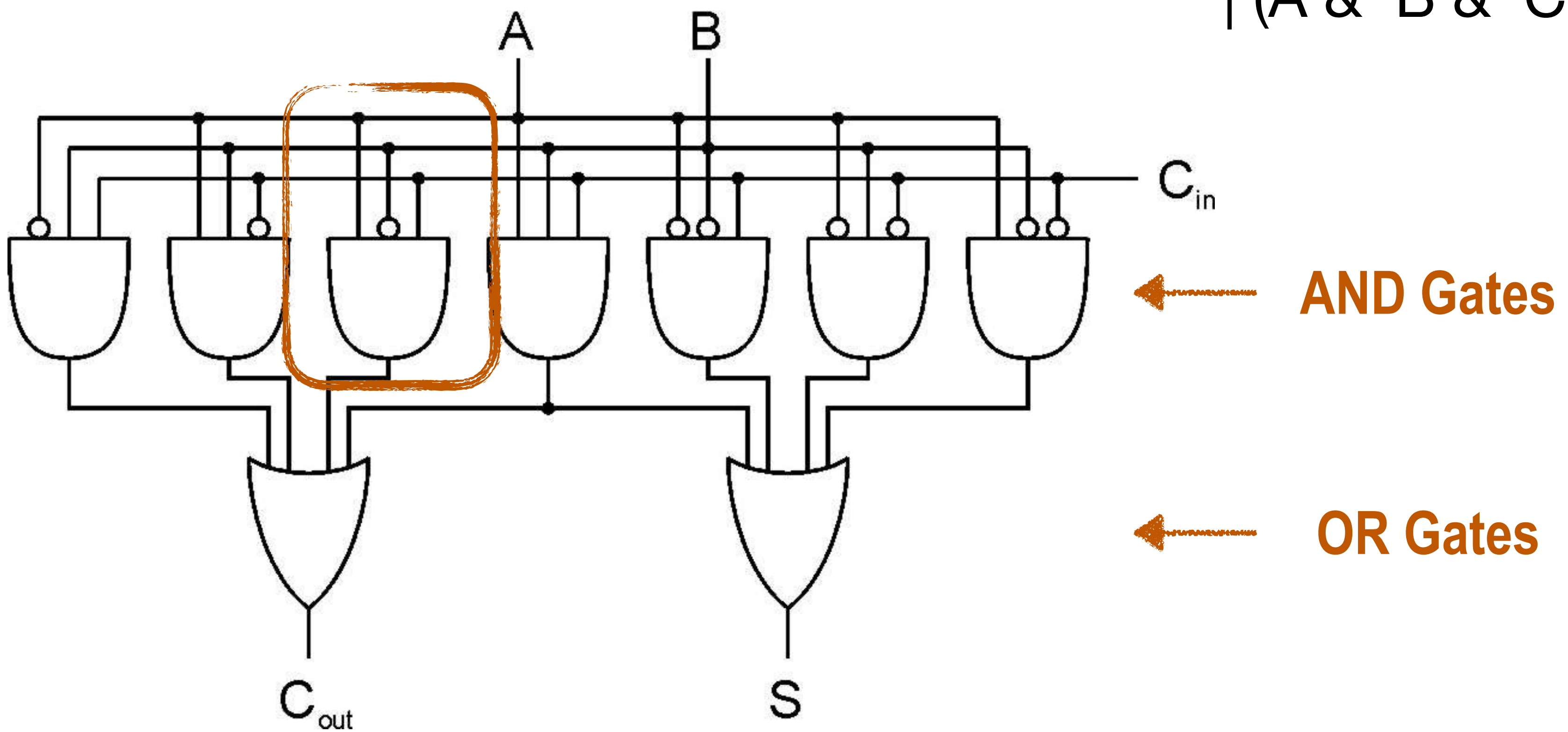

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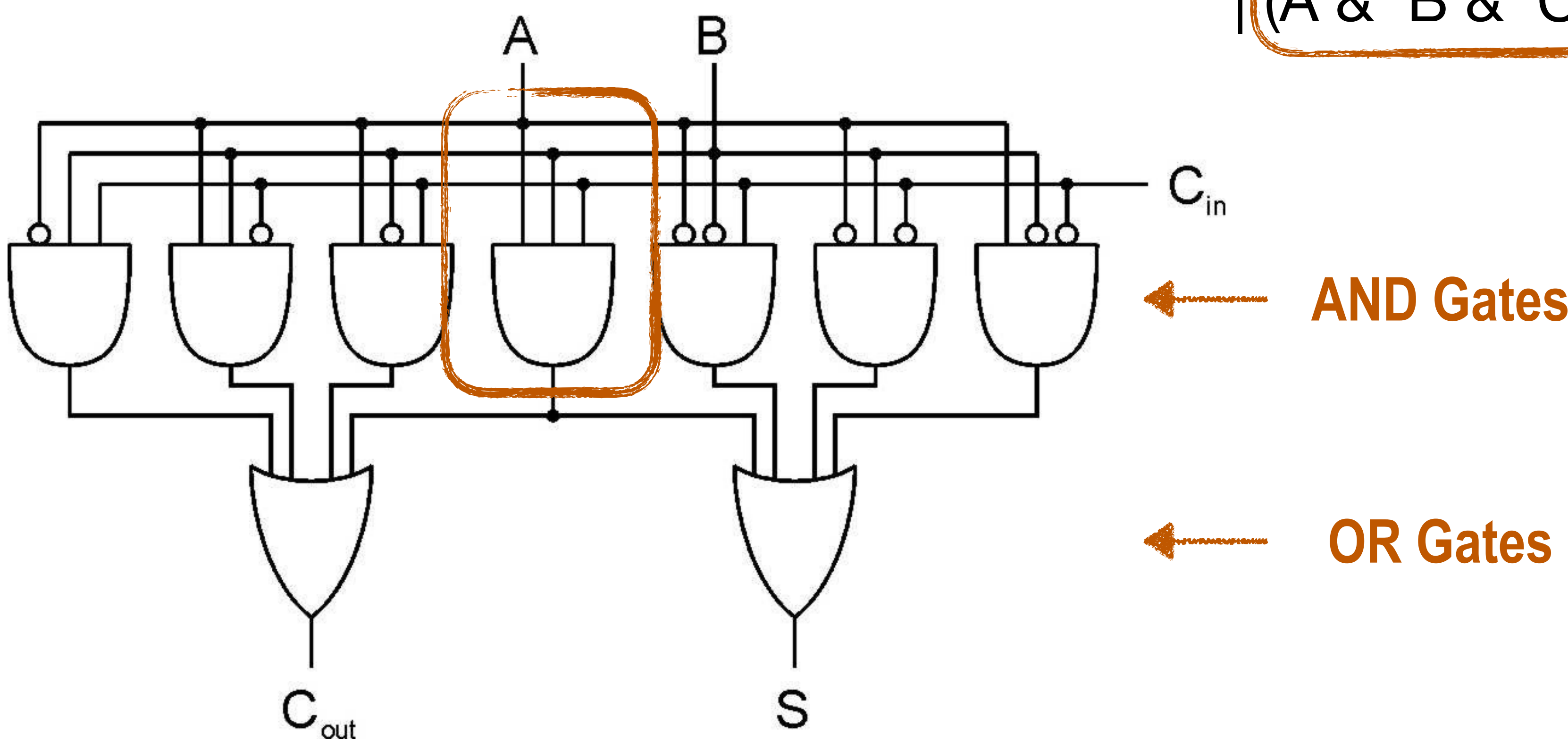
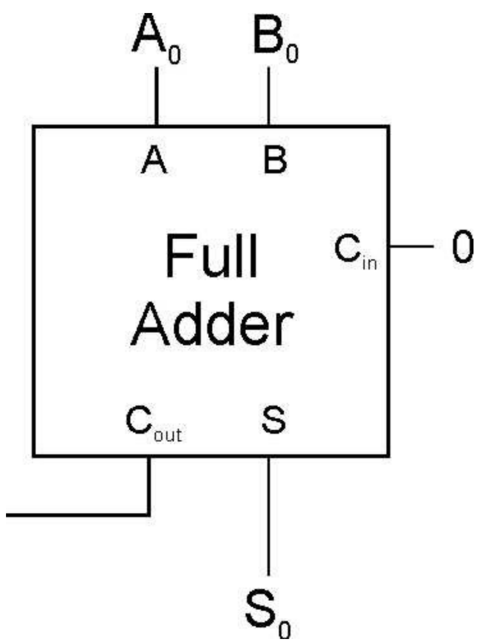
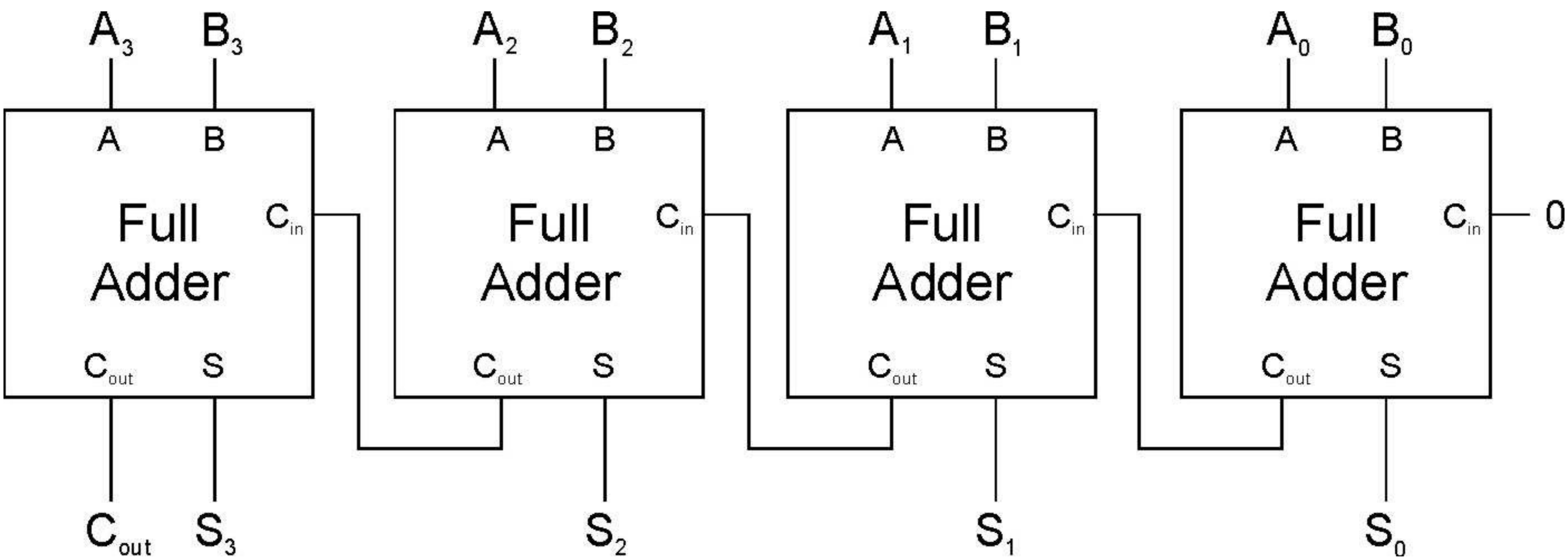


Four-bit Adder

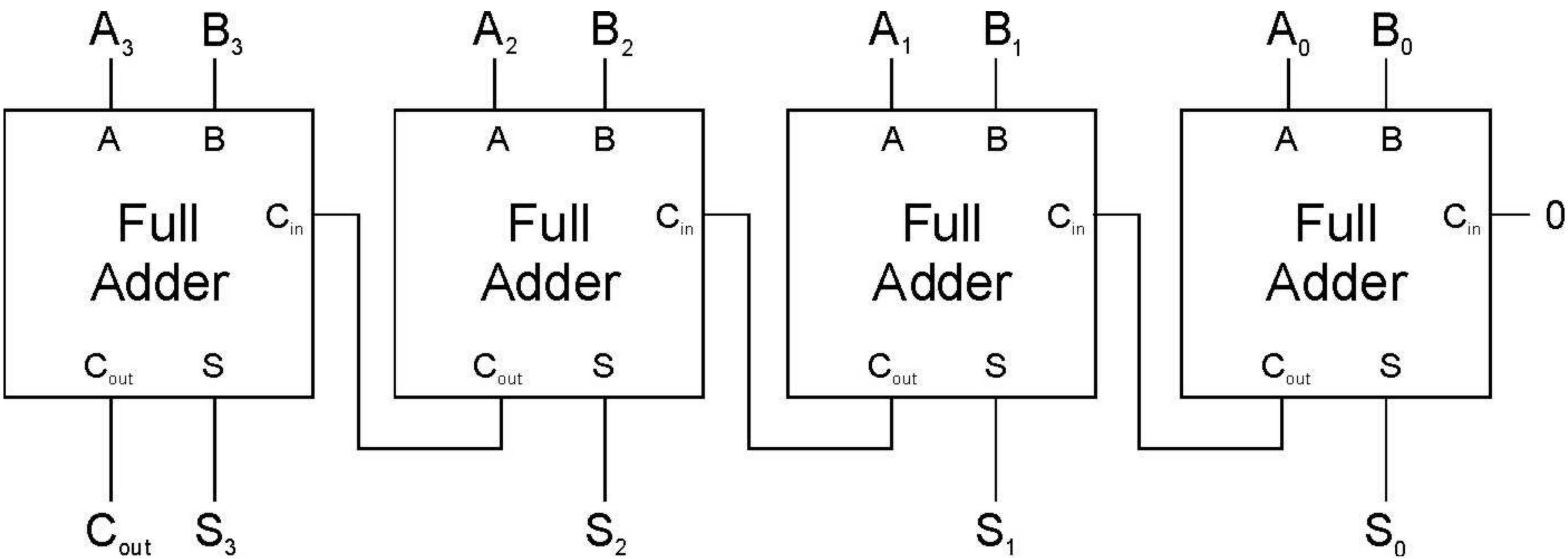


Four-bit Adder



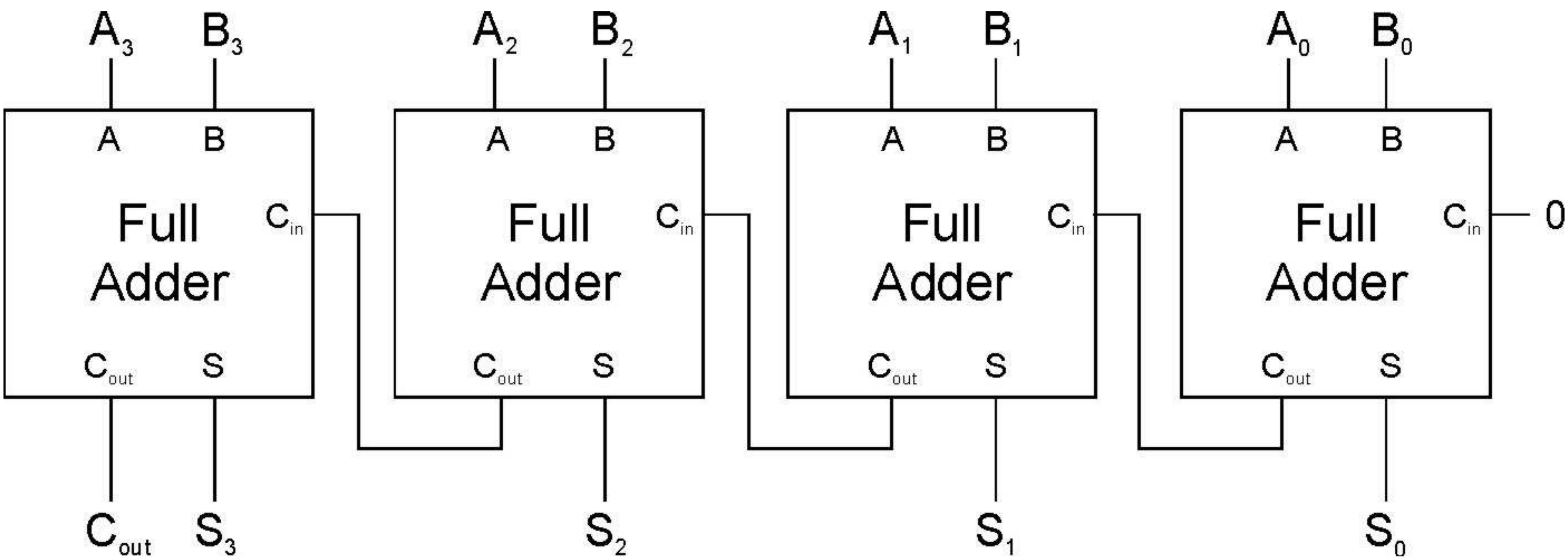
Four-bit Adder

- Ripple-carry Adder
  - Simple, but performance linear to bit width



Four-bit Adder

- Ripple-carry Adder
  - Simple, but performance linear to bit width
- Carry look-ahead adder (CLA)
  - Generate all carriers simultaneously



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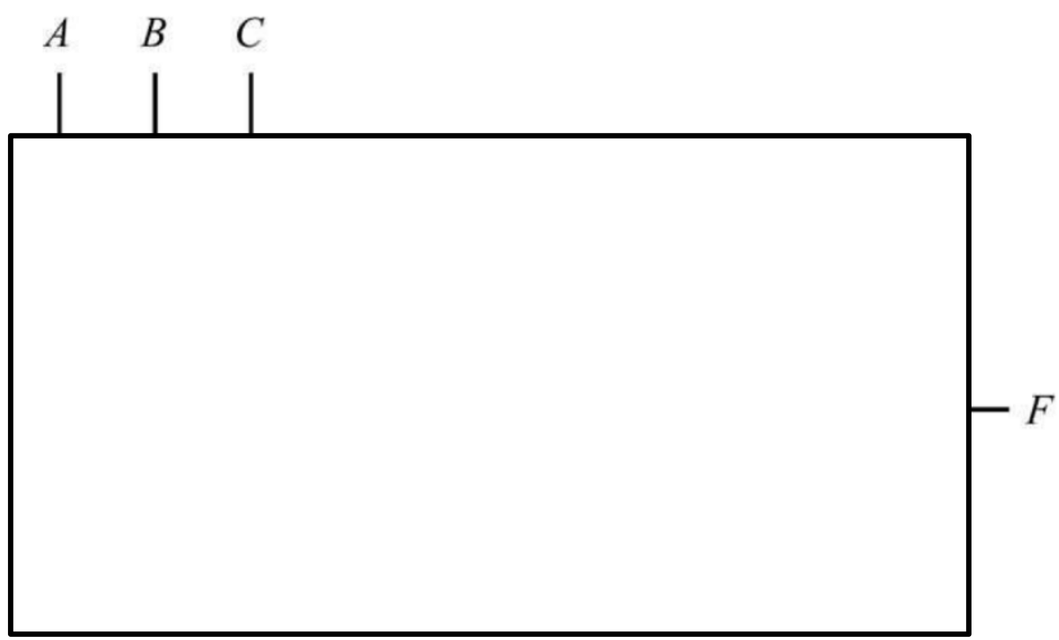

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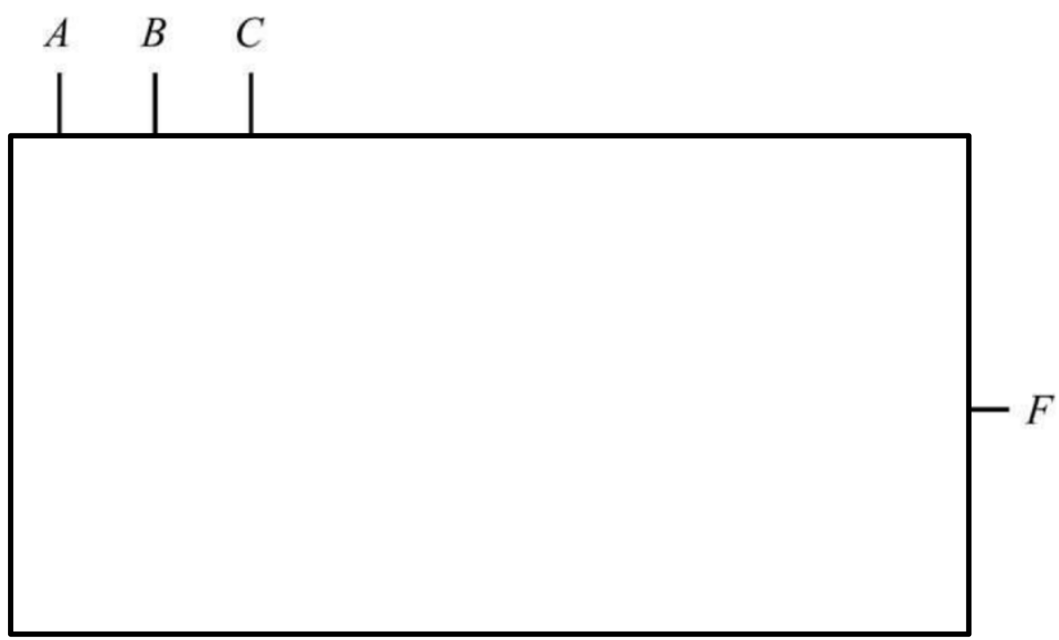

A	B	C	F
0	0	0	**0**
0	0	1	**0**
0	1	0	**0**
0	1	1	**1**
1	0	0	**0**
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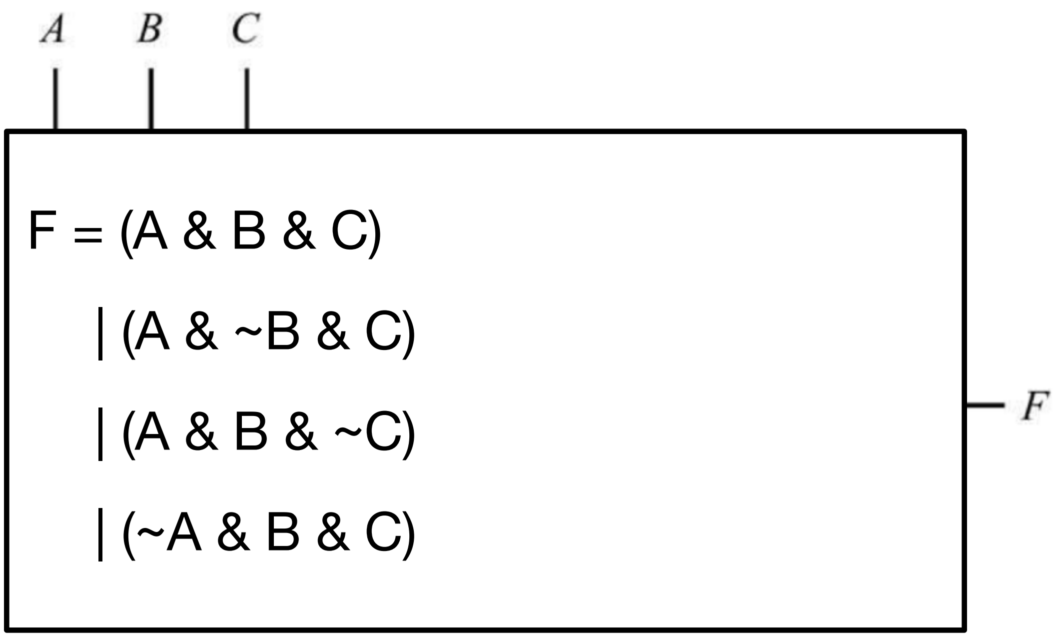

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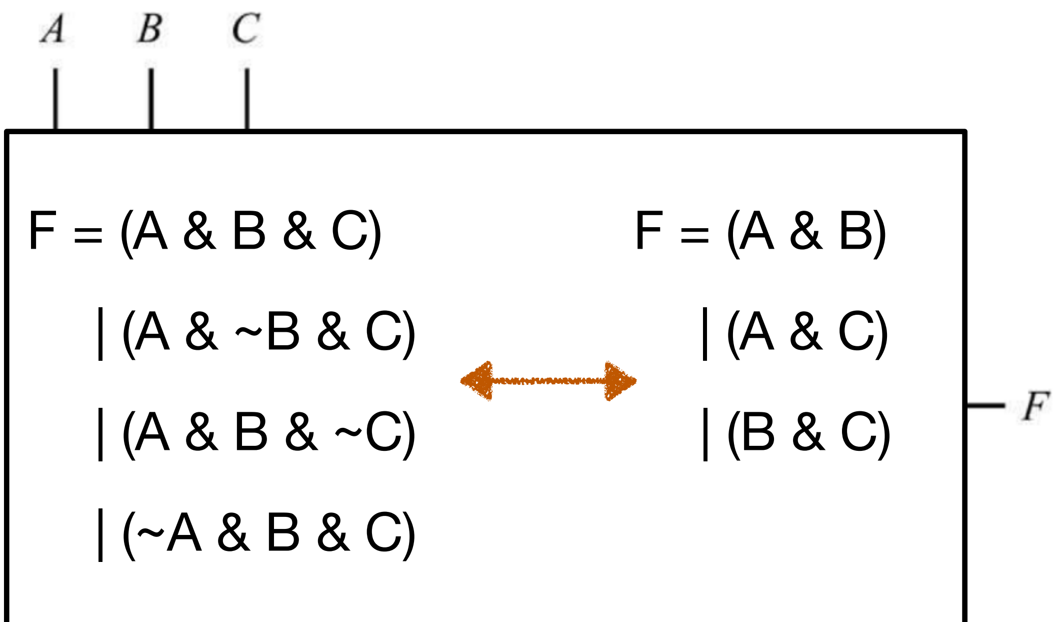

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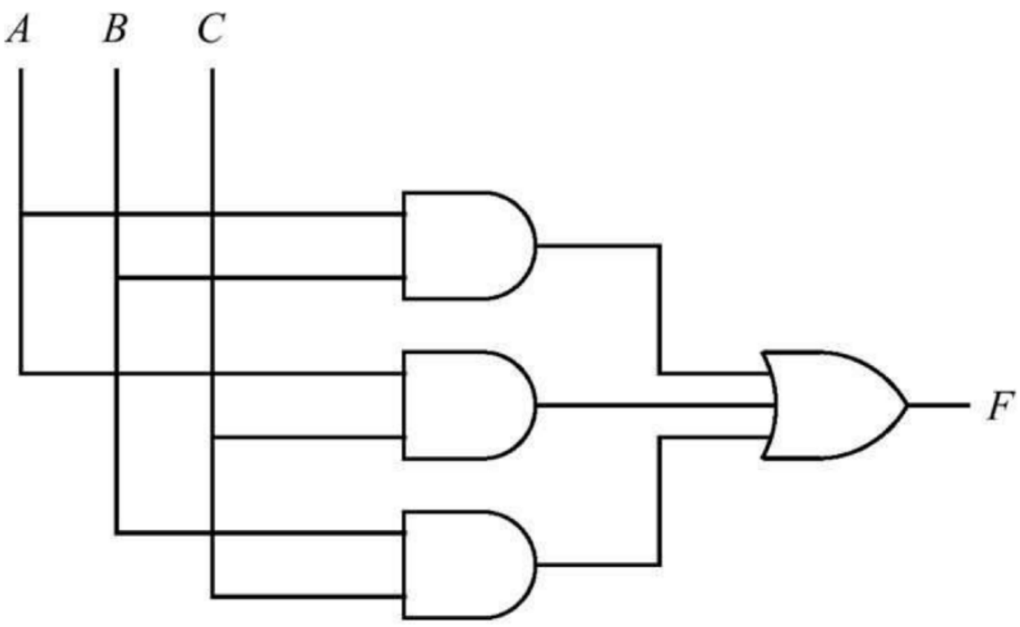

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Multiplication

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- Goal: Computing Product of w-bit numbers x, y

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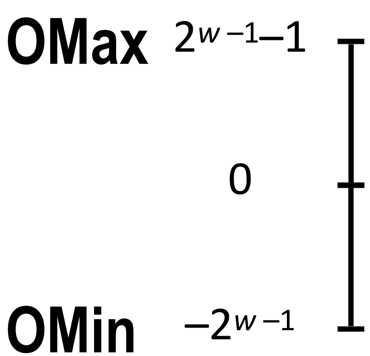
Original Number (w bits)



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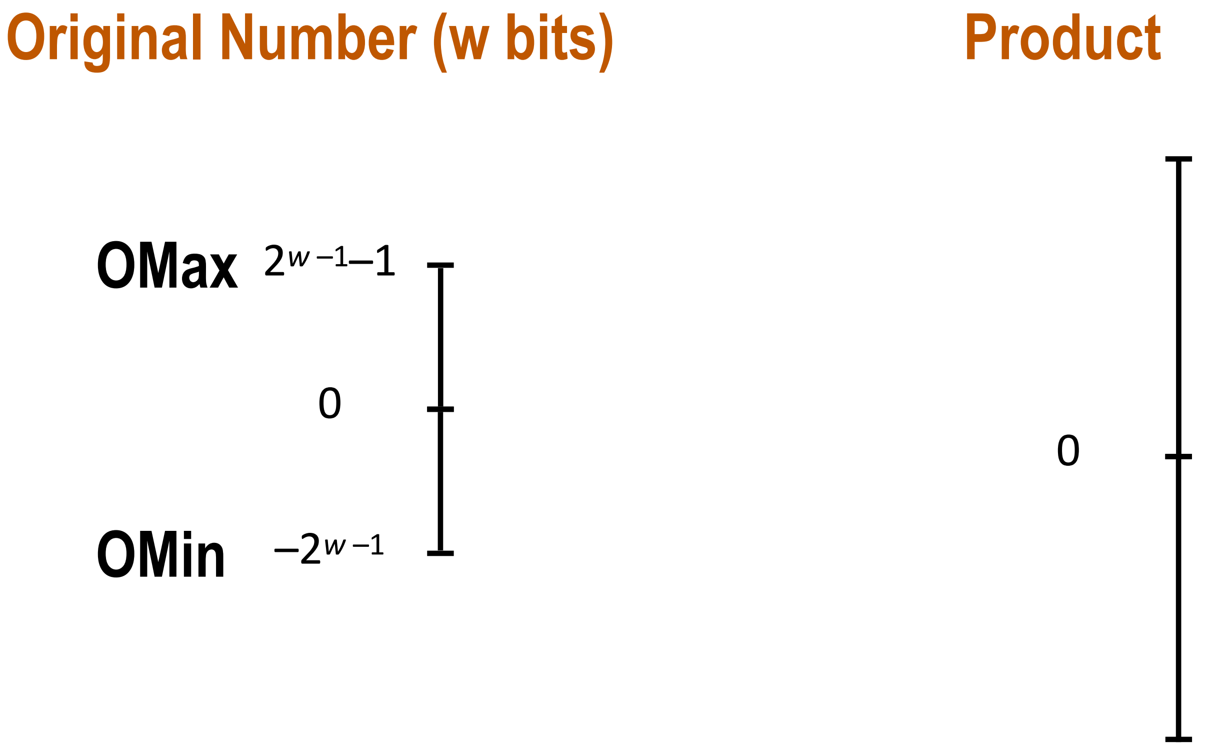

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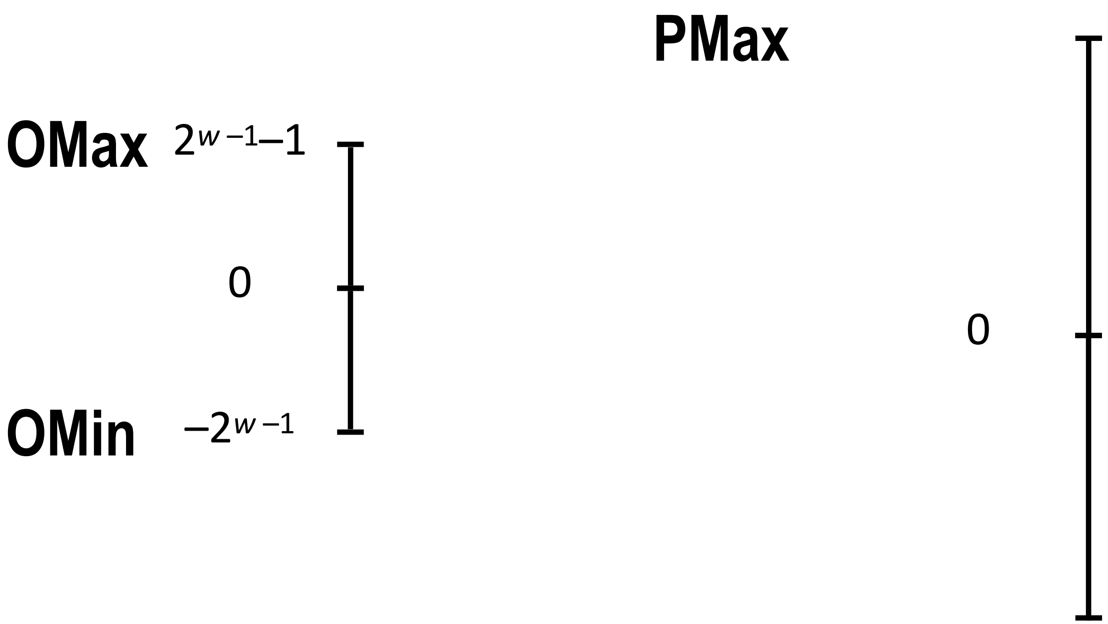

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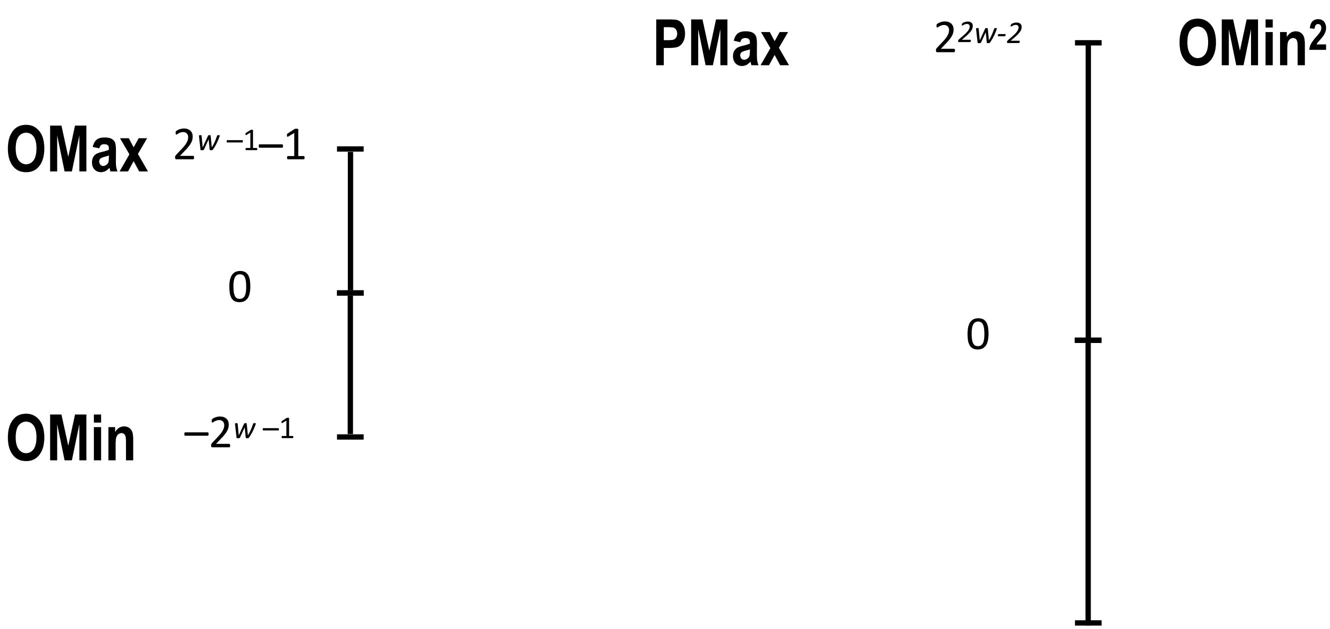

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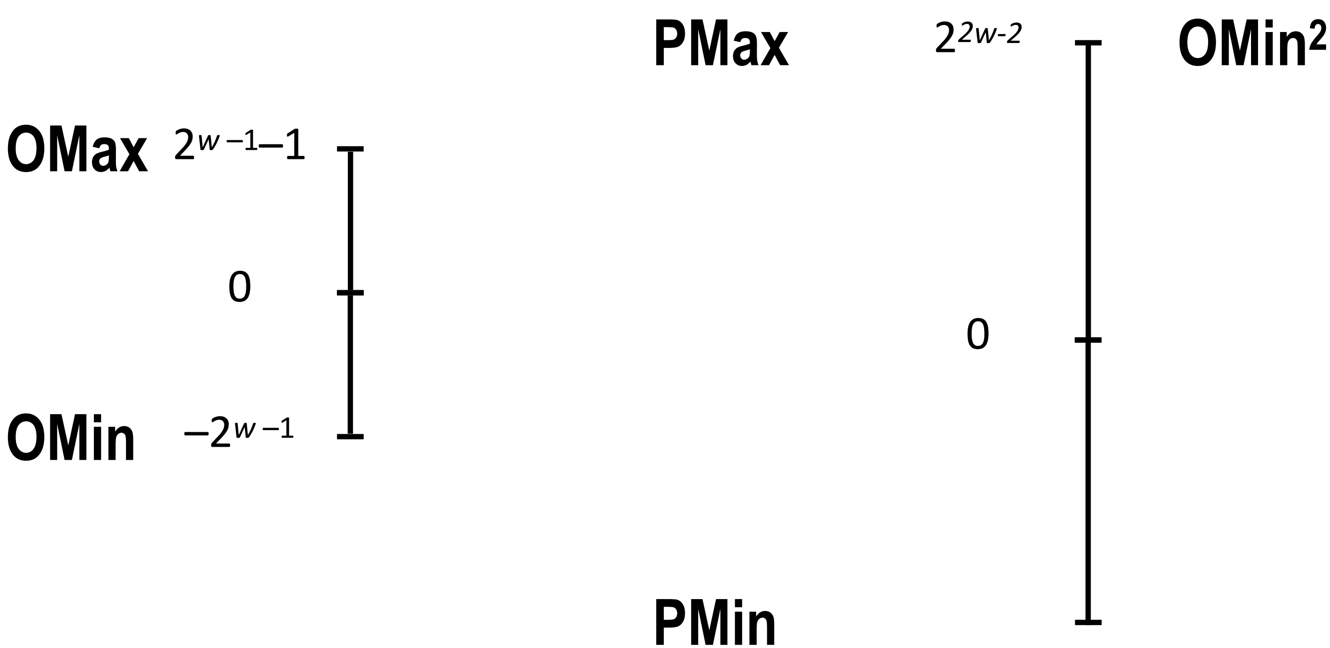

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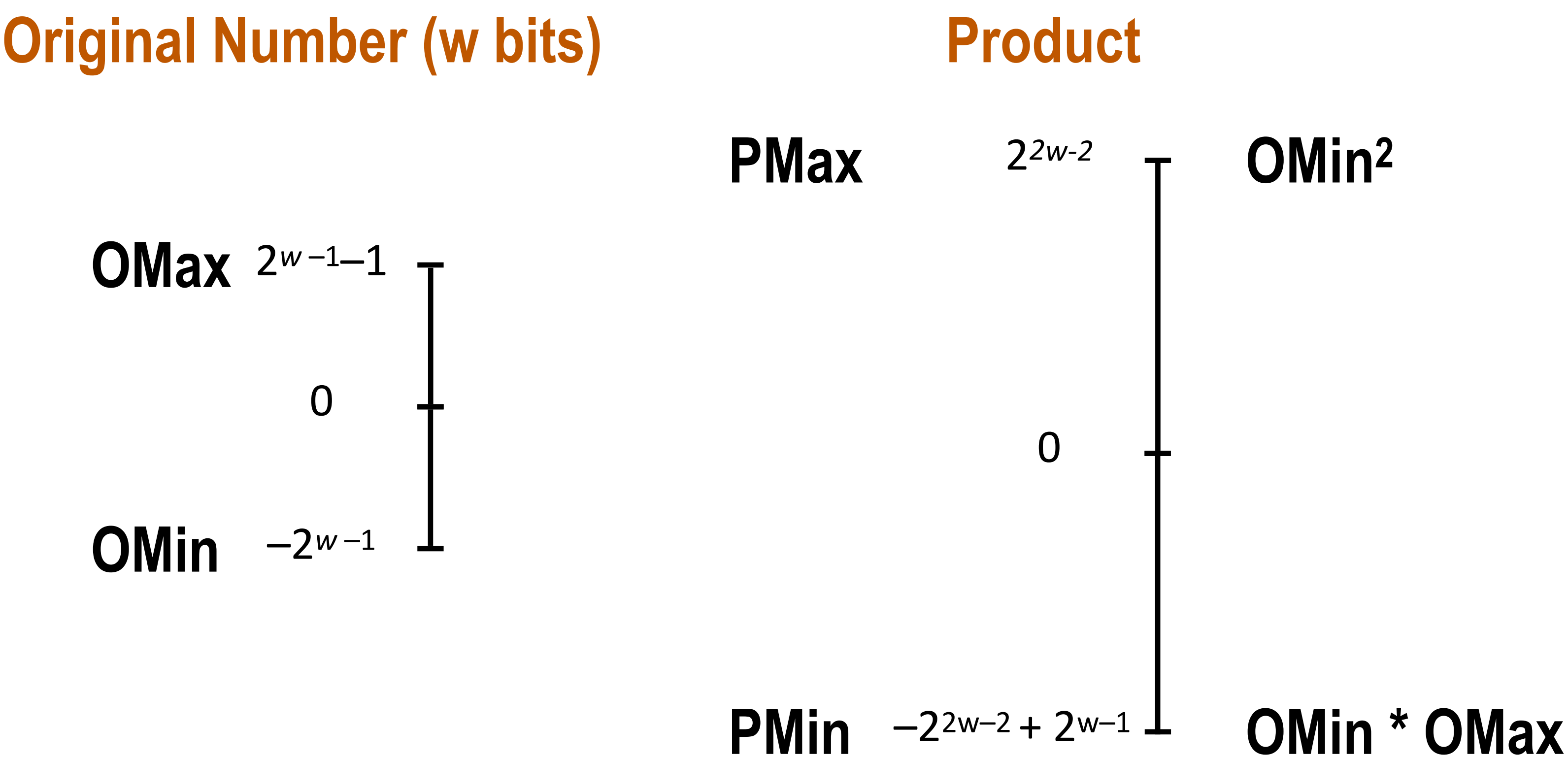

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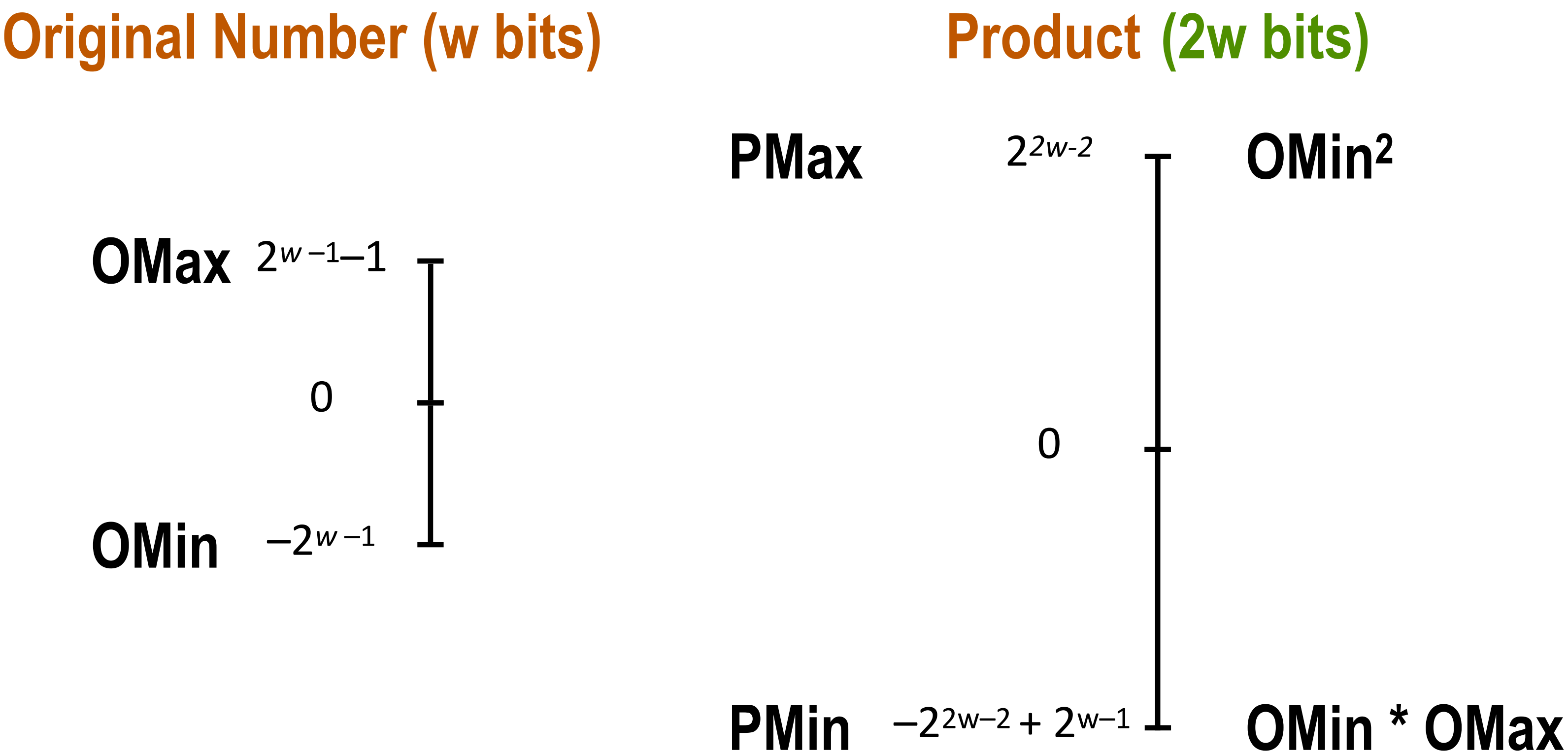
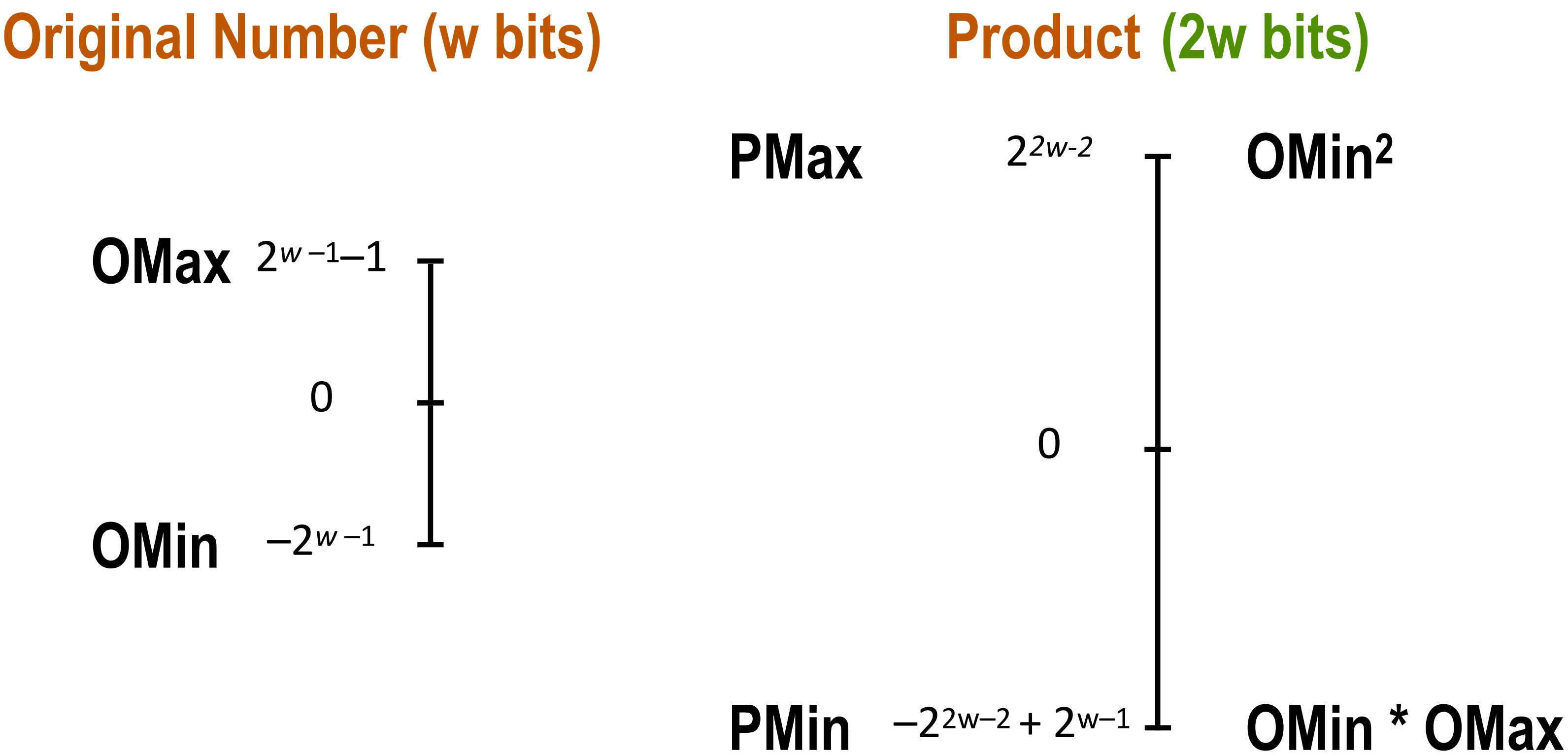


Multiplication

- Goal: Computing Product of *w*-bit numbers *x*, *y*
- Exact results can be bigger than *w* bits
  - Up to *2w* bits (both signed and unsigned)

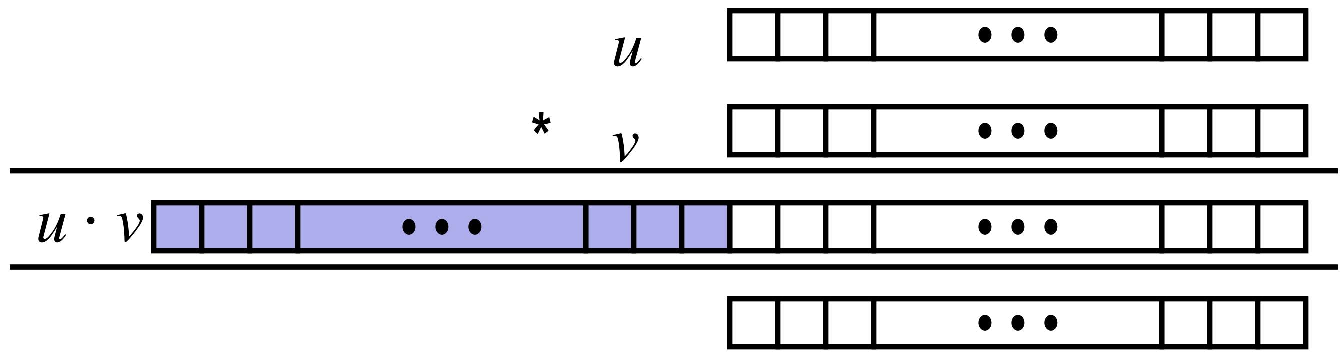


Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



- Standard Multiplication Function
  - Ignores high order *w* bits
- Effectively Implements the following:

$\mathrm{UMult}_w(u, v) = u \cdot v \bmod 2^w$