

Lecture 3: Fourier Analysis

Yuhao Zhu

<http://yuhaozhu.com>
yzhu@rochester.edu

CSC 292/572, Fall 2022
Mobile Visual Computing

Logistics

- Written assignment 1 is up and is due Sept. 16 11:30 AM.
- You can work in groups of 2.
- Course schedule: <https://www.cs.rochester.edu/courses/572/fall2022/schedule.html>. You will find reading assignments and slides.
- Start thinking and talking to me about your final project idea.

The Roadmap

Theoretical Preliminaries

Human Visual Systems

Color in Nature, Arts, & Tech

(a.k.a., the birth, life, and death of light)

Digital Camera Imaging

Modeling and Rendering

Applications



Geometric Transformations

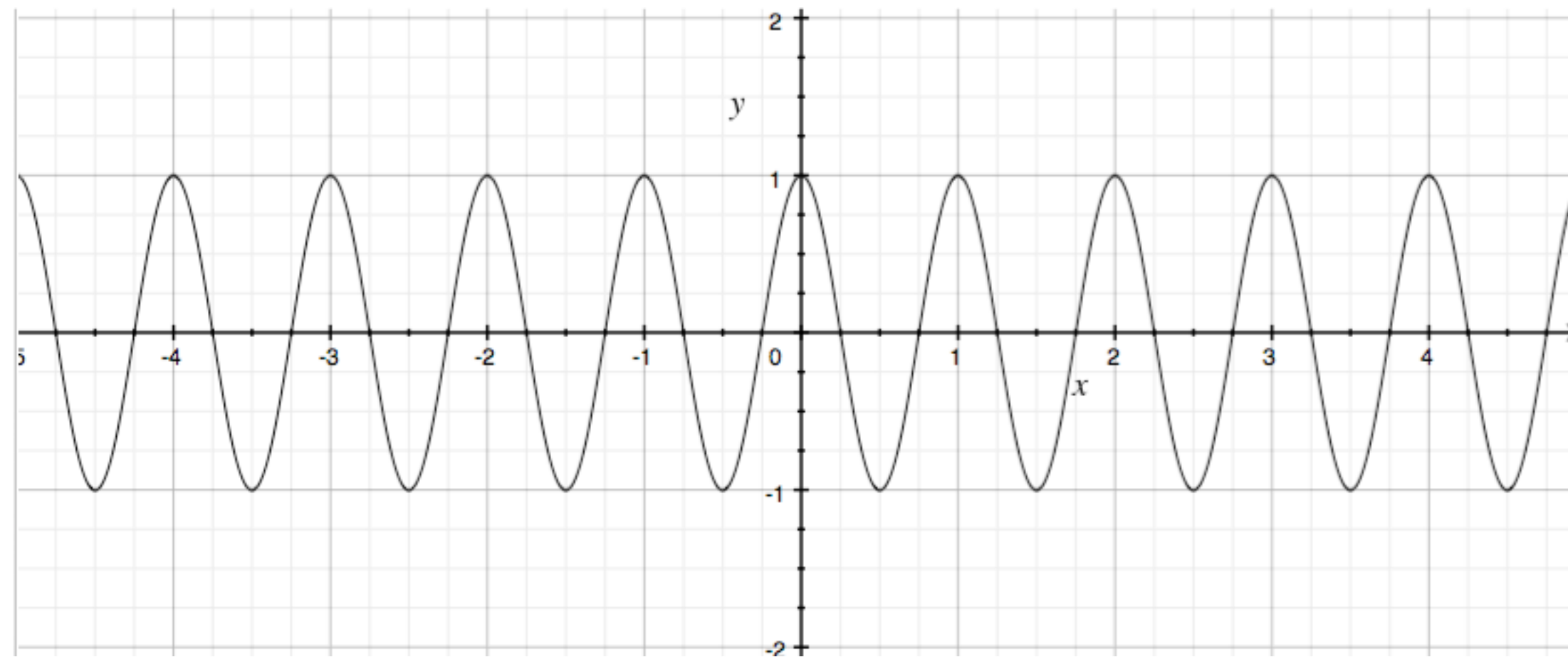
Fourier Series & Transforms

Sampling & Reconstruction

Building Intuitions

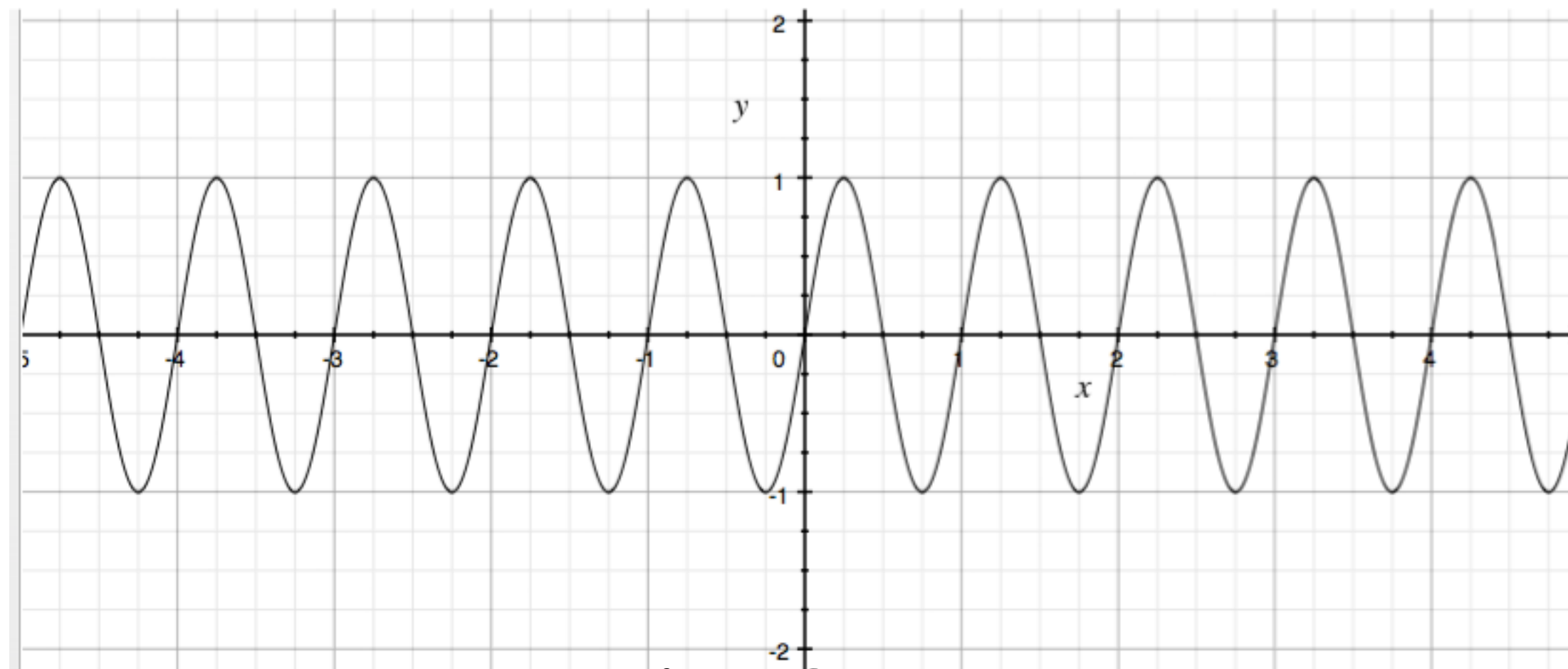
(1D)

Sinusoidal Function



$$\cos 2\pi x$$

Repeating at an interval of 1,
i.e., period (T) = 1
"Even" function

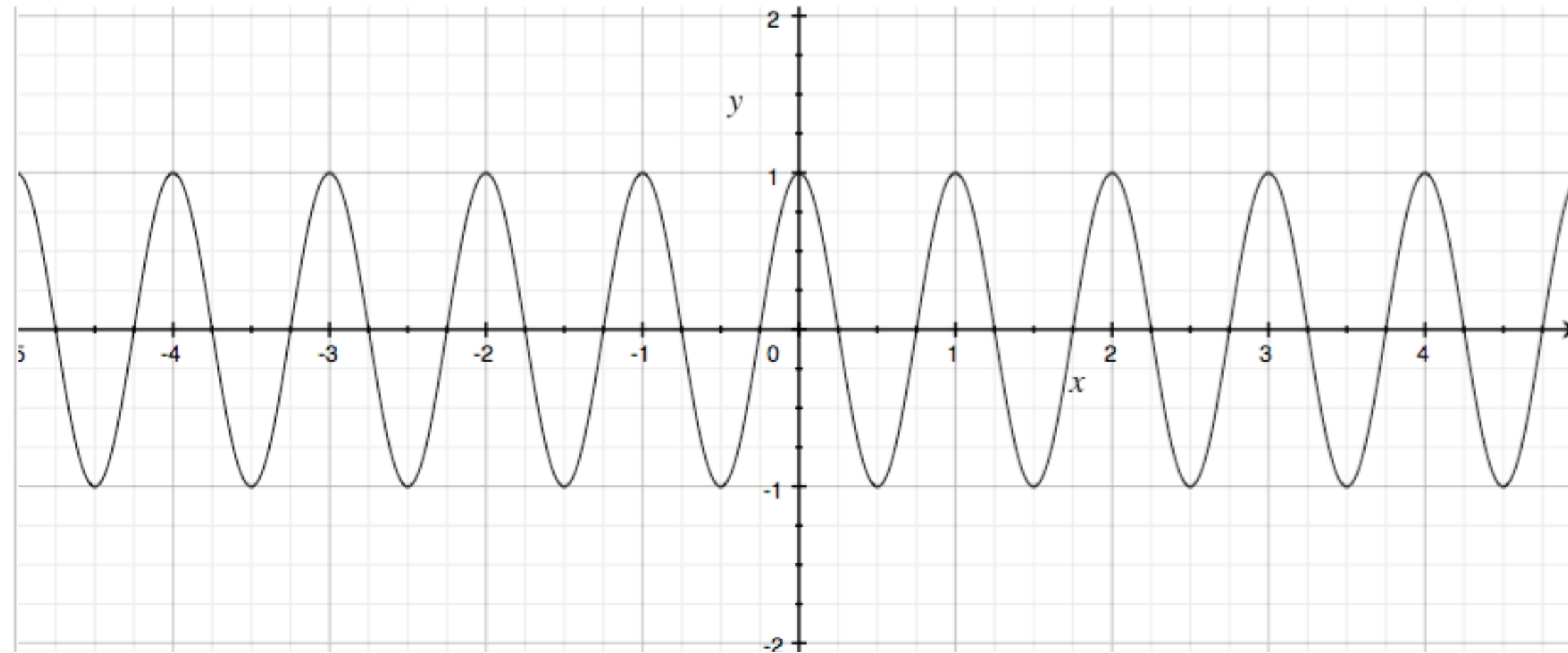


$$\sin 2\pi x$$

Repeating at an interval of 1,
i.e., period (T) = 1
"Odd" function

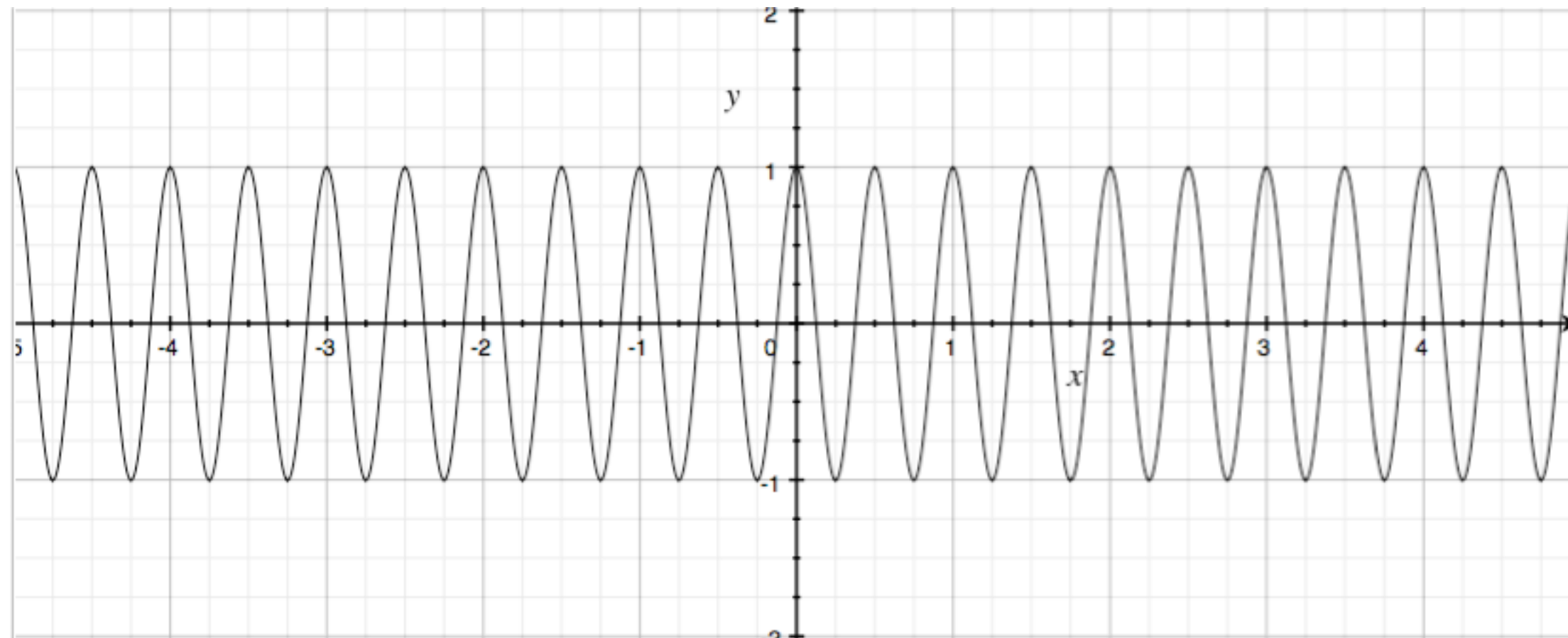
Sinusoidal Function Frequency

$$f = \frac{1}{T}$$



$$\cos 2\pi x$$

$$f = 1$$



$$\cos 4\pi x$$

$$f = 2$$

$$\sin kx, \cos kx$$

$$f_{\text{ordinary}} = \frac{k}{2\pi}$$

$$f_{\text{angular}} = k$$

$$T = \frac{2\pi}{k}$$

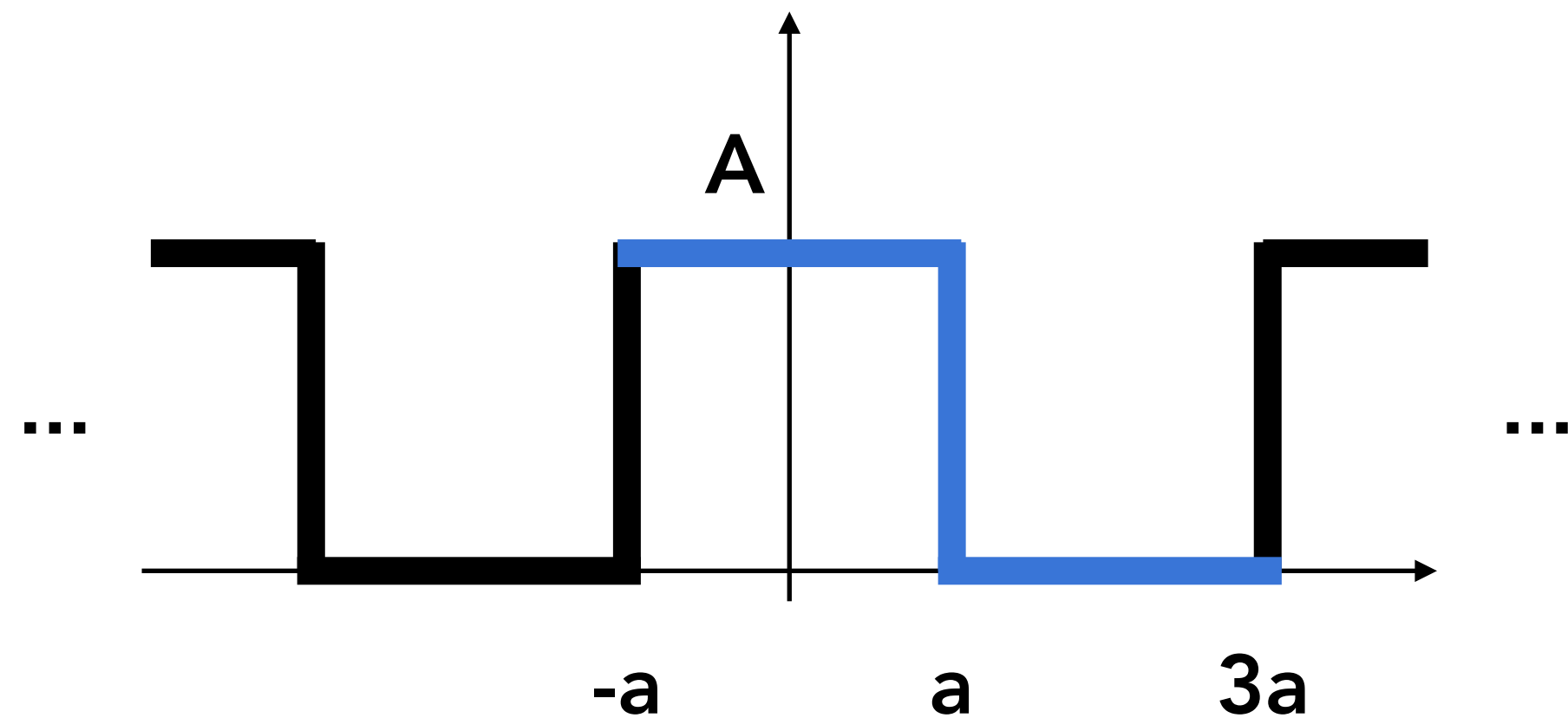
Fourier Series: What It Says



Joseph Fourier (1768 – 1830)

Any **periodic** function can be represented as an **infinite** weighted sum of sines and cosines with **increasing frequencies**.

Two Examples to Build Intuition

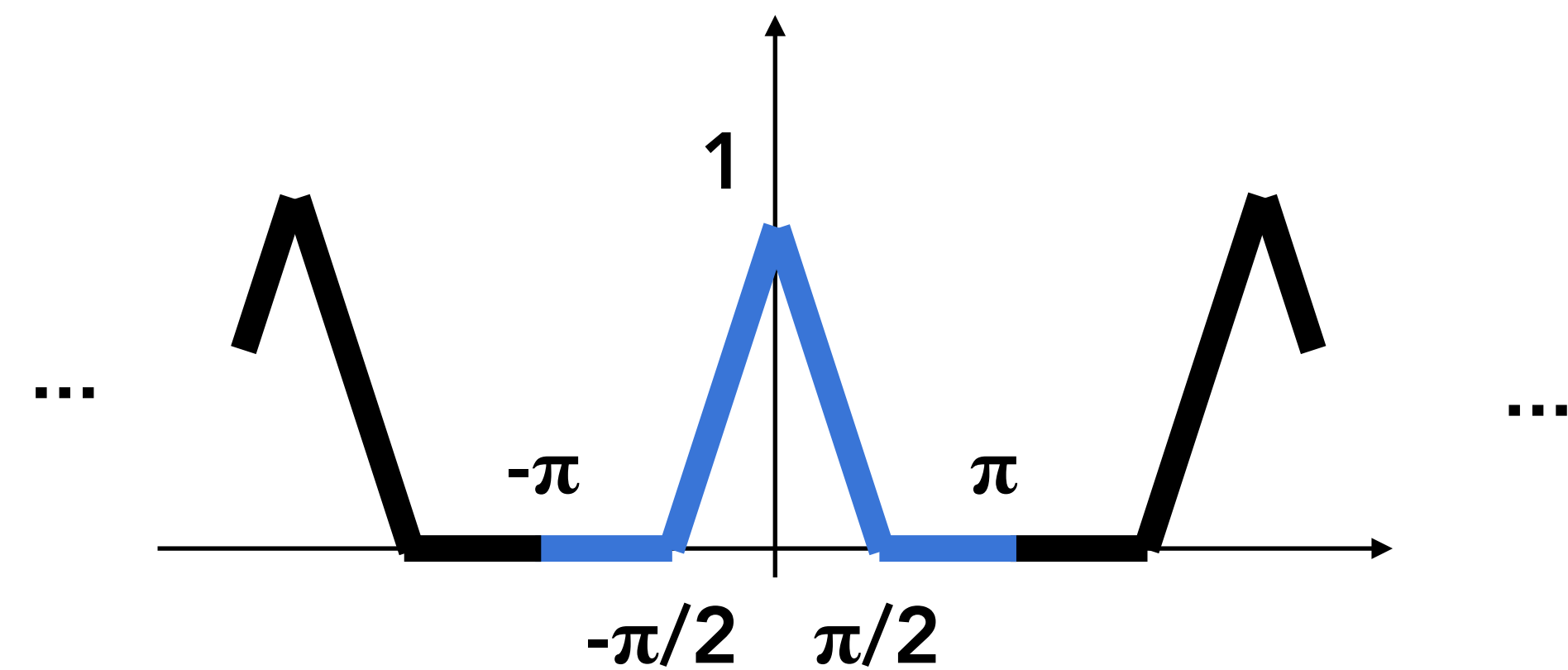


A square function

Period $T = 4a$

Frequency is $1/4a$

Amplitude is A



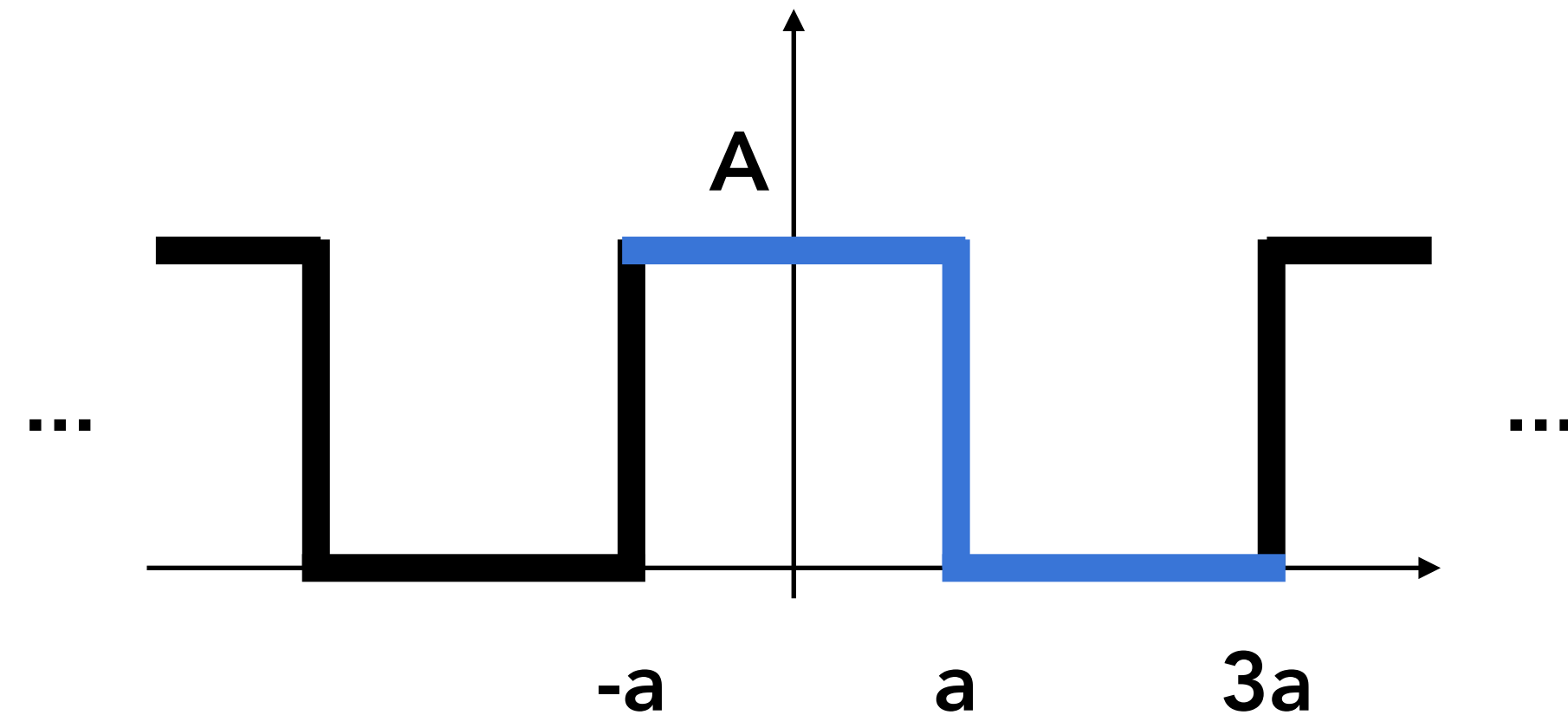
A hat function

Period $T = 2\pi$

Frequency is $1/2\pi$

Amplitude is 1

First Example



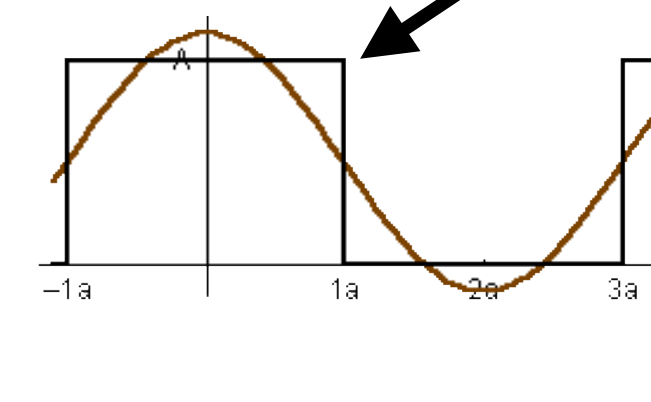
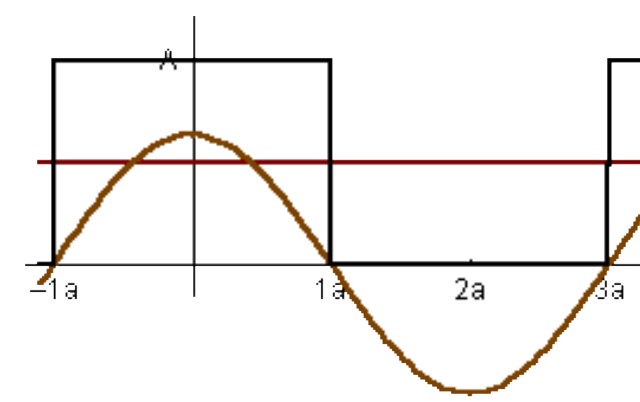
A square function

Period $T = 4a$

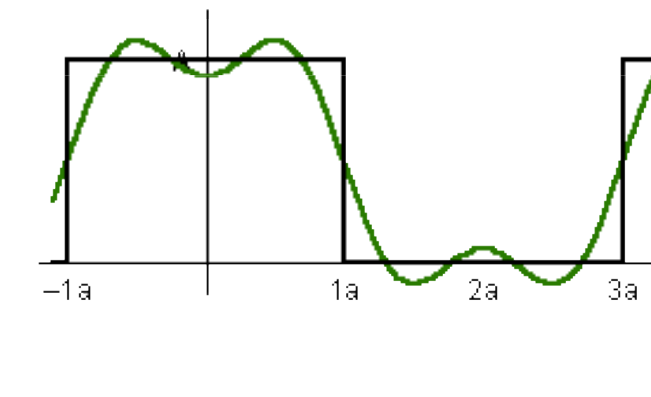
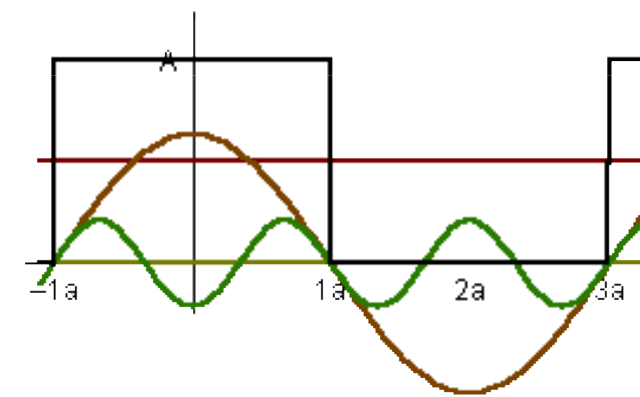
Frequency is $1/4a$

Amplitude is A

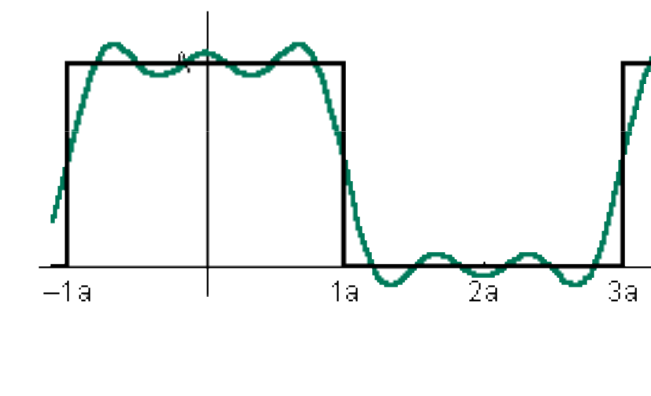
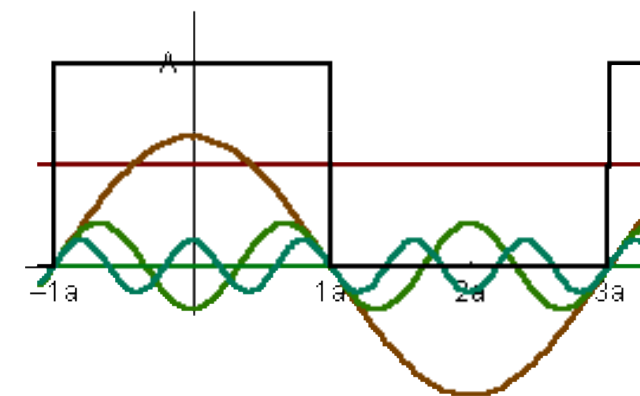
The periodic square
function to be represented



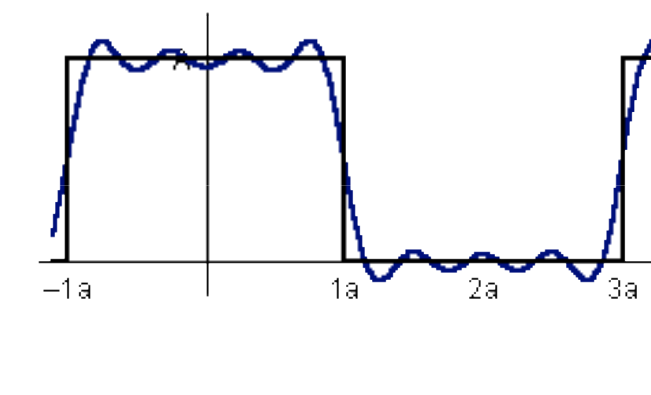
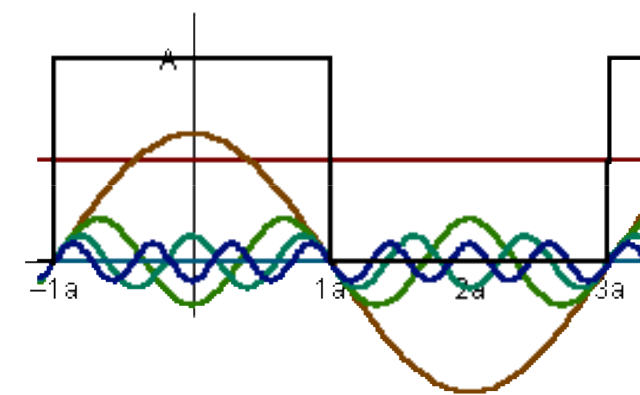
Add first 2 waves



Add first 3 waves



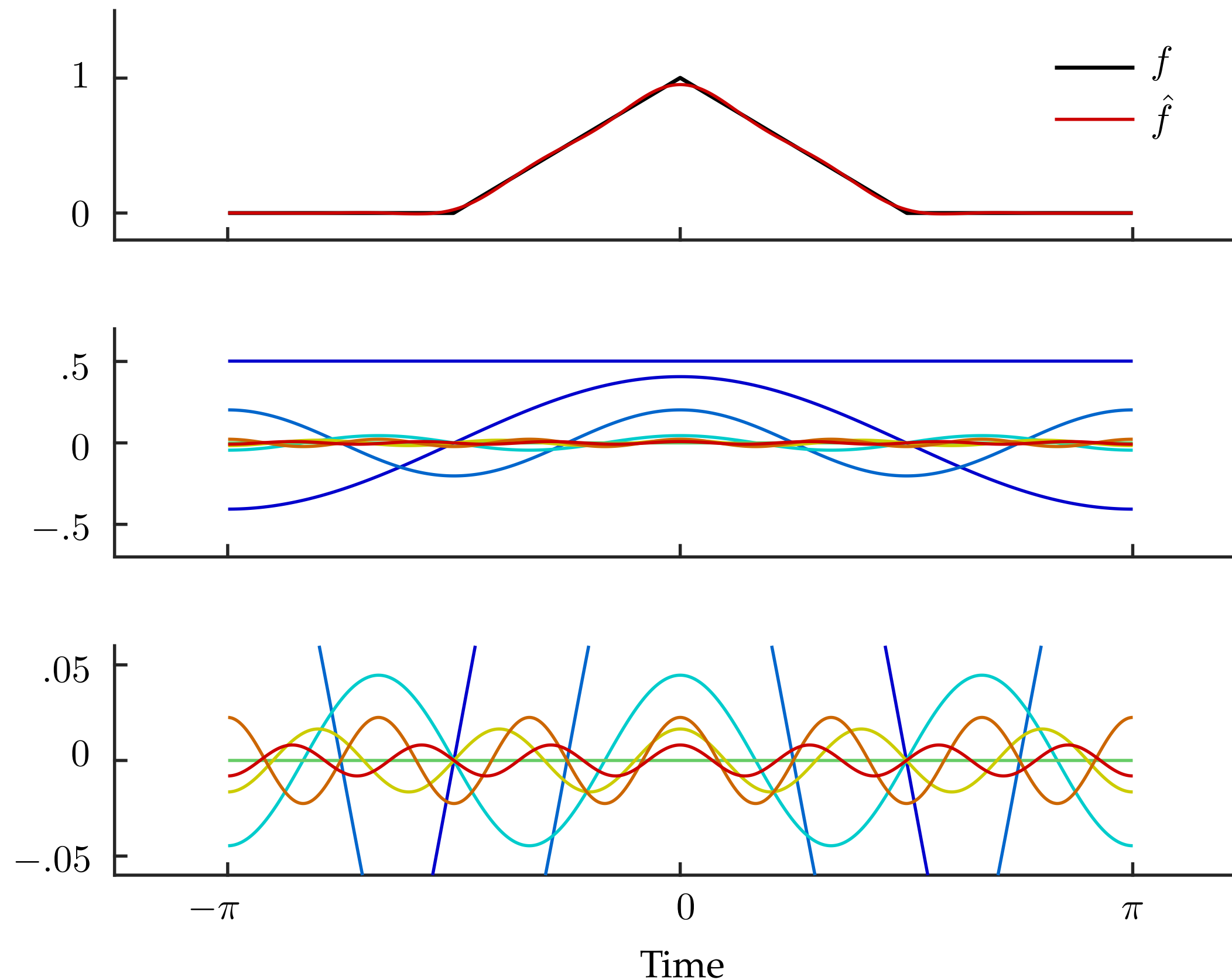
Add first 4 waves



Add first 5 waves

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

Second Example



The periodic hat function and the sum of the first seven terms

The first seven terms

Zoom in so you see the high-frequency part

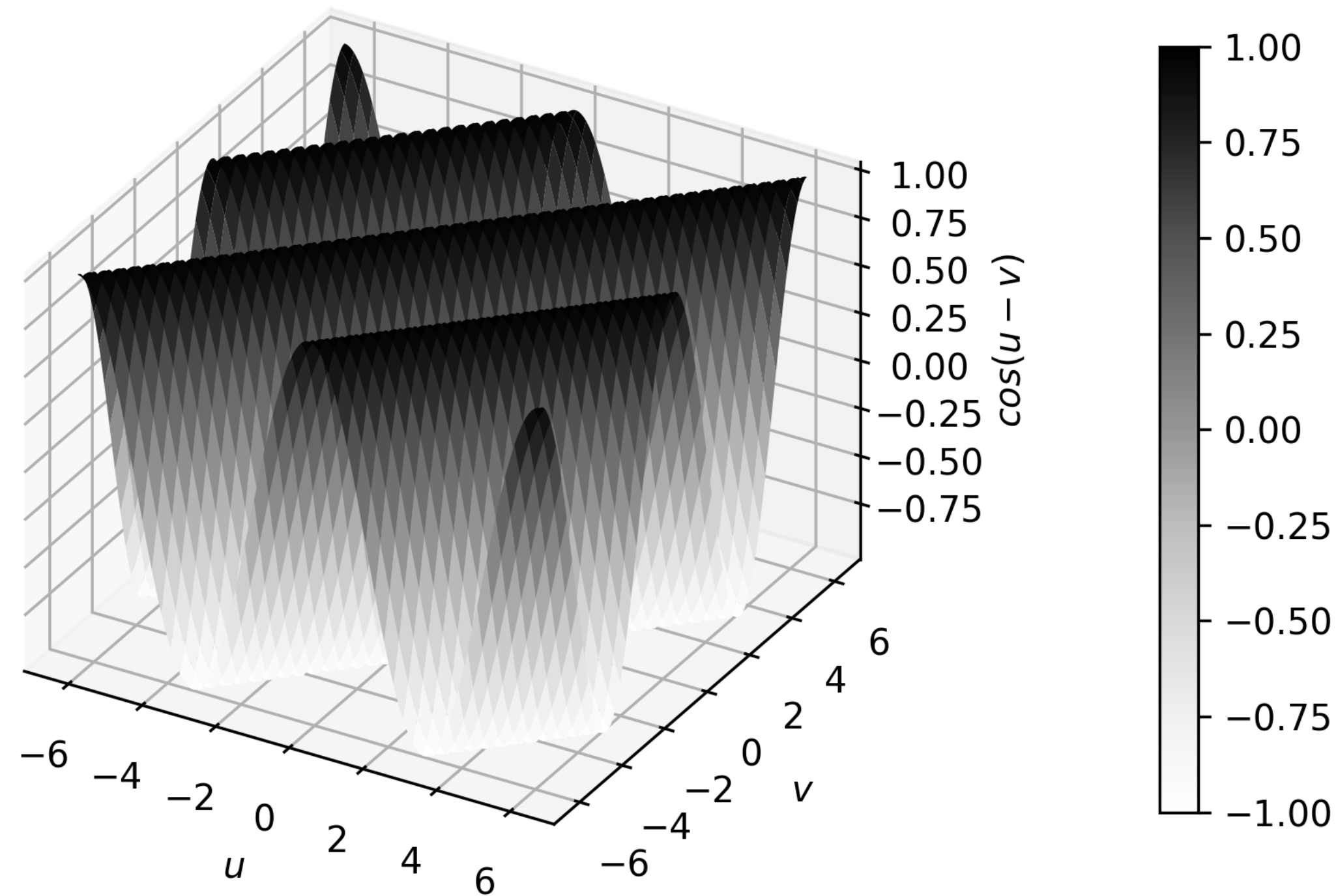
$$f(x) = \frac{1}{2} + \frac{4}{\pi^2} \cos(x) + \frac{4}{(3\pi)^2} \cos(3x) + \frac{4}{(5\pi)^2} \cos(5x) + \frac{4}{(7\pi)^2} \cos(7x) + \dots$$

Building Intuitions

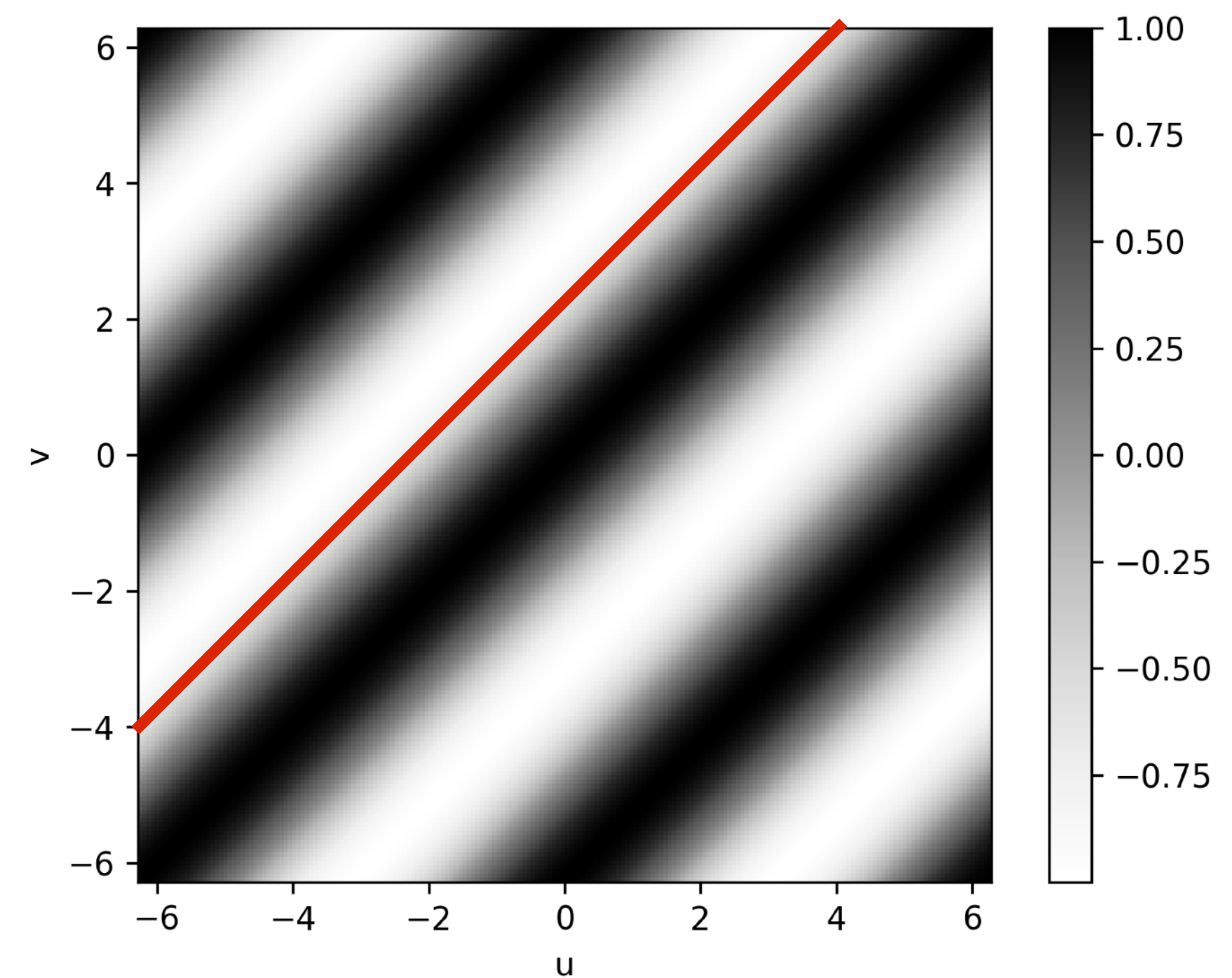
(2D)

2D Sinusoidal Functions

$\cos(u-v)$

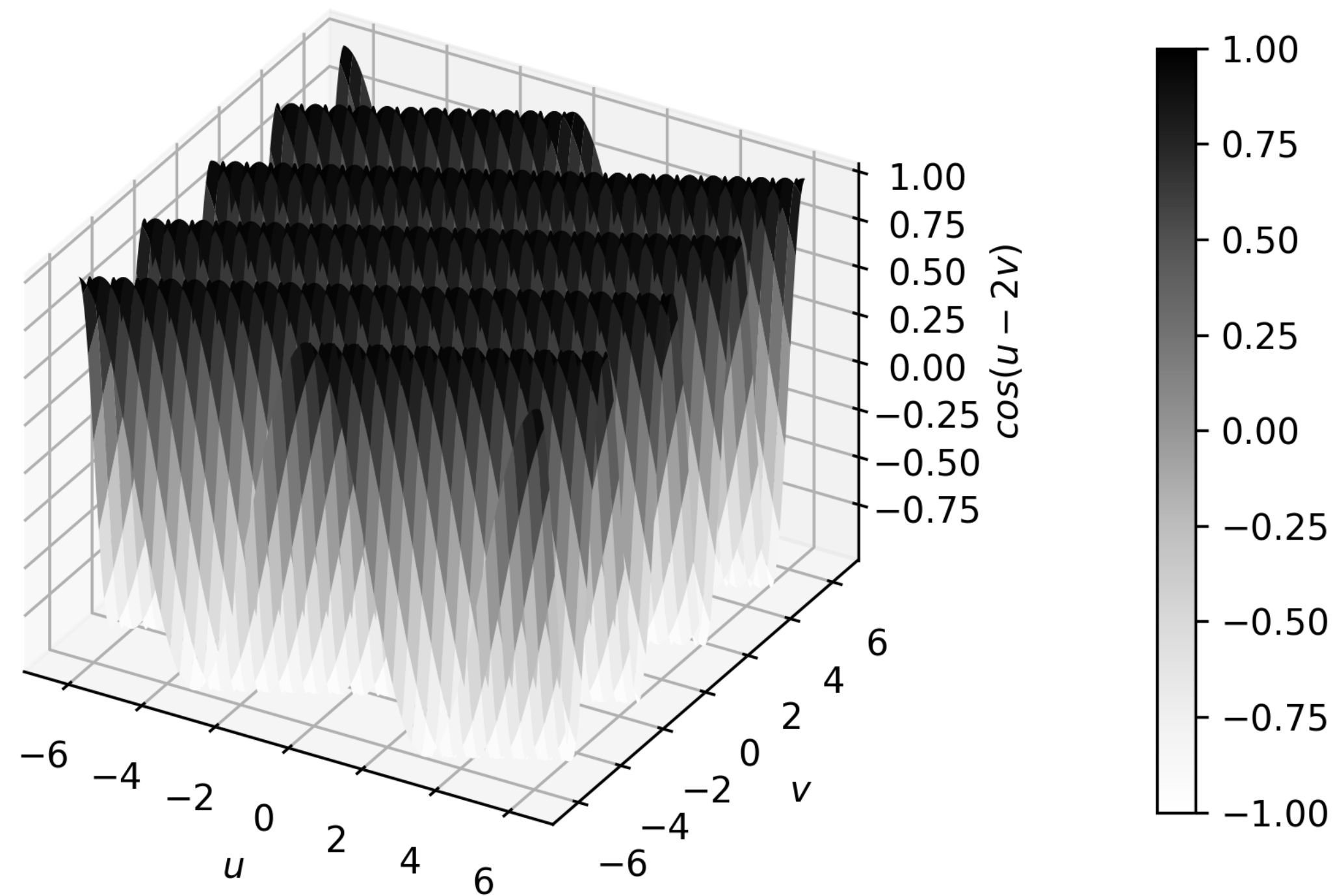


$\cos(u-v)$ is constant where $u-v$
is constant

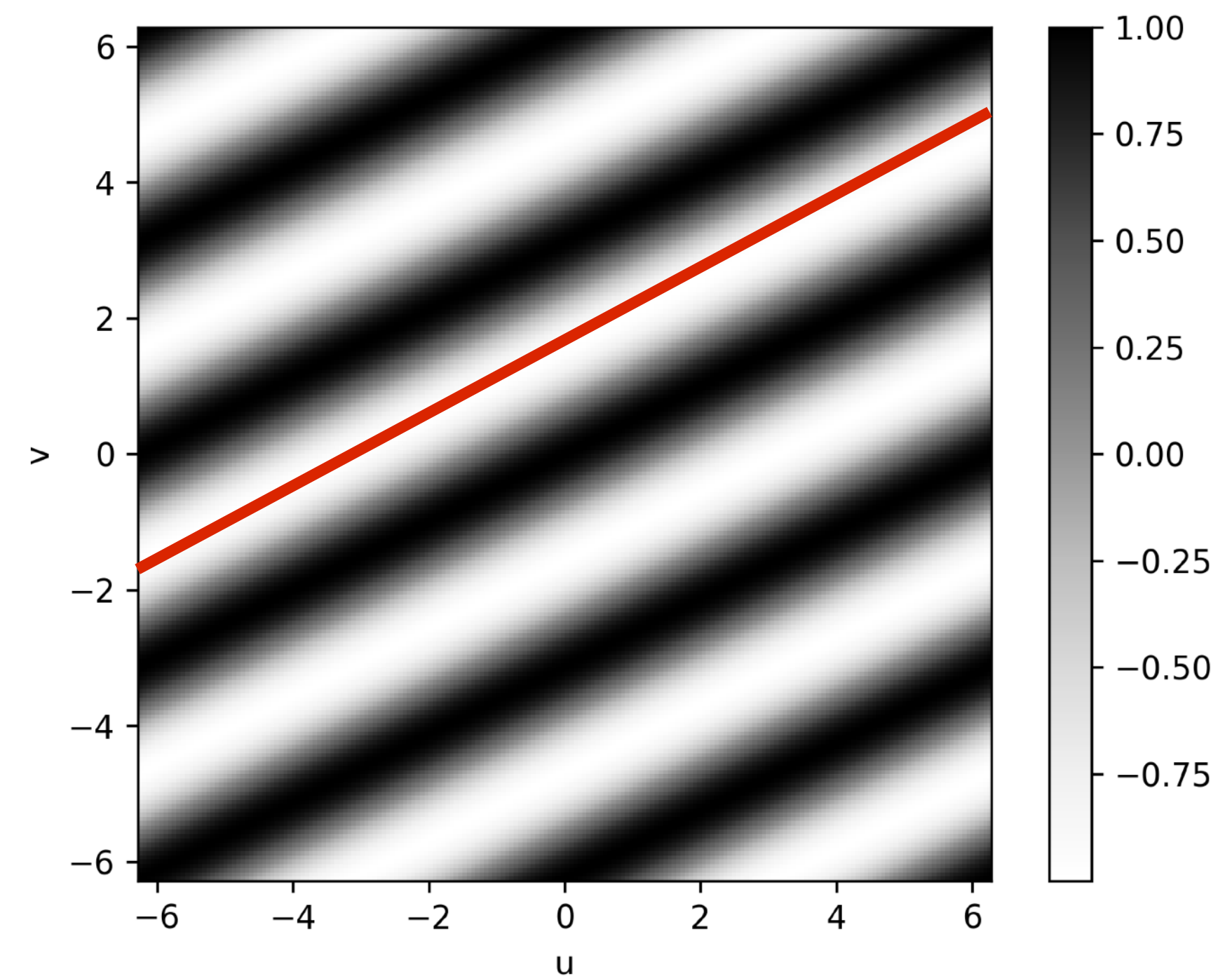


2D Sinusoidal Functions

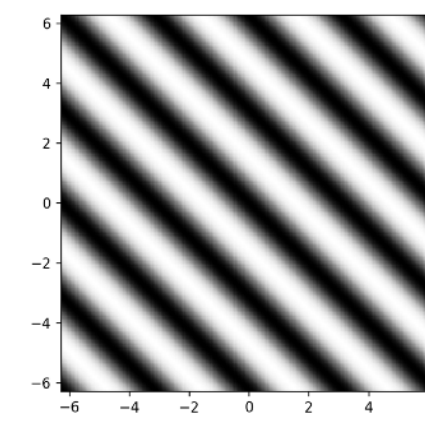
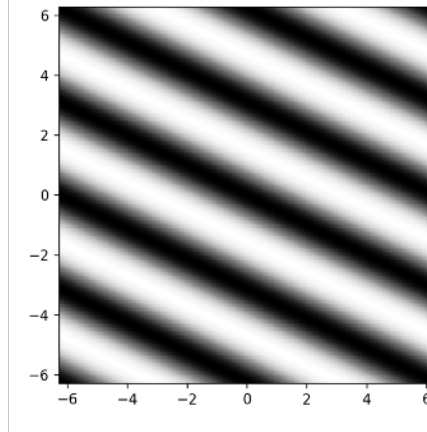
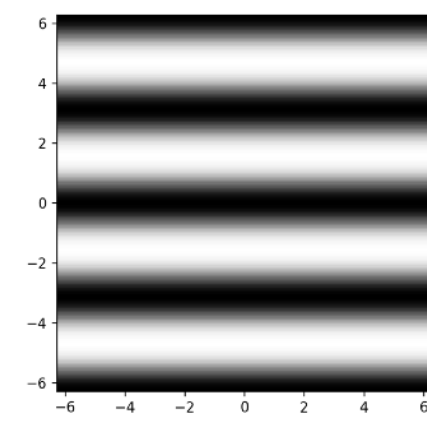
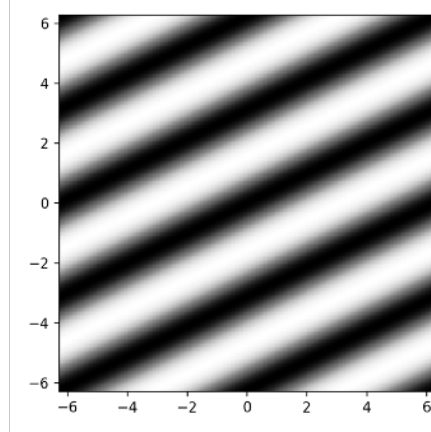
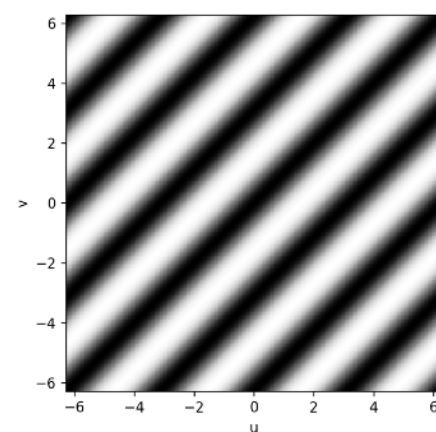
$\cos(u-2v)$



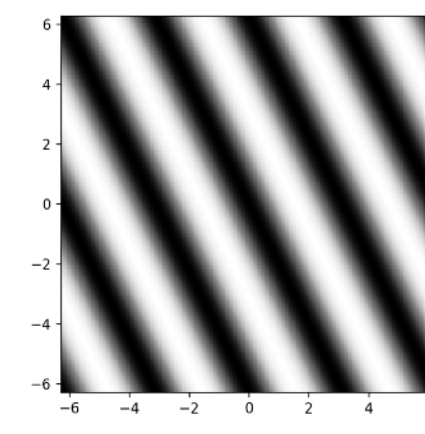
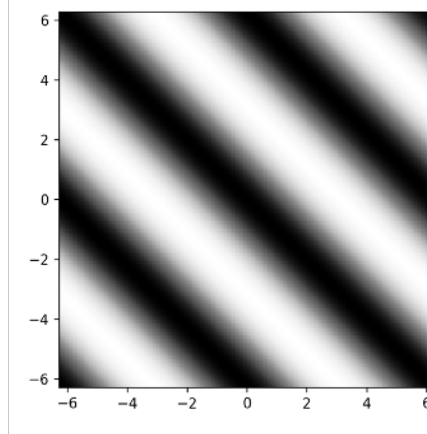
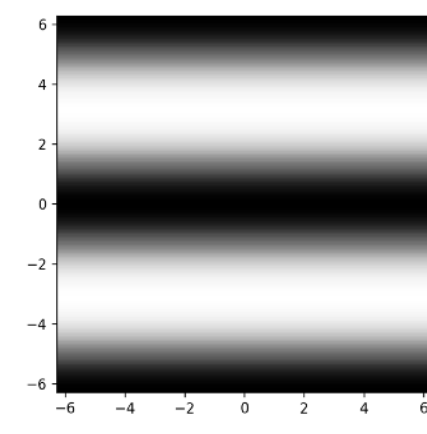
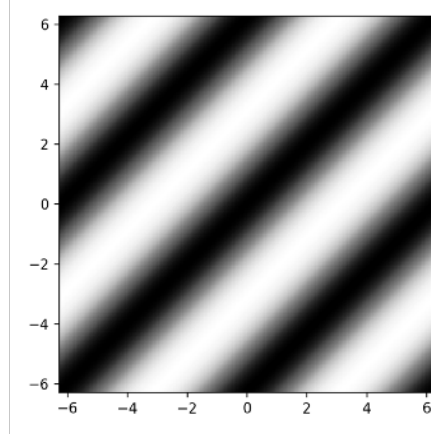
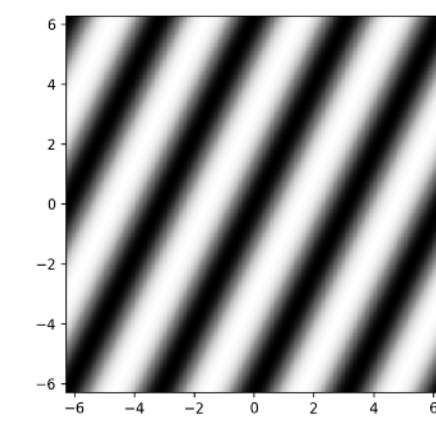
$\cos(u-2v)$ has the same value
with $u-2v$ is constant



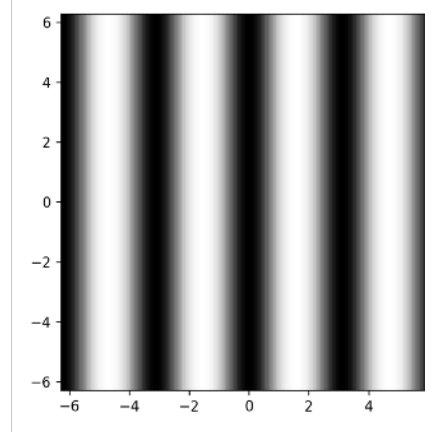
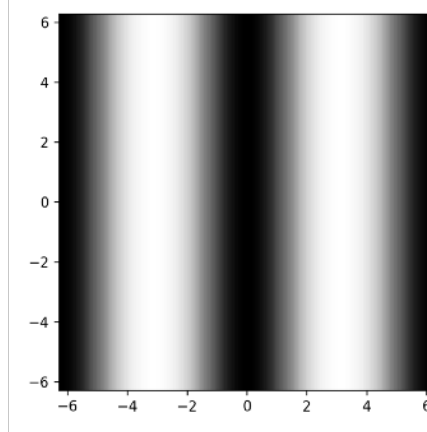
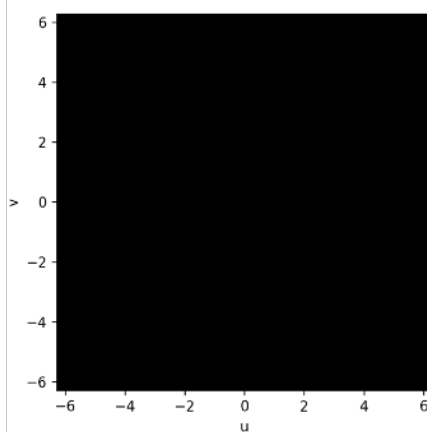
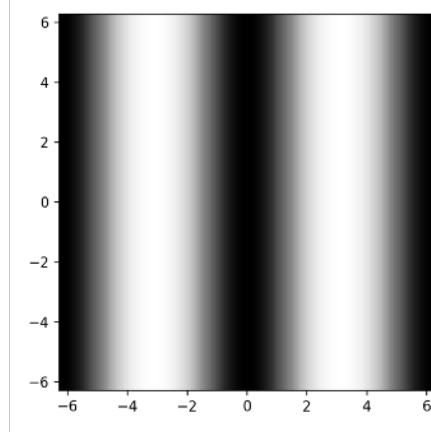
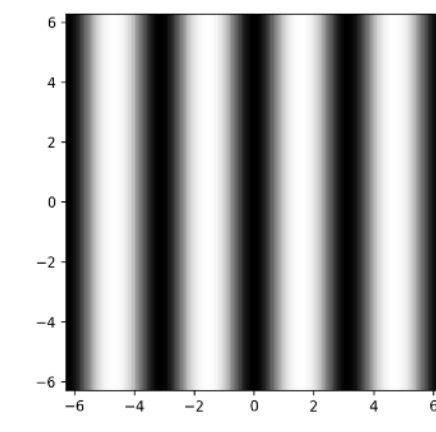
$$\cos(-2u+2v)$$



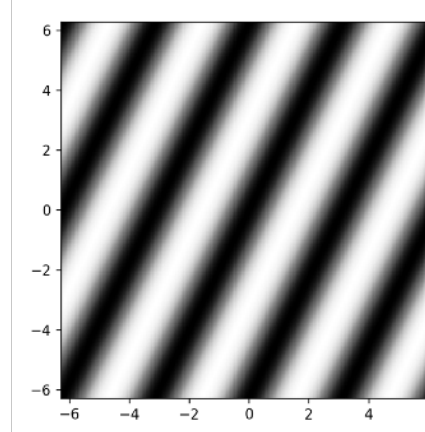
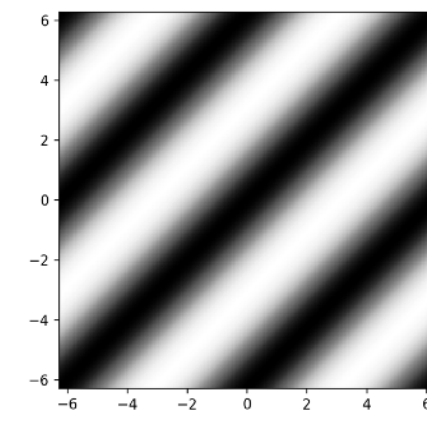
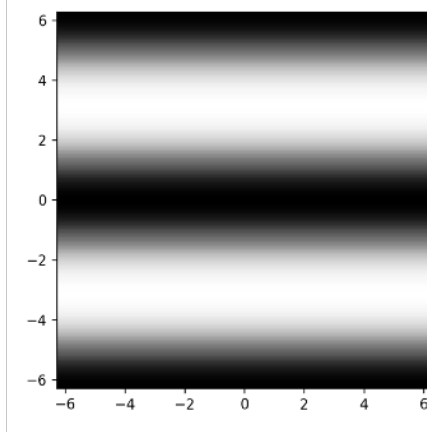
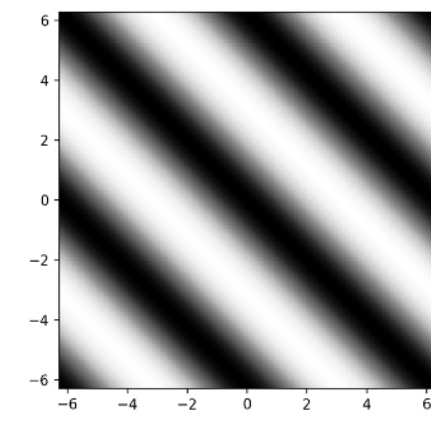
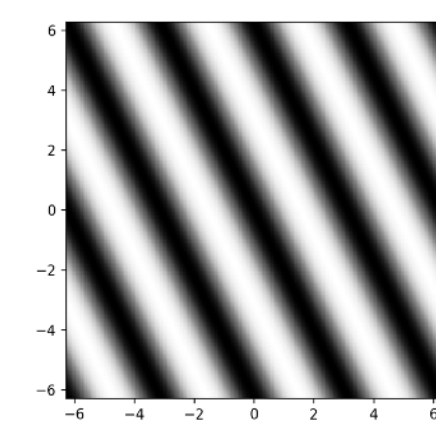
$$\cos(2u+2v)$$



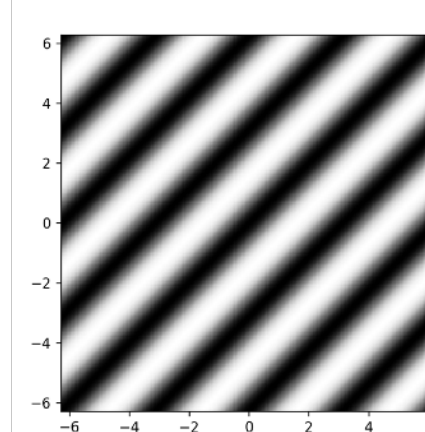
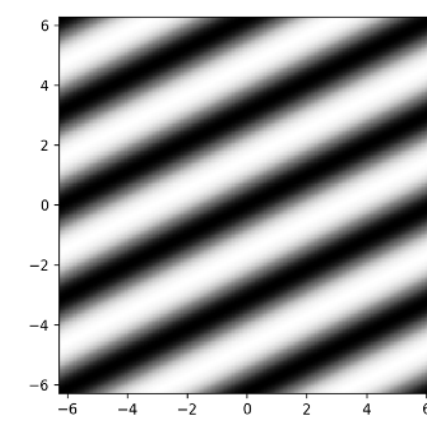
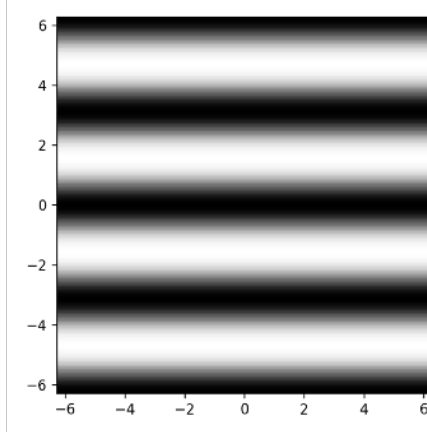
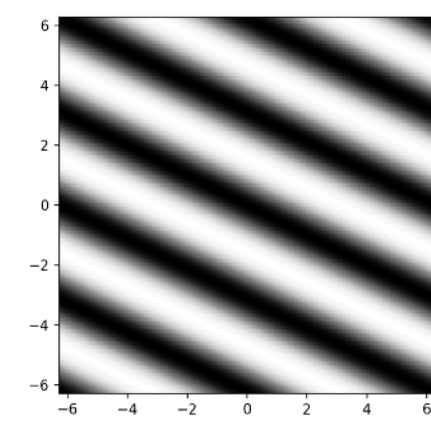
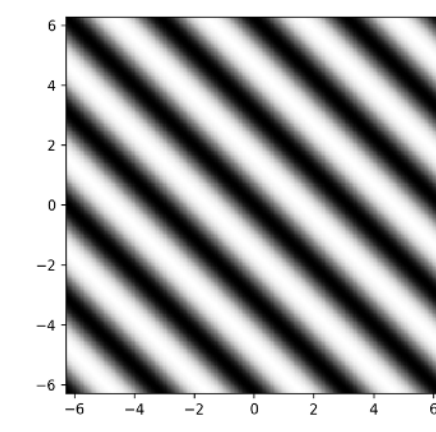
$$\cos(-2u+0v)$$



→ **u**



$$\cos(-2u-2v)$$

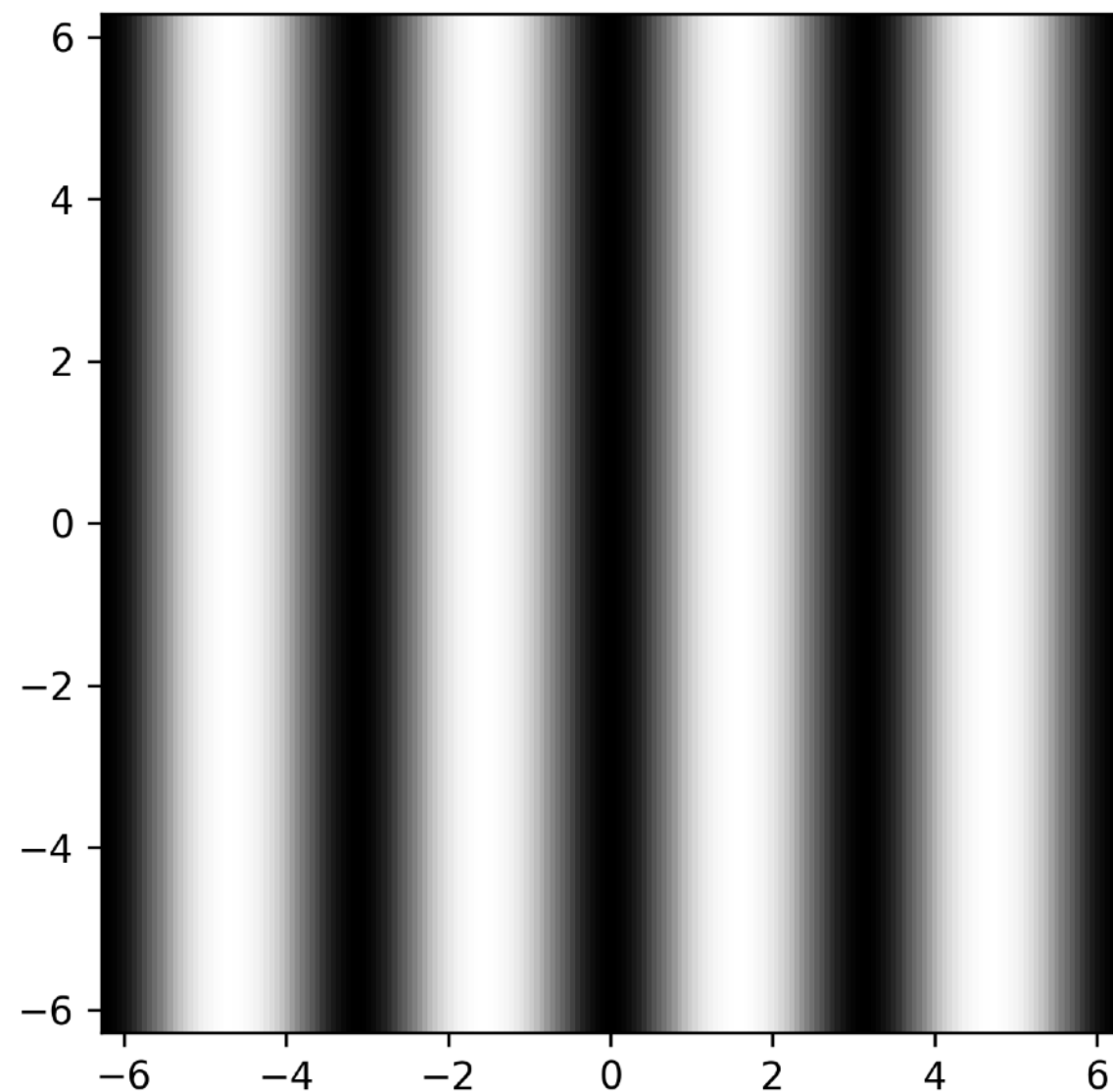


$$\cos(2u-2v)$$

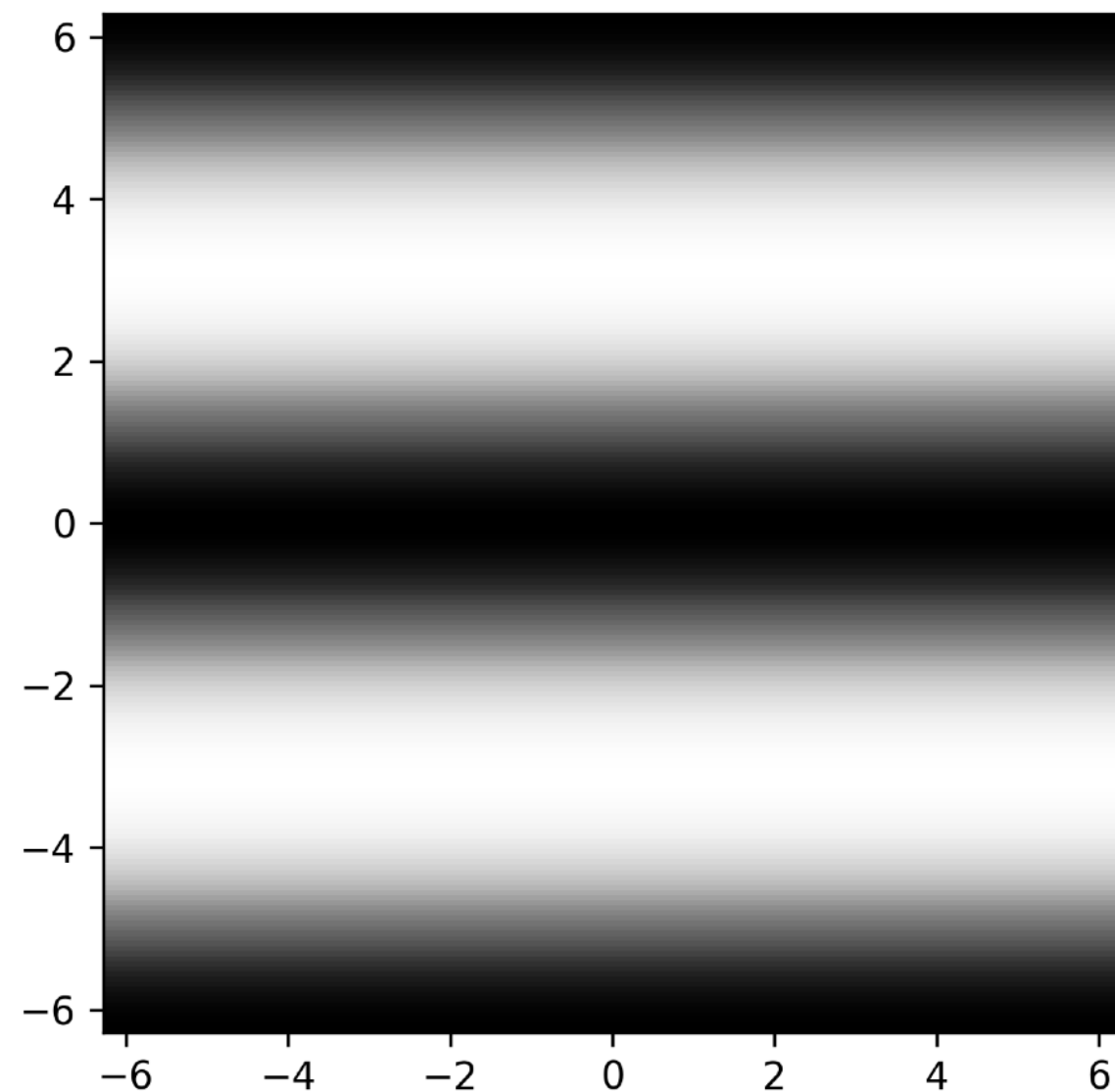
$$\cos(0u-2v)$$

2D Frequency Intuitions

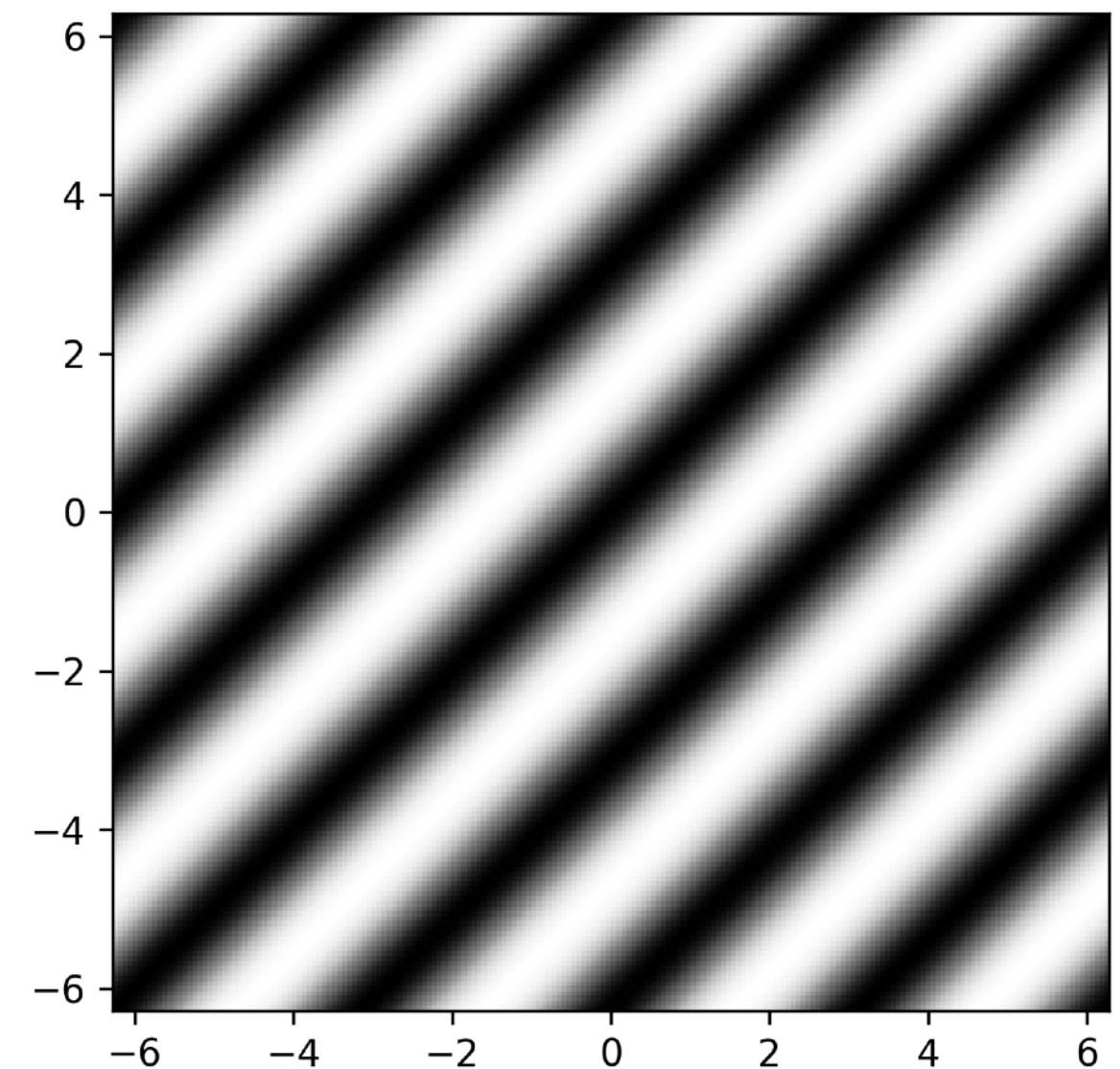
- 2D frequencies characterize the image spatial changes in horizontal and vertical directions
 - Smooth changes \rightarrow low frequencies
 - Sharp changes \rightarrow high frequencies



$\cos(2u+0v)$



$\cos(0u-v)$



$\cos(2u-2v)$

Image = 2D Signal

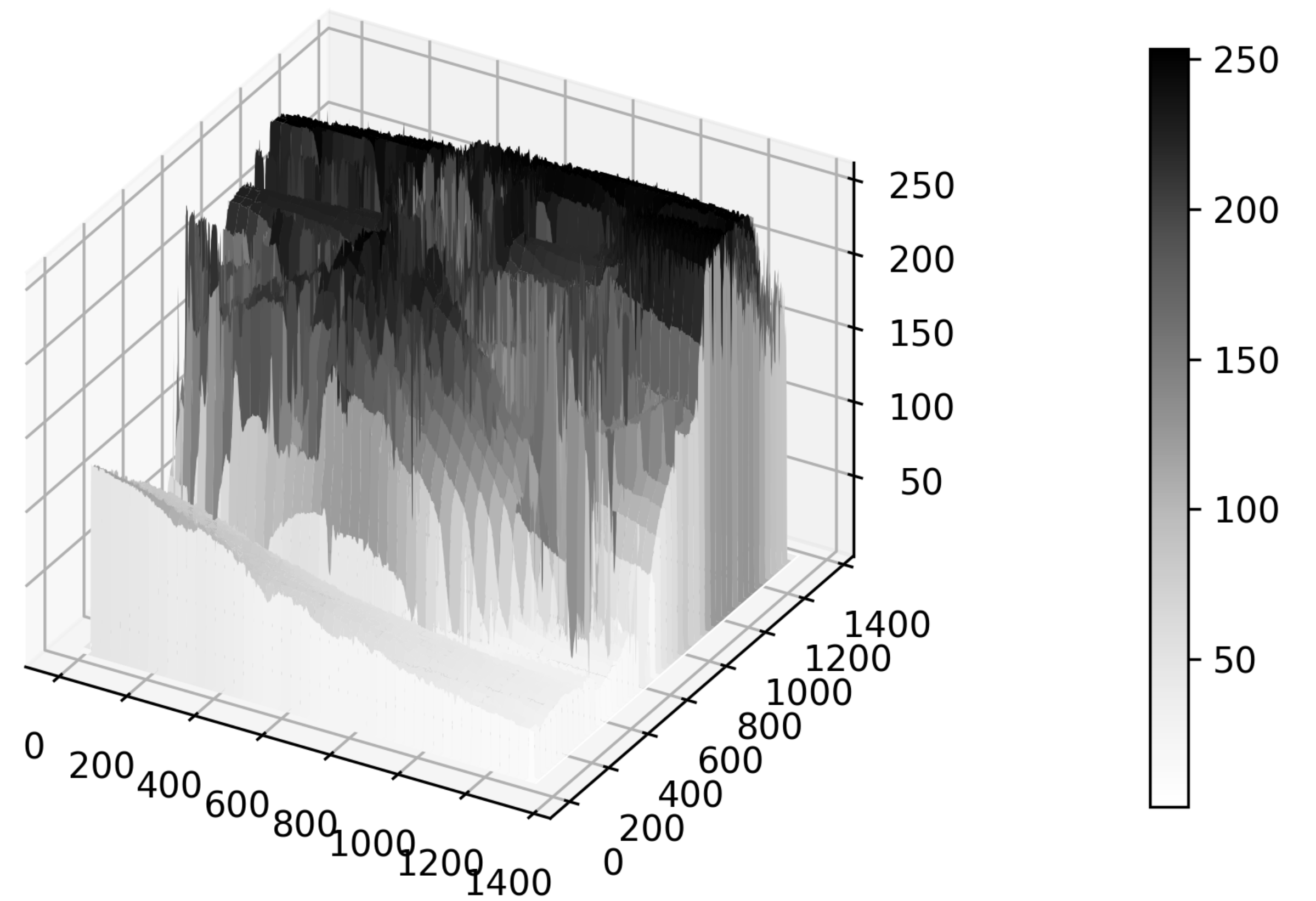
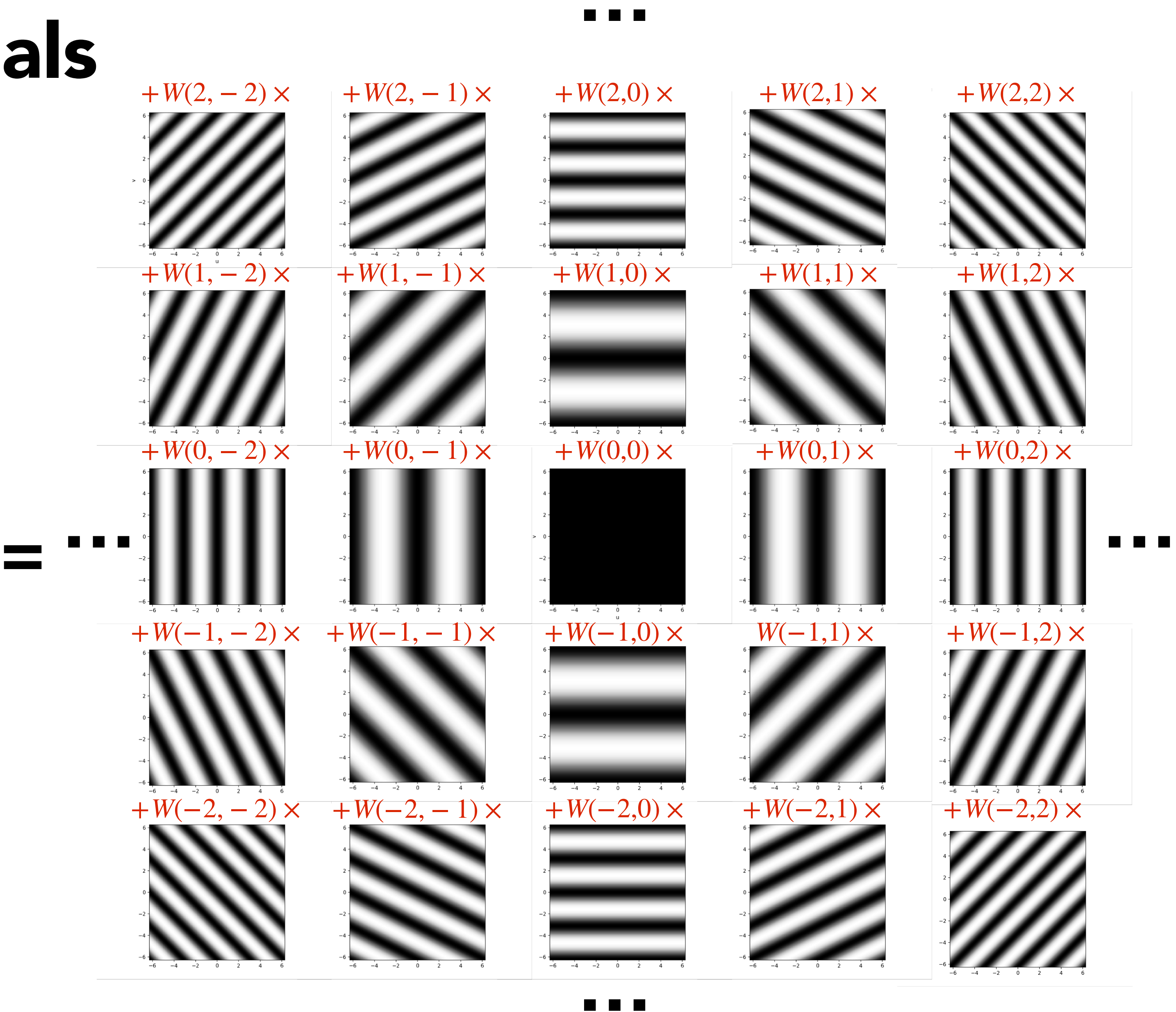


Image = 2D Signal

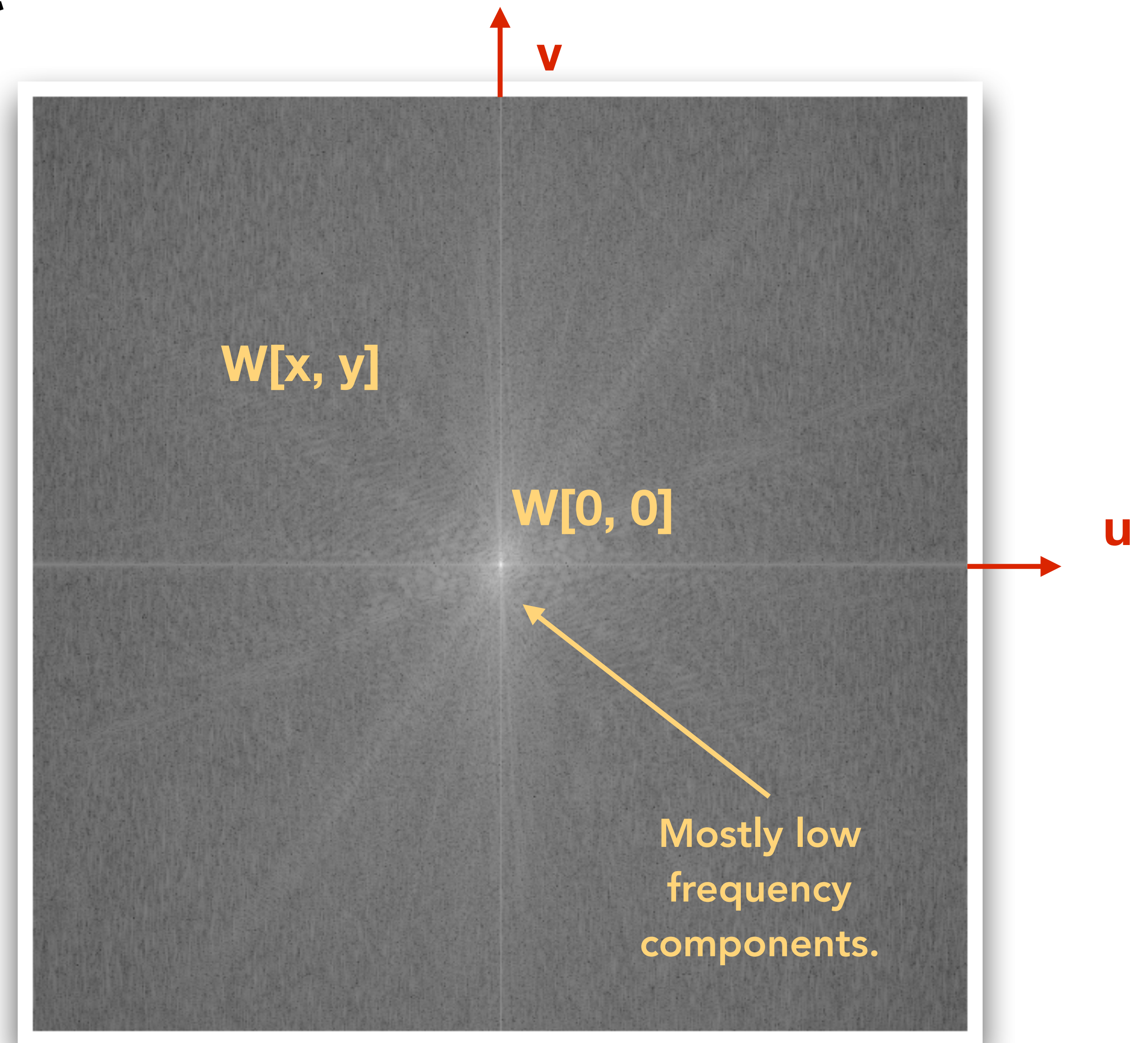


- An image is nothing more than a (complicated) 2D function $f(x, y)$
- $f(x, y)$ maps a pixel coordinates $[x, y]$ to its pixel value
- Can a 2D signal be expressed as a weighted sum of a set of 2D sinusoidal functions?

Decomposing 2D Signals



Plotting the 2D Weights*

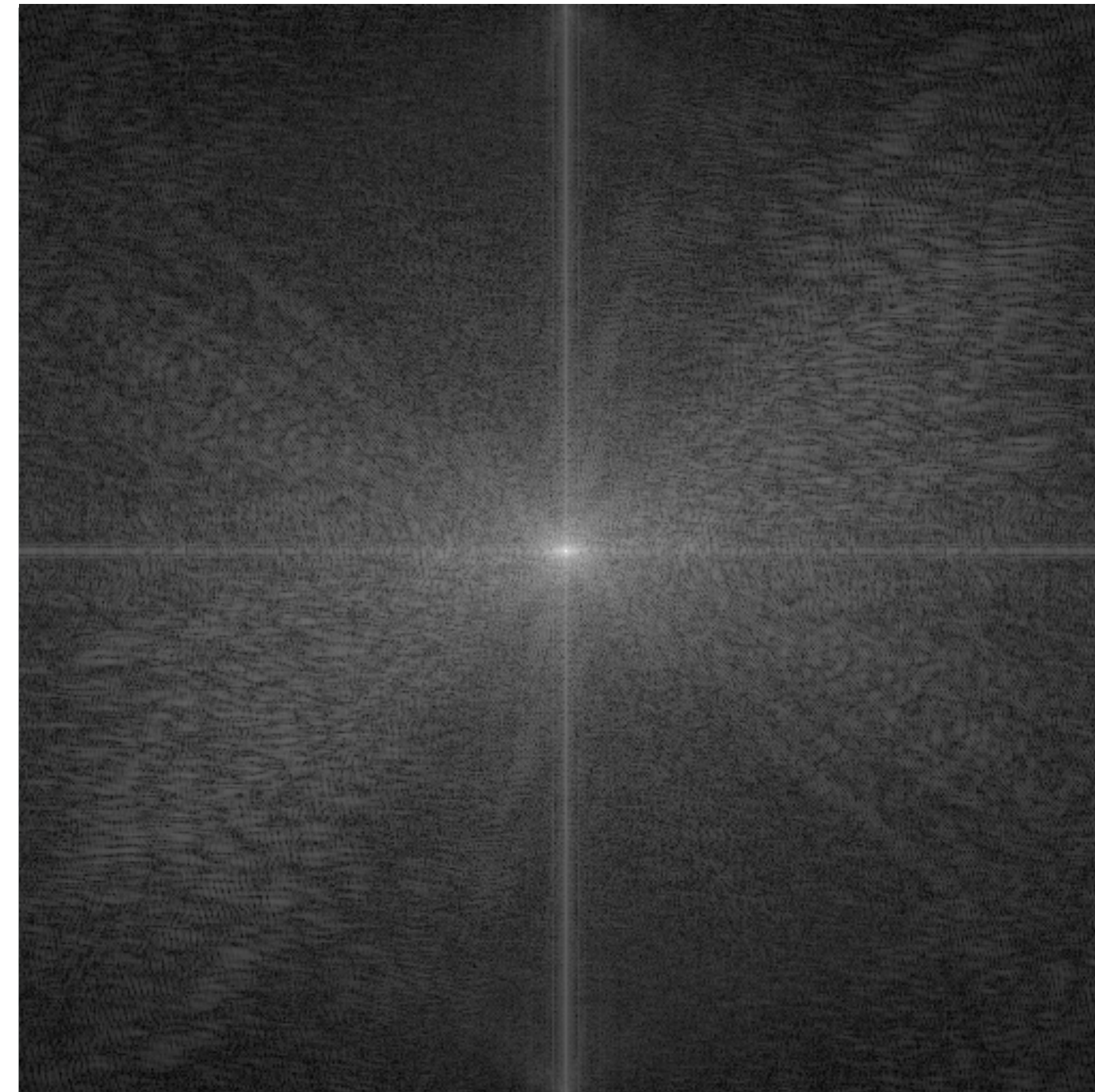


* the weights are actually complex values, so we plot their magnitudes here.

Manipulating the Weights



Spatial Domain

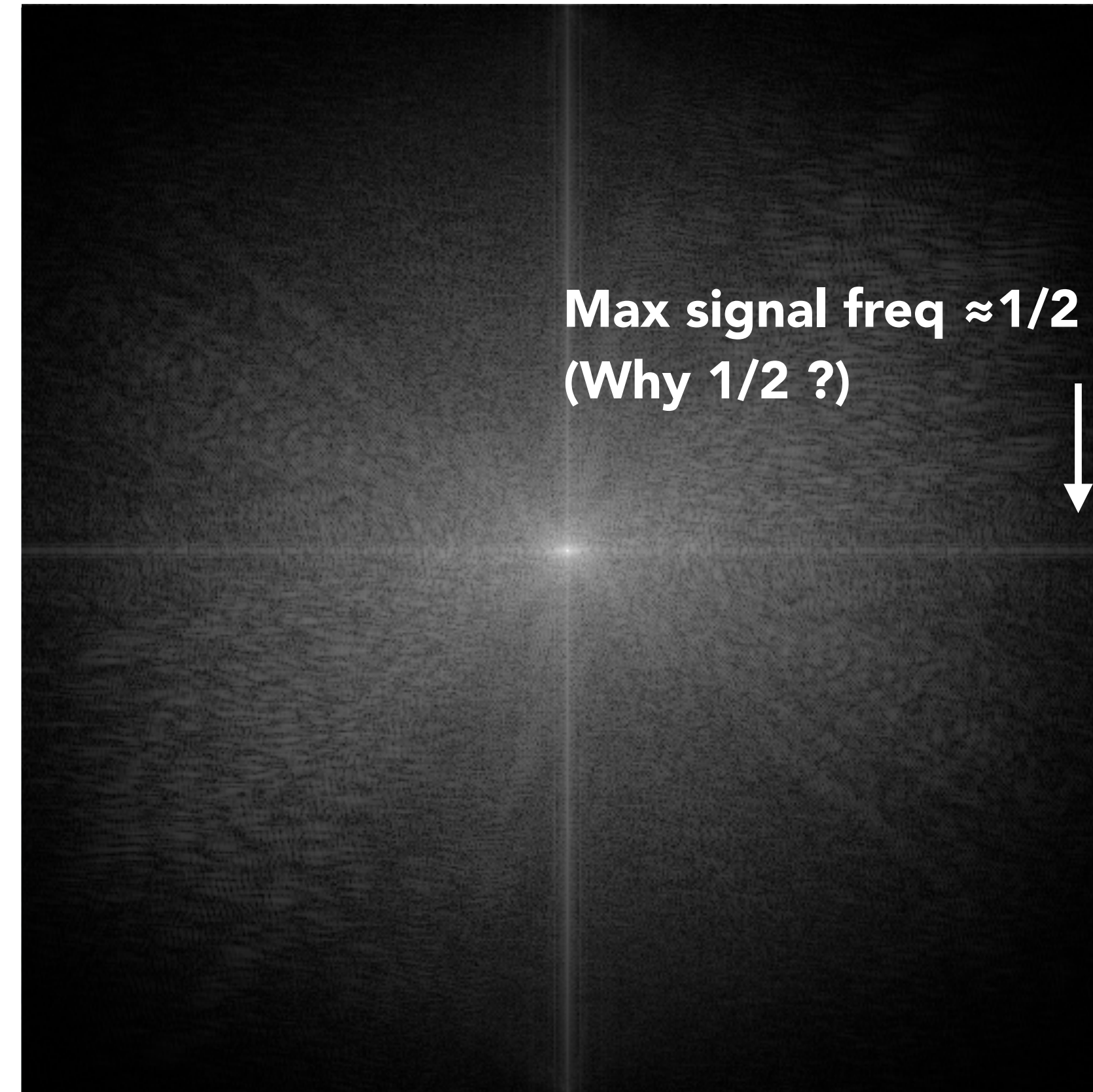


Frequency Domain

Manipulating the Weights



Spatial Domain

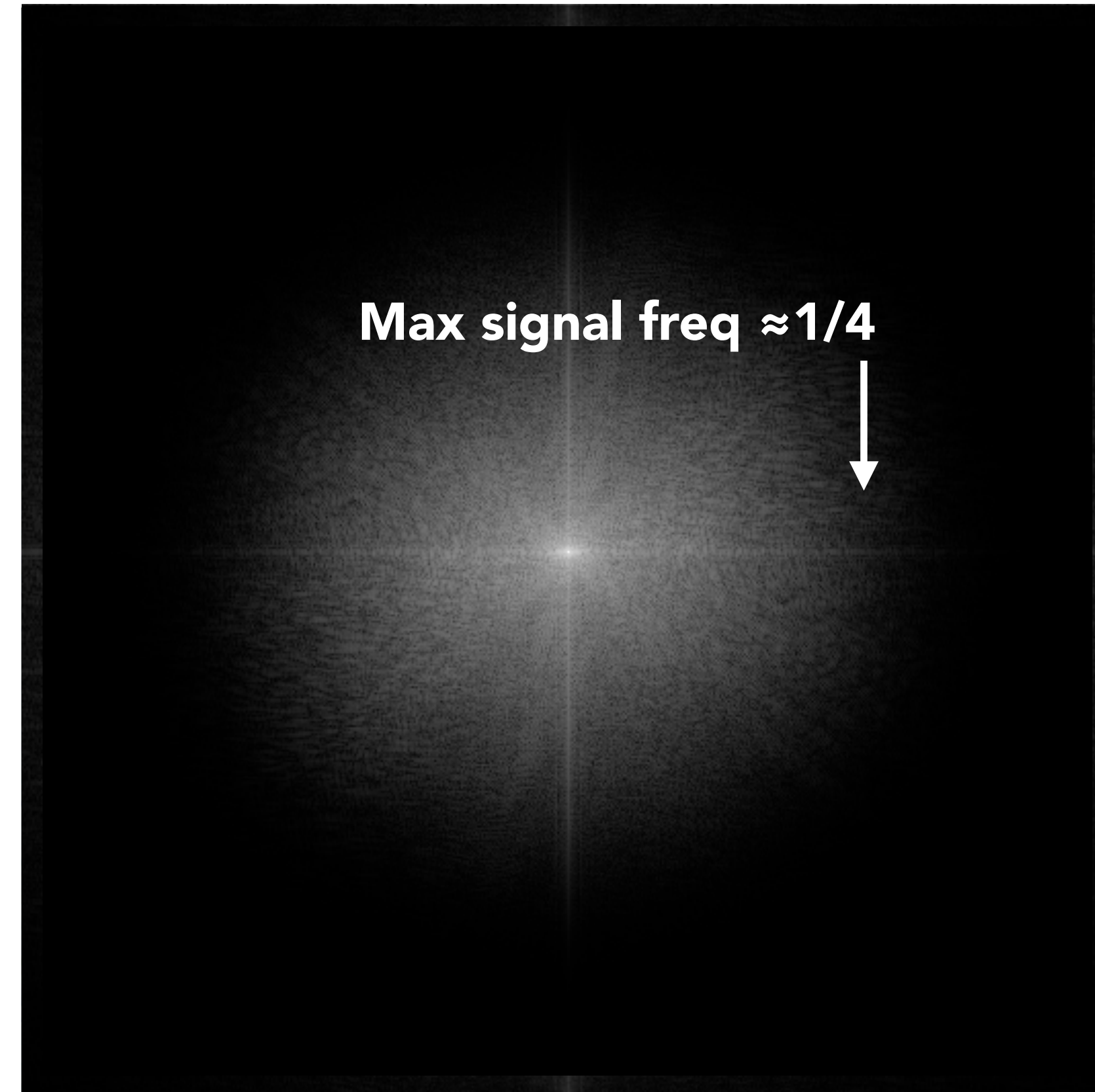


Frequency Domain

Manipulating the Weights



Spatial Domain

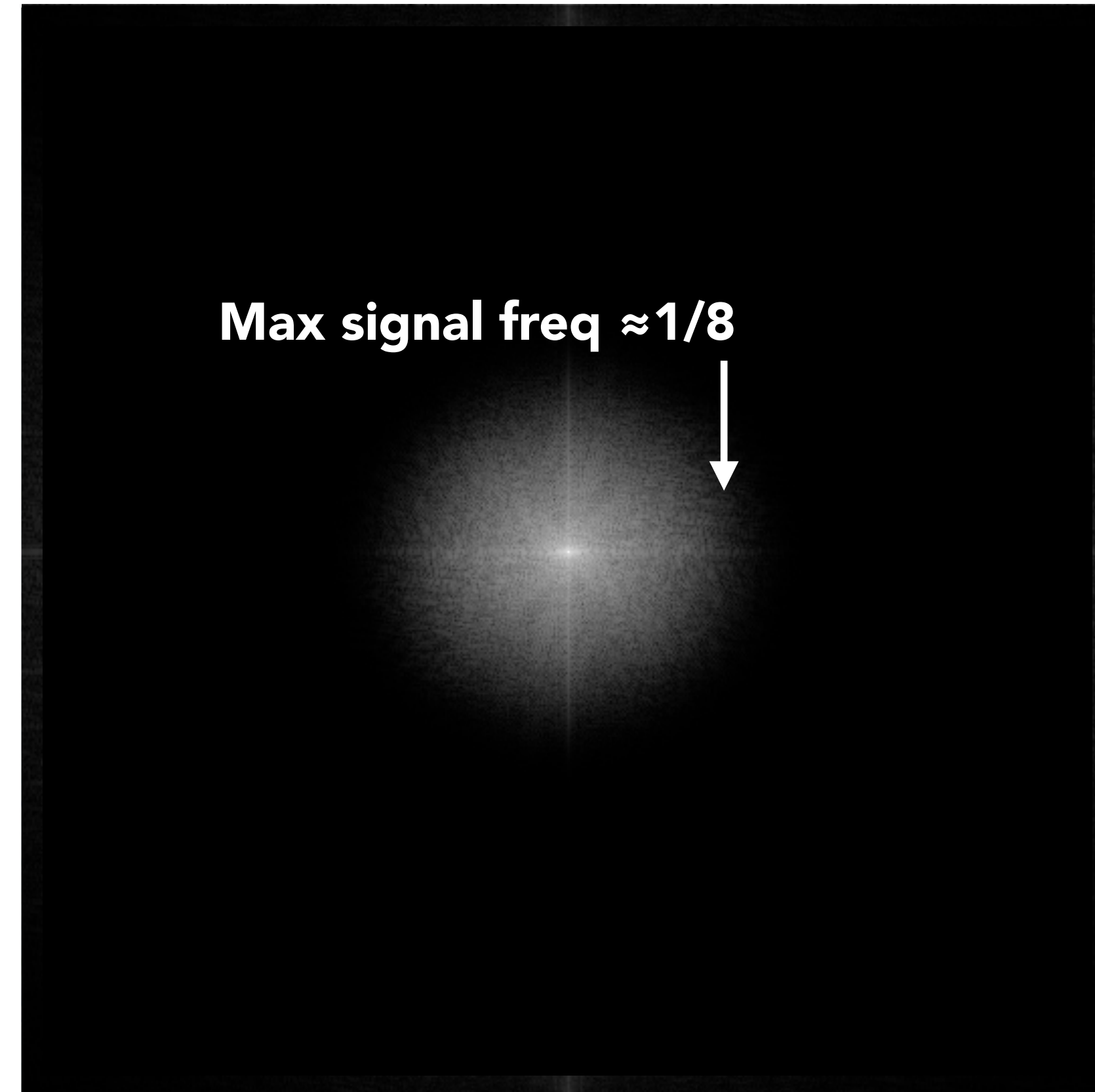


Frequency Domain

Manipulating the Weights



Spatial Domain

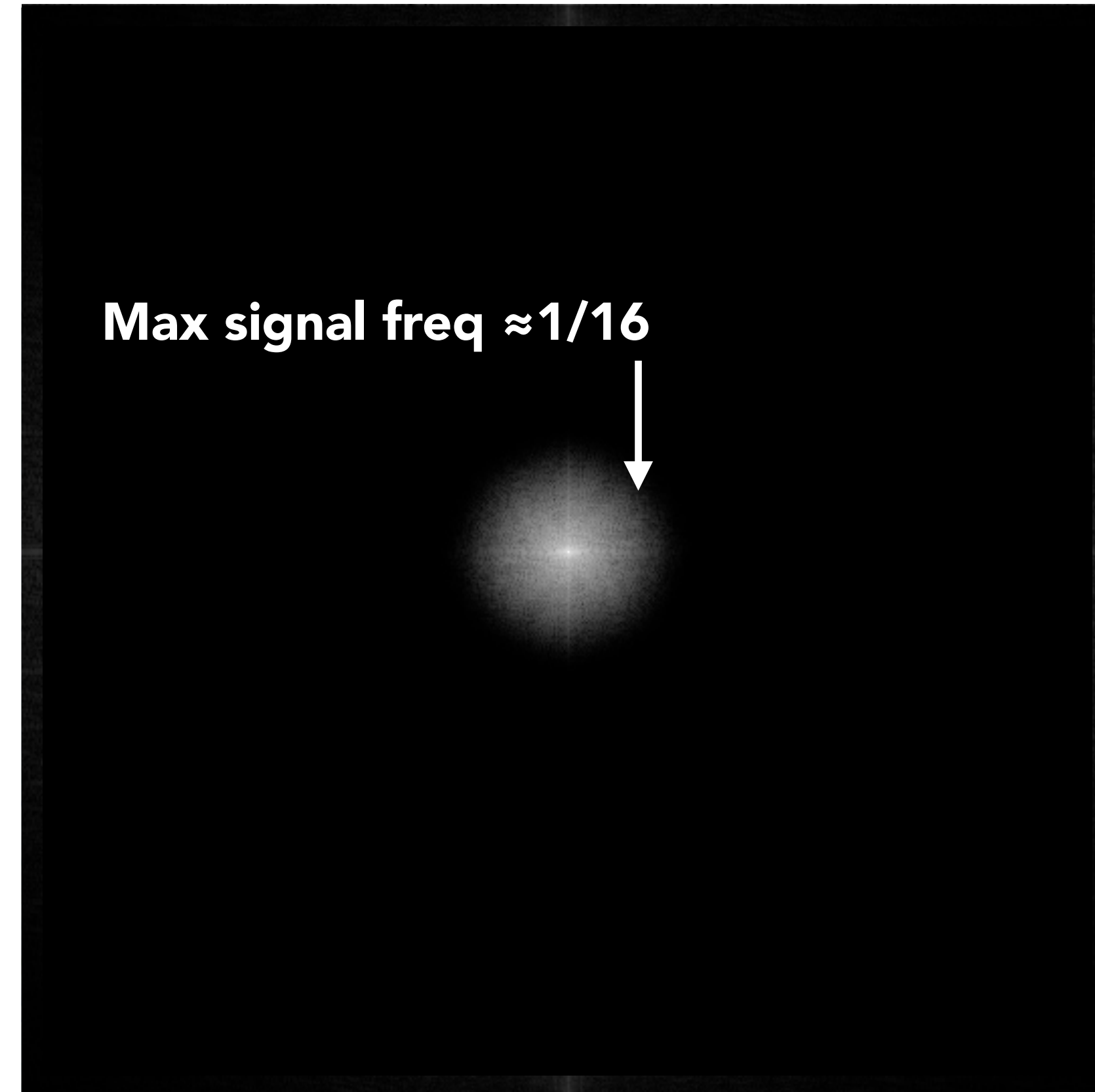


Frequency Domain

Manipulating the Weights



Spatial Domain

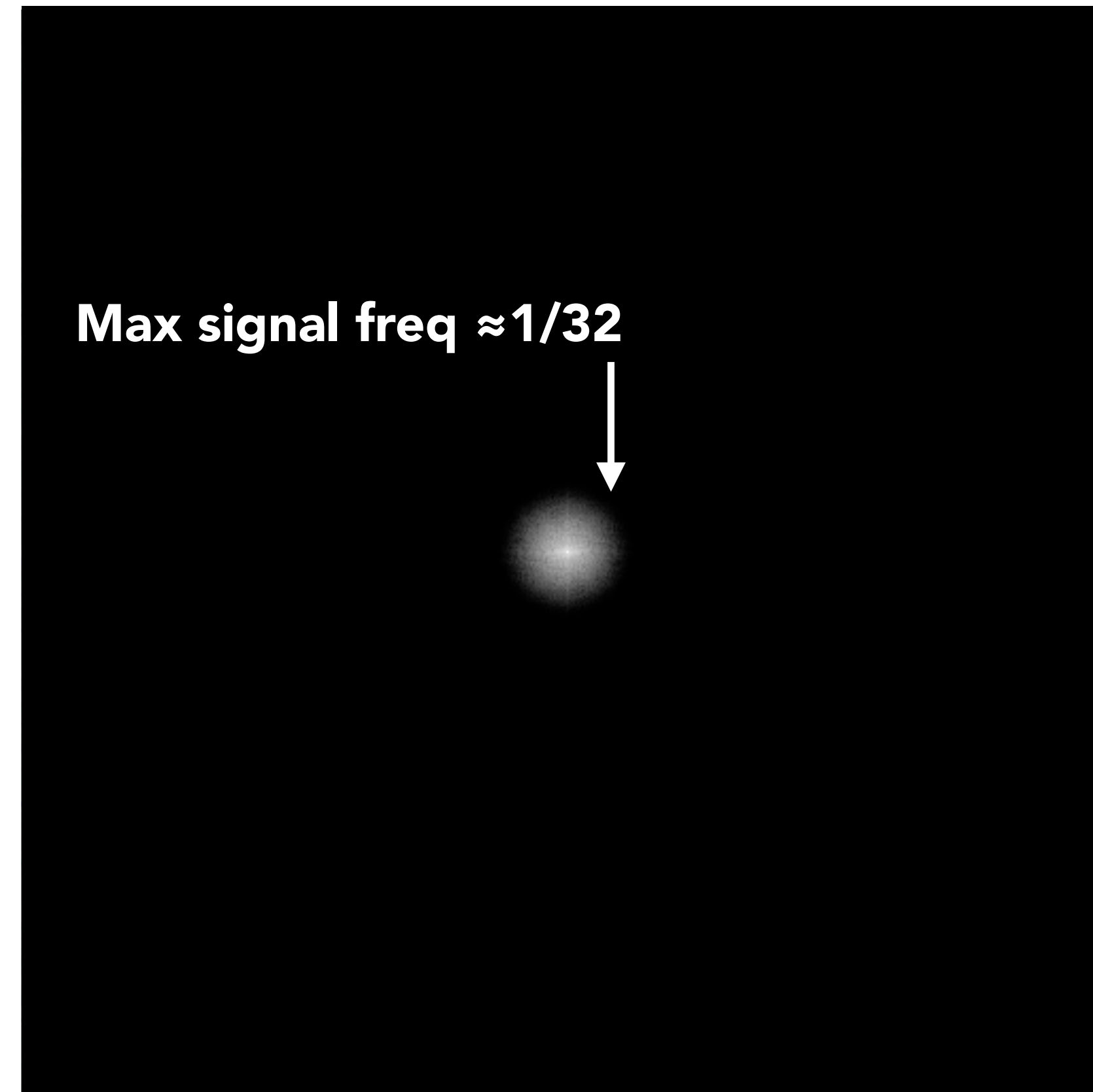


Frequency Domain

Manipulating the Weights



Spatial Domain

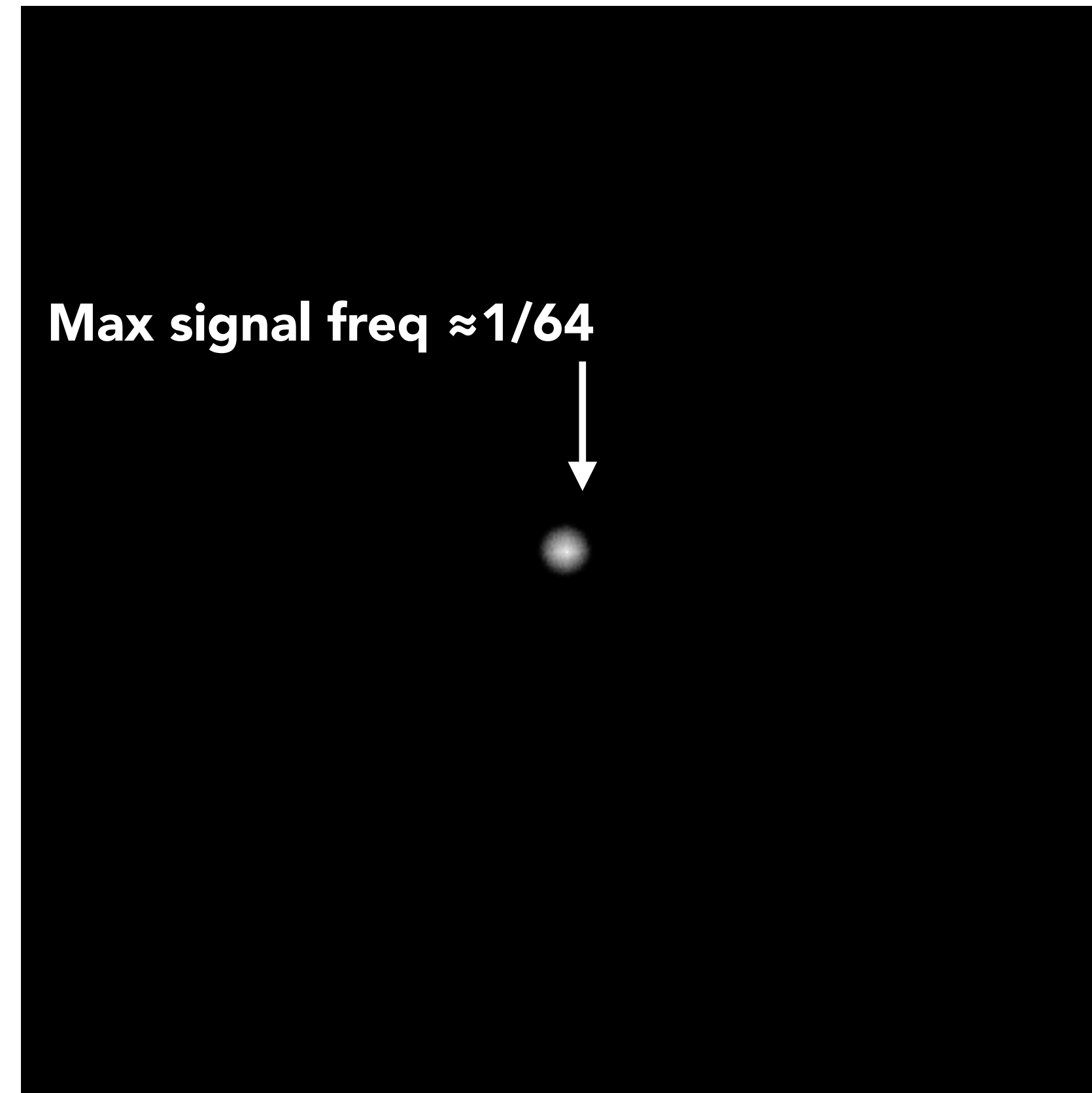


Frequency Domain

Manipulating the Weights



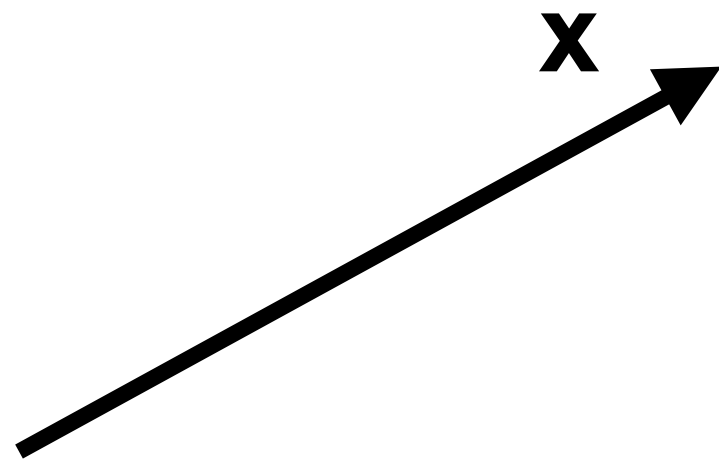
Spatial Domain



Frequency Domain

Why Does It Work?

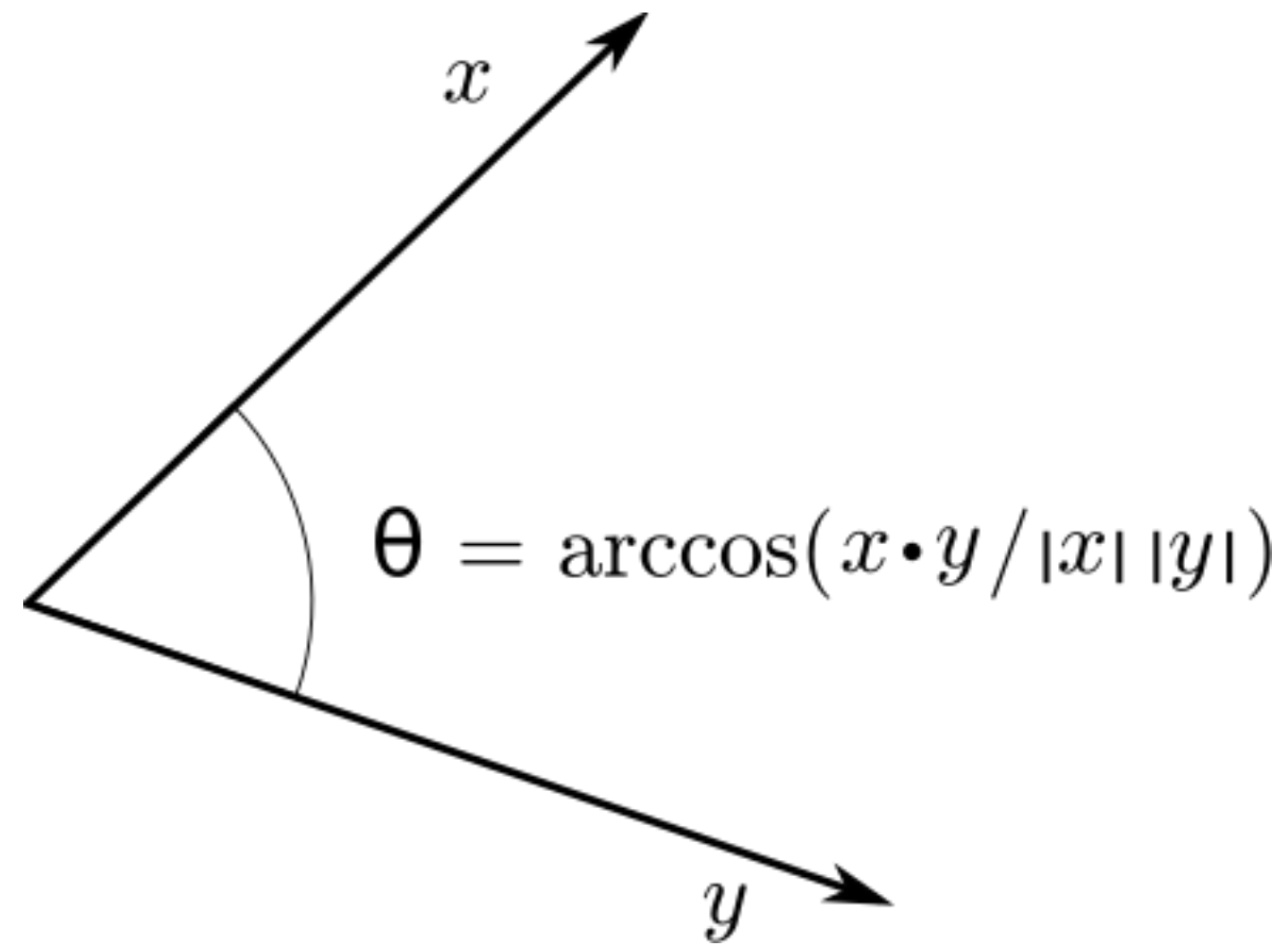
Vector Norm



Vector norm $||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$

Unit vector $\hat{\mathbf{x}} = \frac{\mathbf{x}}{||\mathbf{x}||}$

Inner/Dot Product



$$\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2]$$

$$\mathbf{y} = [\mathbf{y}_1, \mathbf{y}_2]$$

Geometric
definition

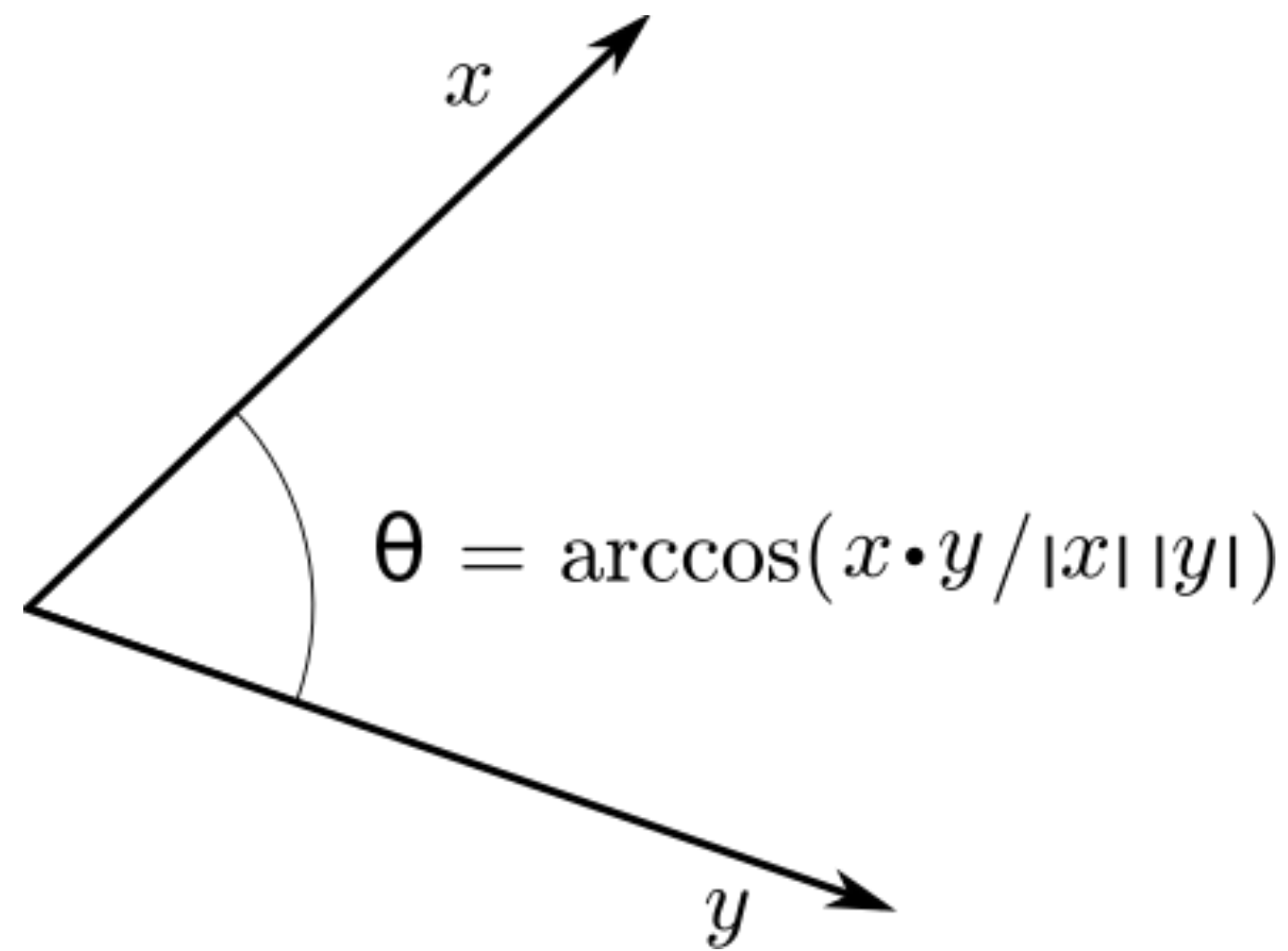
$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta$$

Length (norm) of \mathbf{x}

Algebraic
definition

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}_1 \mathbf{y}_1 + \mathbf{x}_2 \mathbf{y}_2$$

Inner/Dot Product



$$\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2]$$

$$\mathbf{y} = [\mathbf{y}_1, \mathbf{y}_2]$$

These two definitions are equivalent. Prove it in WA2.

Geometric
definition

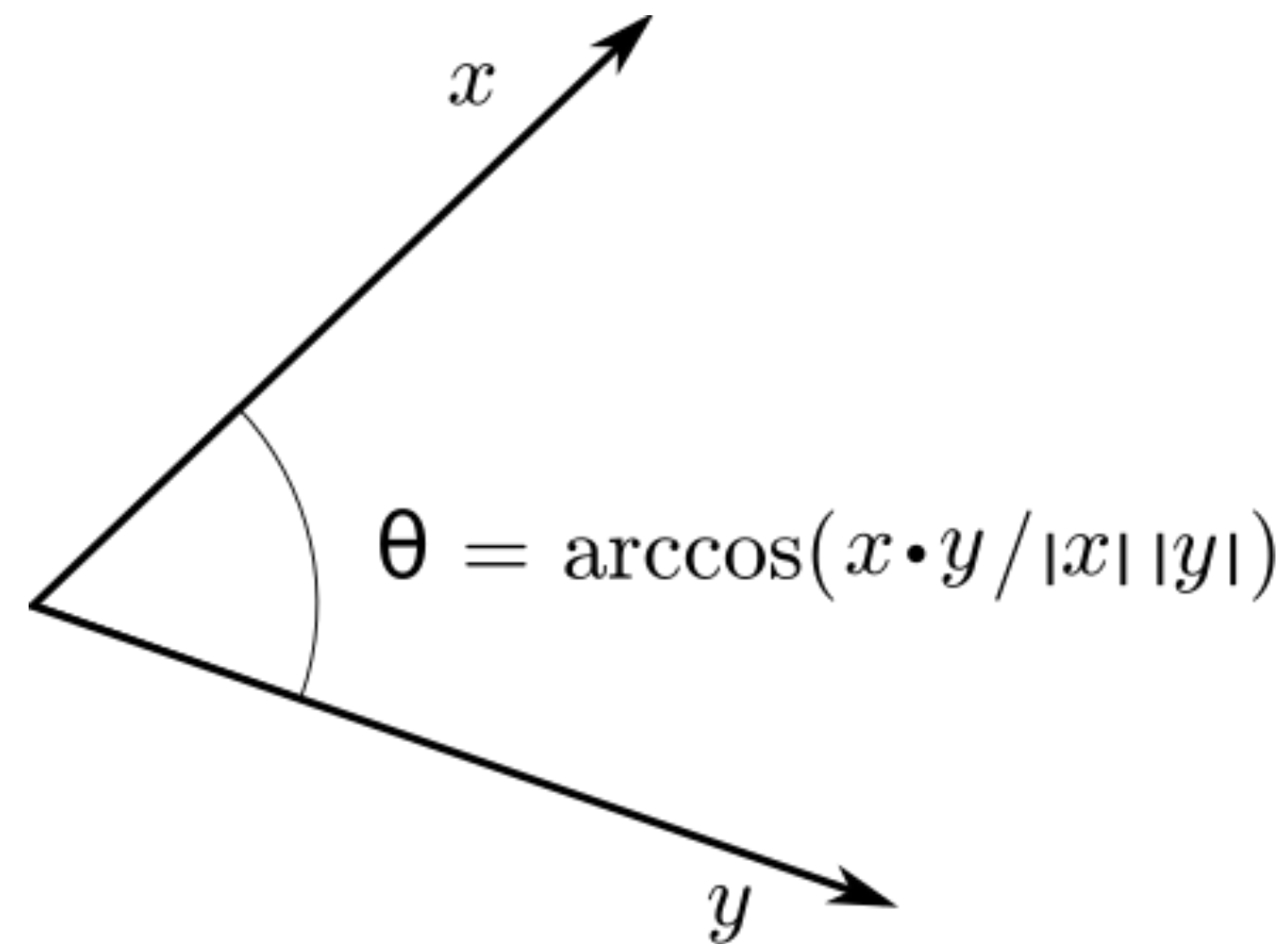
$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta$$

Length (norm) of x

Algebraic
definition

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}_1 \mathbf{y}_1 + \mathbf{x}_2 \mathbf{y}_2$$

Inner/Dot Product



$$\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2]$$

$$\mathbf{y} = [\mathbf{y}_1, \mathbf{y}_2]$$

These two definitions are equivalent. Prove it in WA2.

Geometric definition

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta$$

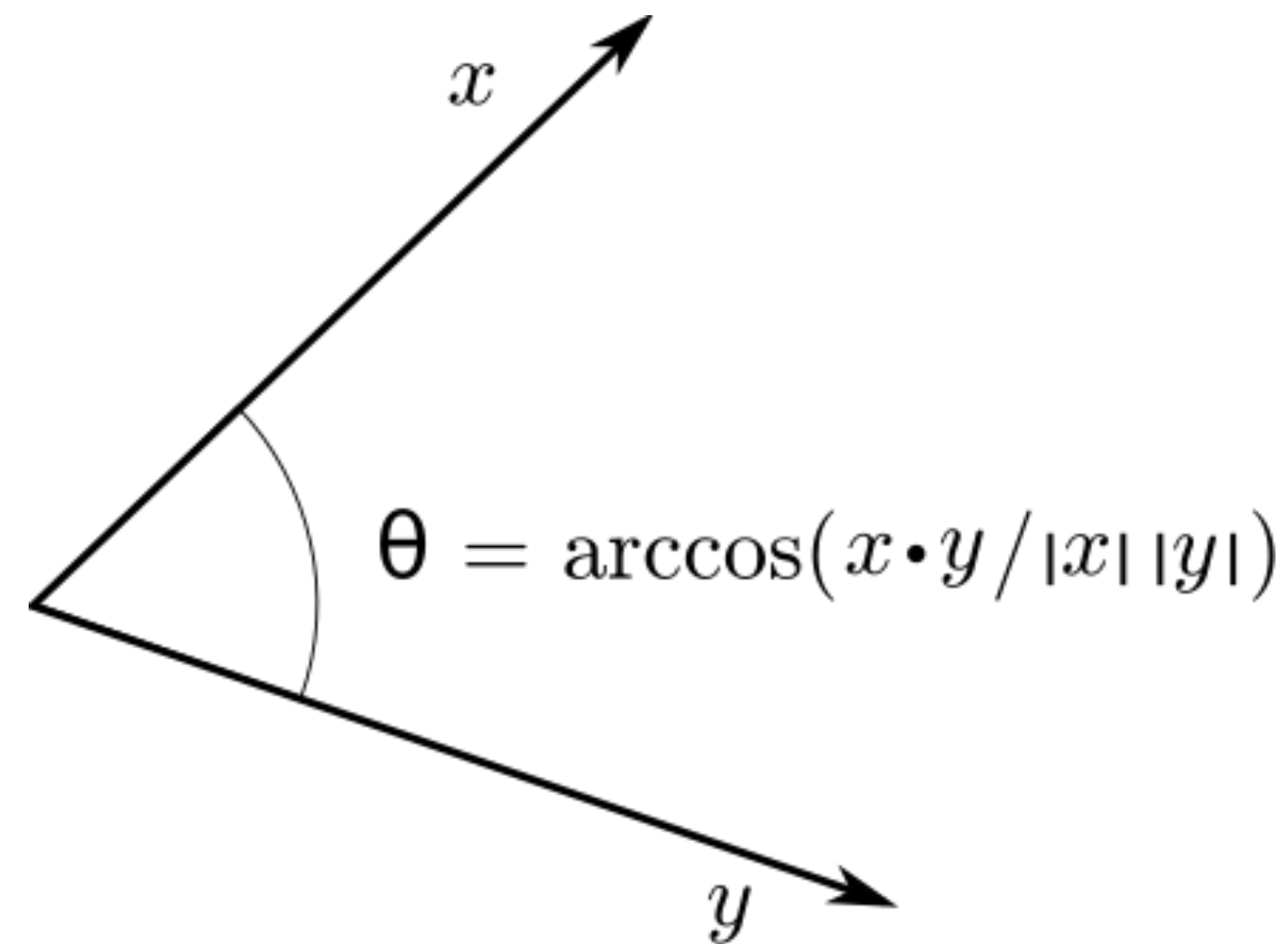
Length (norm) of x

Algebraic definition

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}_1 \mathbf{y}_1 + \mathbf{x}_2 \mathbf{y}_2$$

- Inner product of two vectors tells us how "similar" two vectors are.

Inner/Dot Product



$$\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2]$$

$$\mathbf{y} = [\mathbf{y}_1, \mathbf{y}_2]$$

These two definitions are equivalent. Prove it in WA2.

Geometric
definition

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta$$

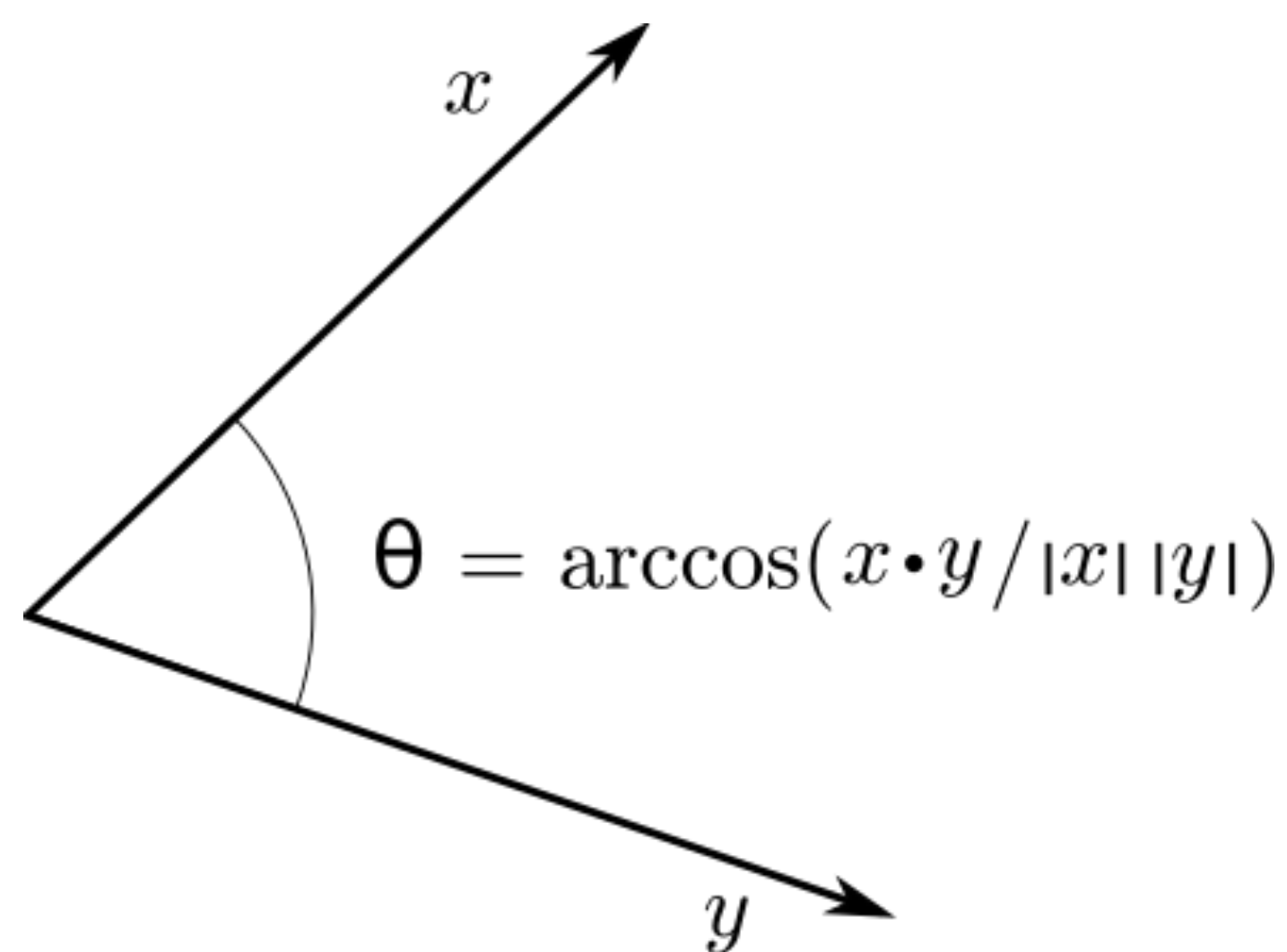
Length (norm) of \mathbf{x}

Algebraic
definition

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}_1 \mathbf{y}_1 + \mathbf{x}_2 \mathbf{y}_2$$

- Inner product of two vectors tells us how "similar" two vectors are.
- 0 if they are orthogonal;

Inner/Dot Product



$$\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2]$$

$$\mathbf{y} = [\mathbf{y}_1, \mathbf{y}_2]$$

These two definitions are equivalent. Prove it in WA2.

Geometric
definition

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta$$

Length (norm) of x

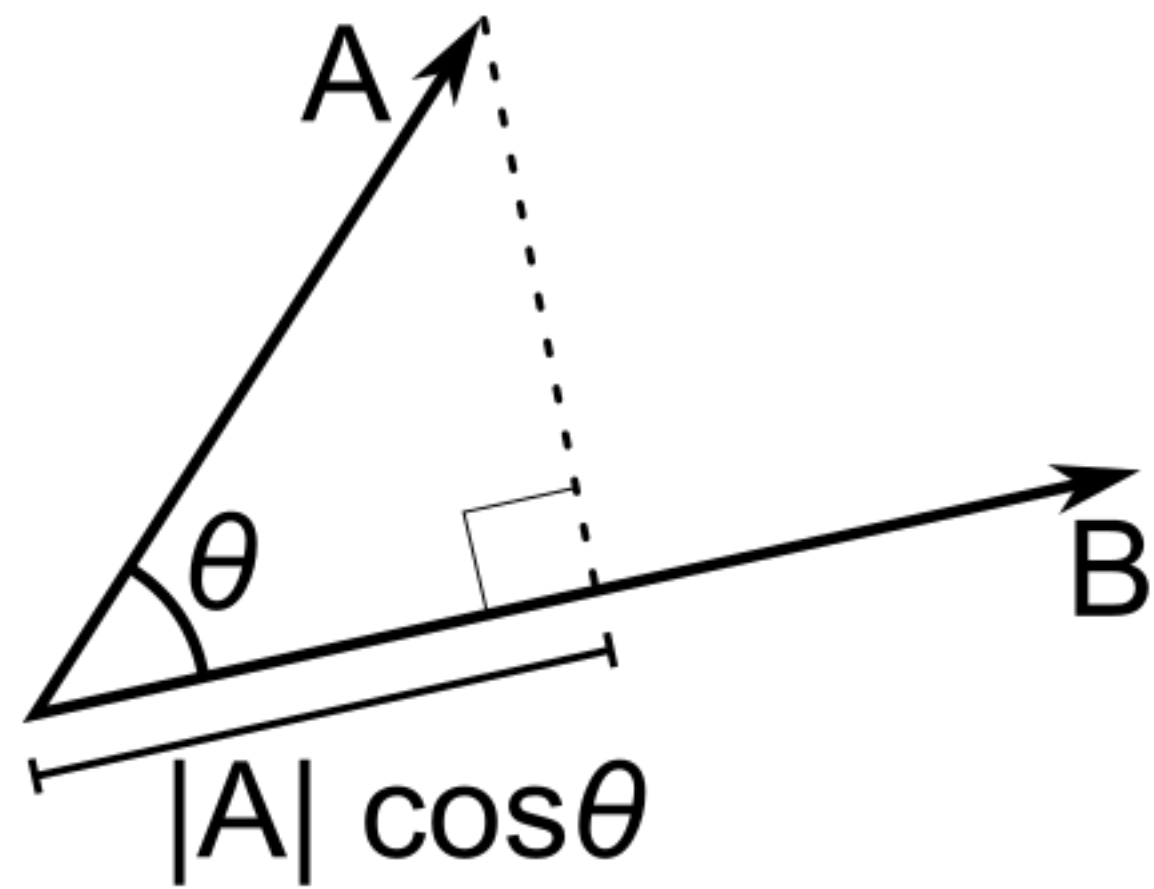
Algebraic
definition

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}_1 \mathbf{y}_1 + \mathbf{x}_2 \mathbf{y}_2$$

- Inner product of two vectors tells us how "similar" two vectors are.
 - 0 if they are orthogonal;
 - Maximal if they have the same direction.

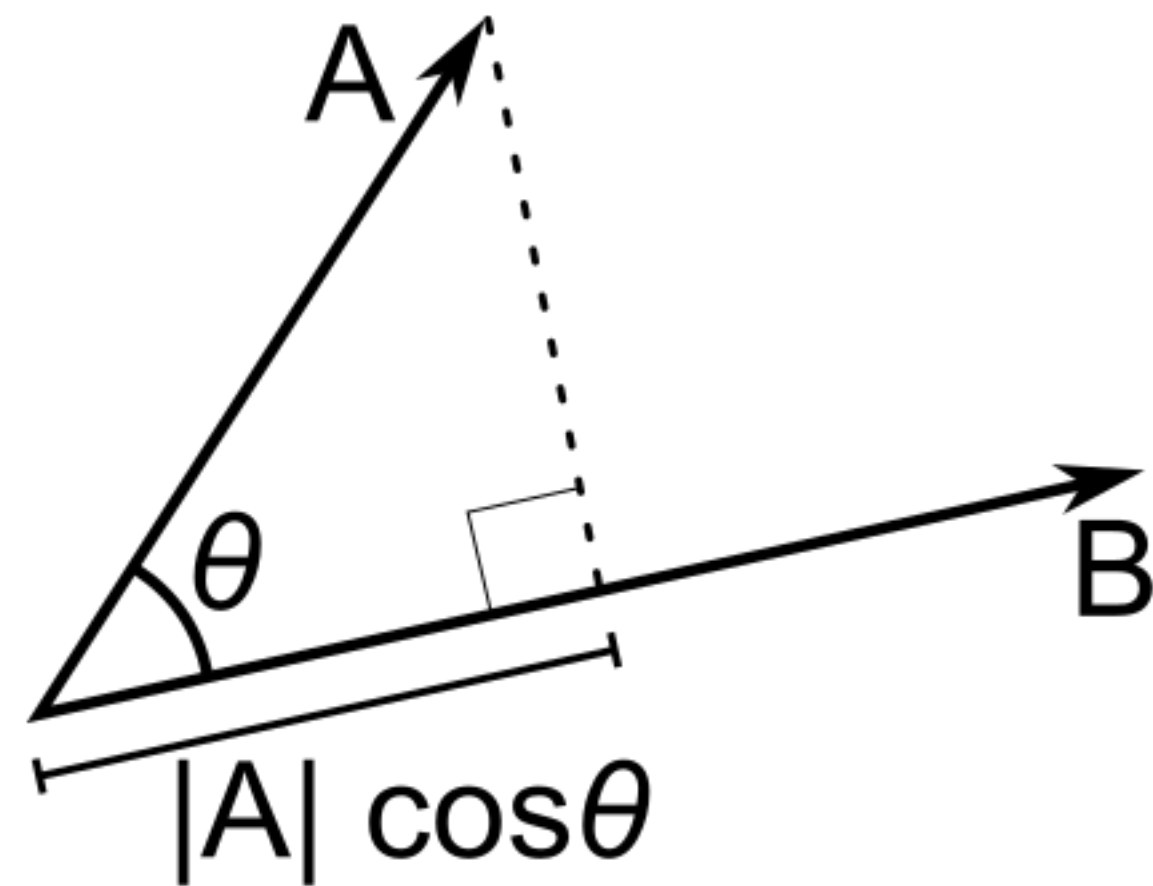
Vector Projection

$$\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\| \|\mathbf{B}\| \cos \theta$$



Vector Projection

$$\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\| \|\mathbf{B}\| \cos\theta$$

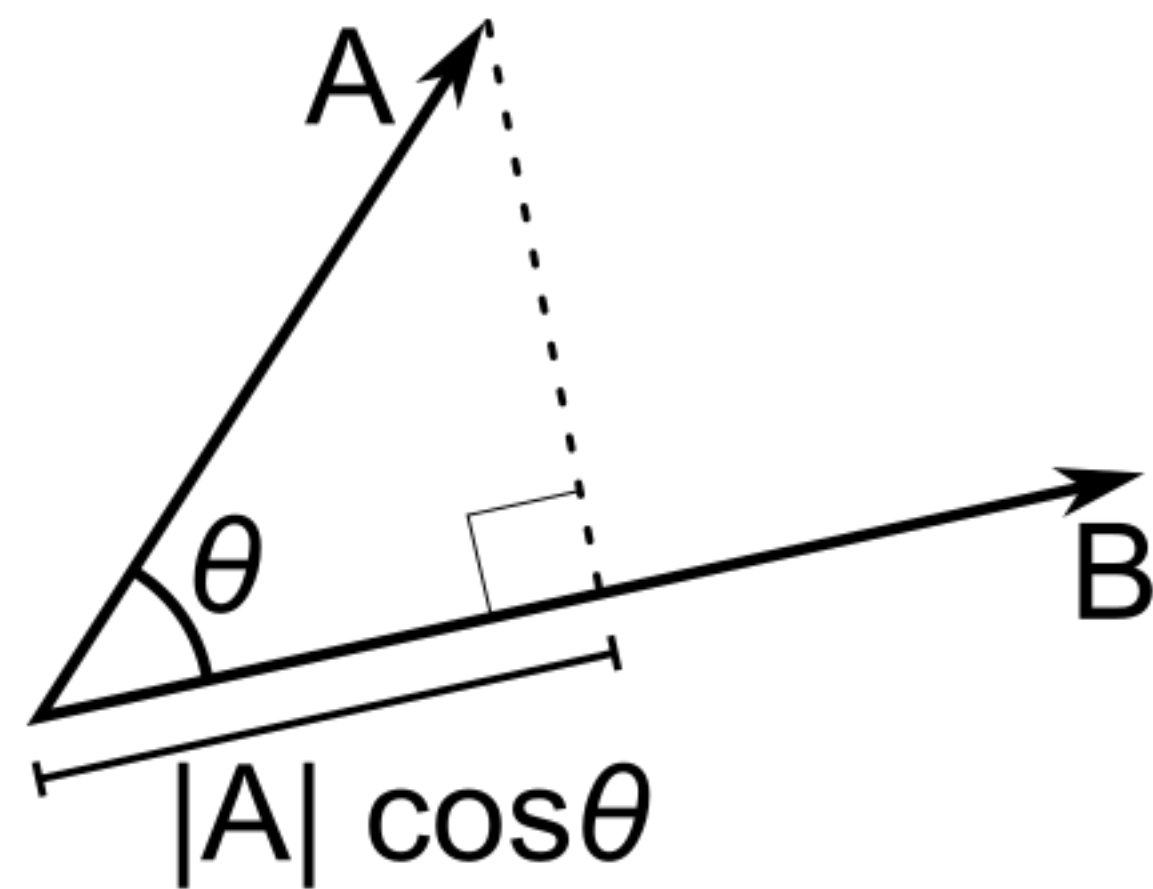


Magnitude of
projection of vector **A**
in the direction of **B**

$$A_B = \|\mathbf{A}\| \cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|}$$

Vector Projection

$$\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\| \|\mathbf{B}\| \cos \theta$$



Magnitude of
projection of vector \mathbf{A}
in the direction of \mathbf{B}

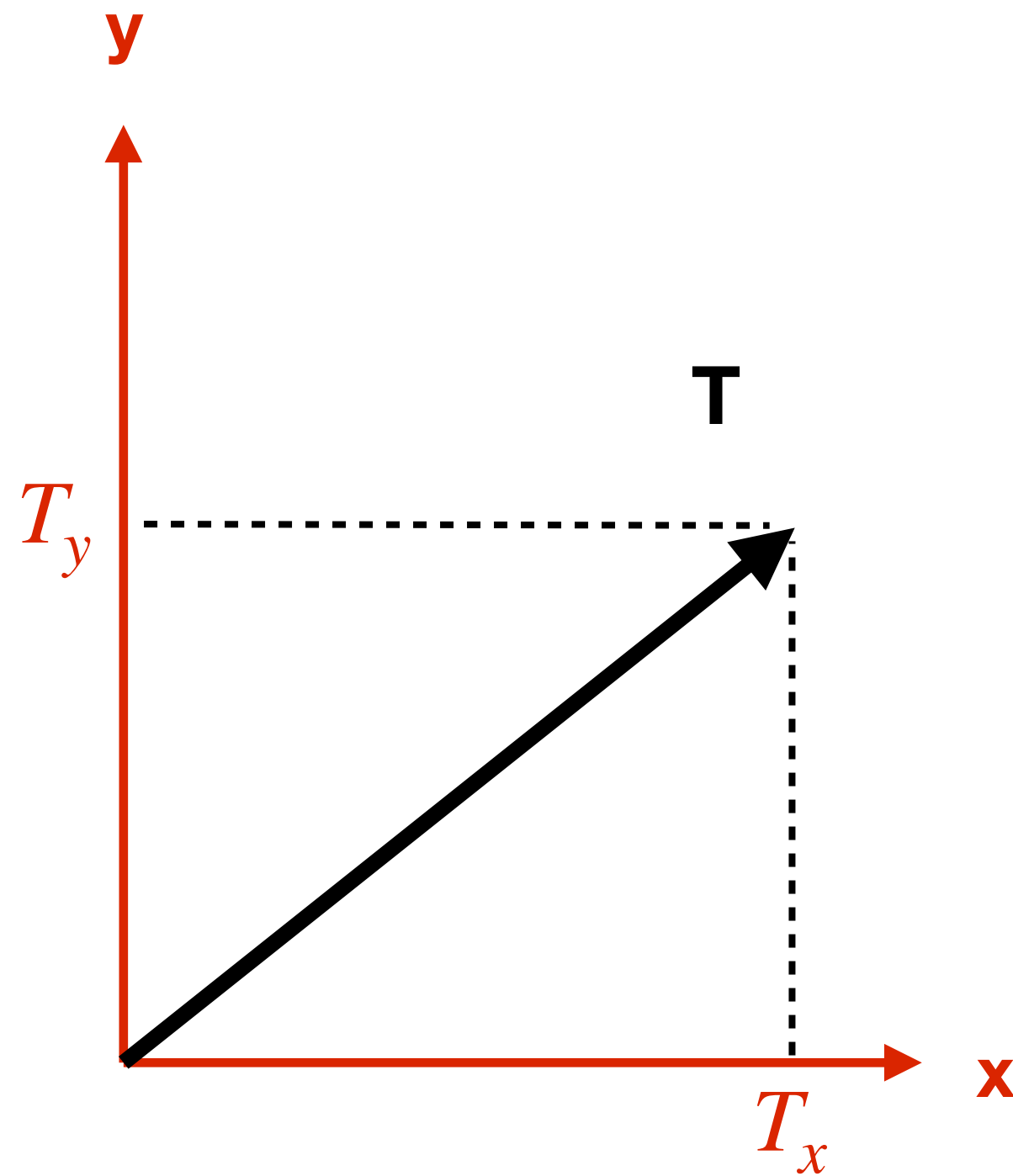
The projected vector

$$A_B = \|\mathbf{A}\| \cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|}$$

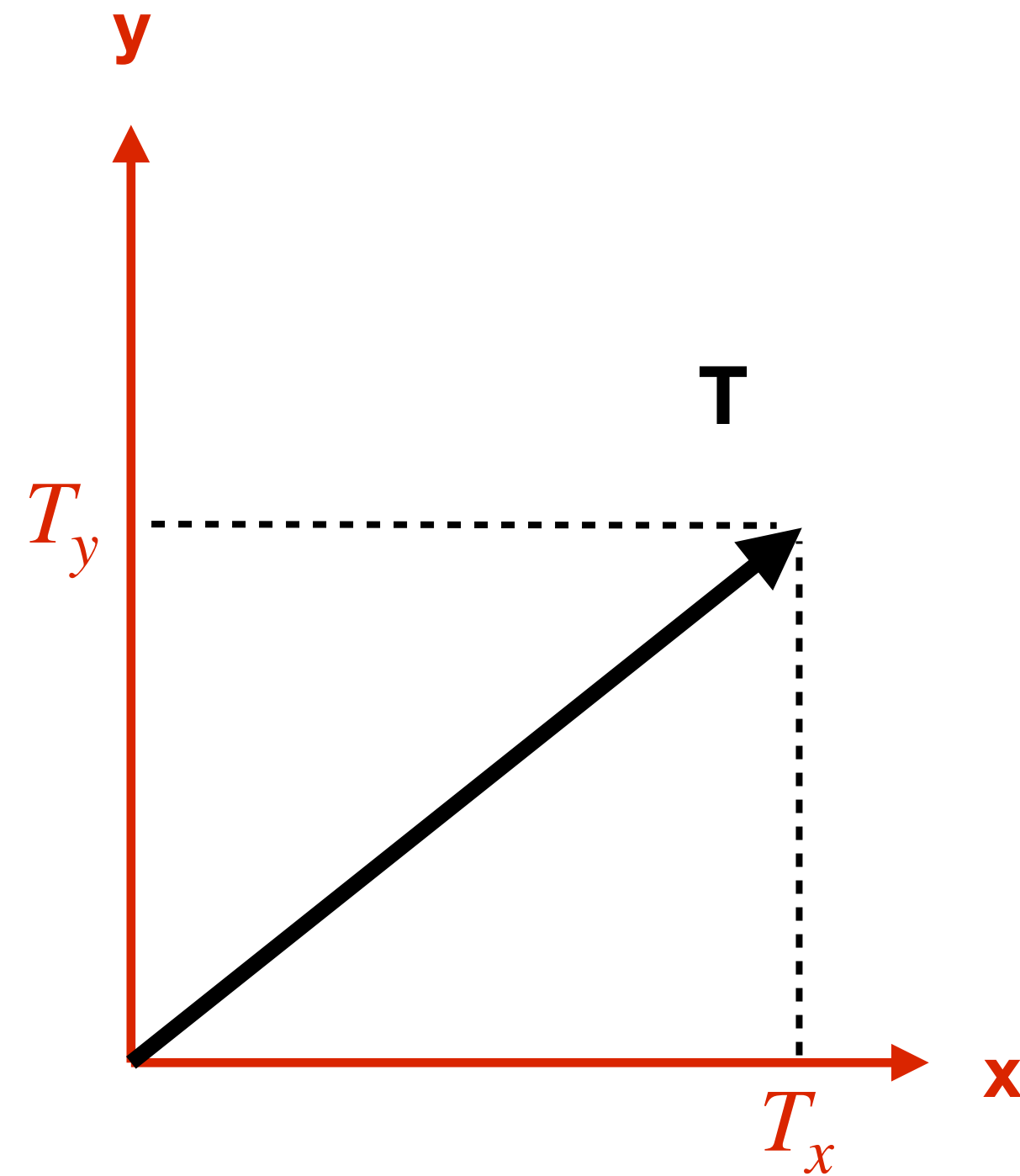
$$A_B \hat{\mathbf{B}} = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|} \frac{\mathbf{B}}{\|\mathbf{B}\|} = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|^2} \mathbf{B}$$

Projecting a Vector to Two Orthogonal Vectors

$$T_x = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|}, \quad T_y = \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|}, \quad \mathbf{T} = [T_x, T_y]$$



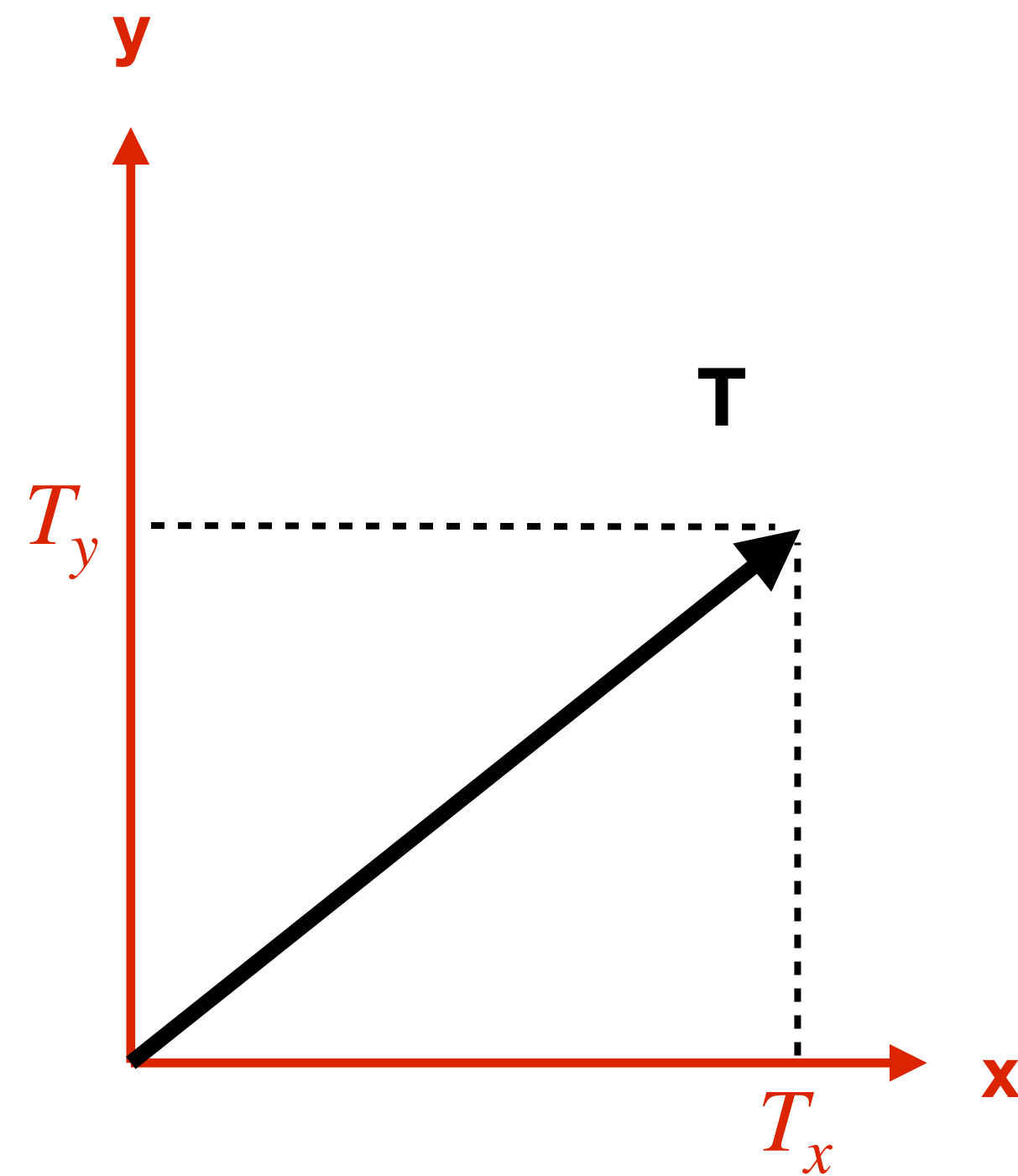
Projecting a Vector to Two Orthogonal Vectors



$$T_x = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|}, \quad T_y = \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|}, \quad \mathbf{T} = [T_x, T_y]$$

$$\mathbf{T} = T_x \hat{\mathbf{x}} + T_y \hat{\mathbf{y}} = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} + \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y}$$

Projecting a Vector to Two Orthogonal Vectors



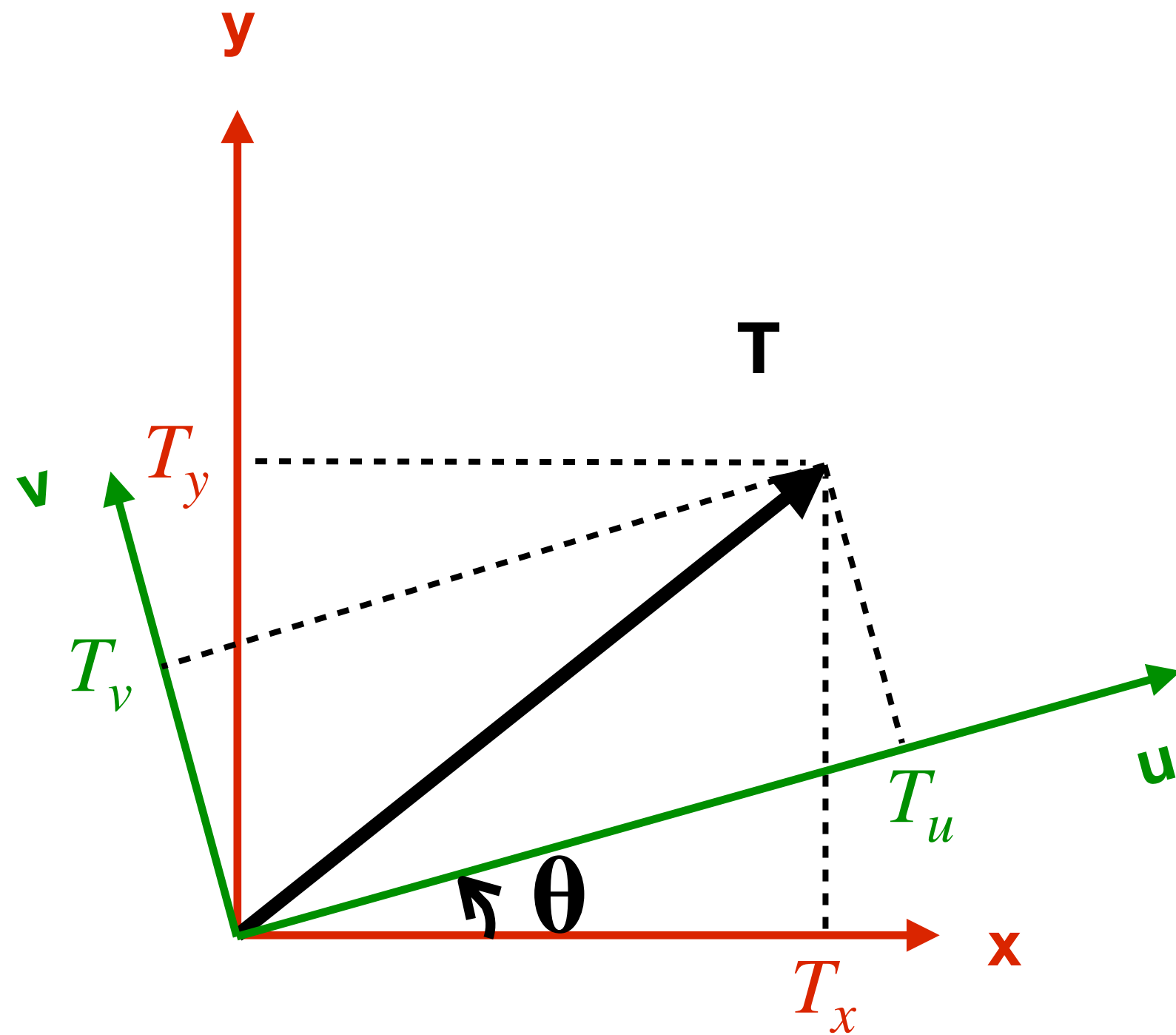
$$T_x = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|}, \quad T_y = \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|}, \quad \mathbf{T} = [T_x, T_y]$$

$$\mathbf{T} = T_x \hat{\mathbf{x}} + T_y \hat{\mathbf{y}} = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} + \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y}$$

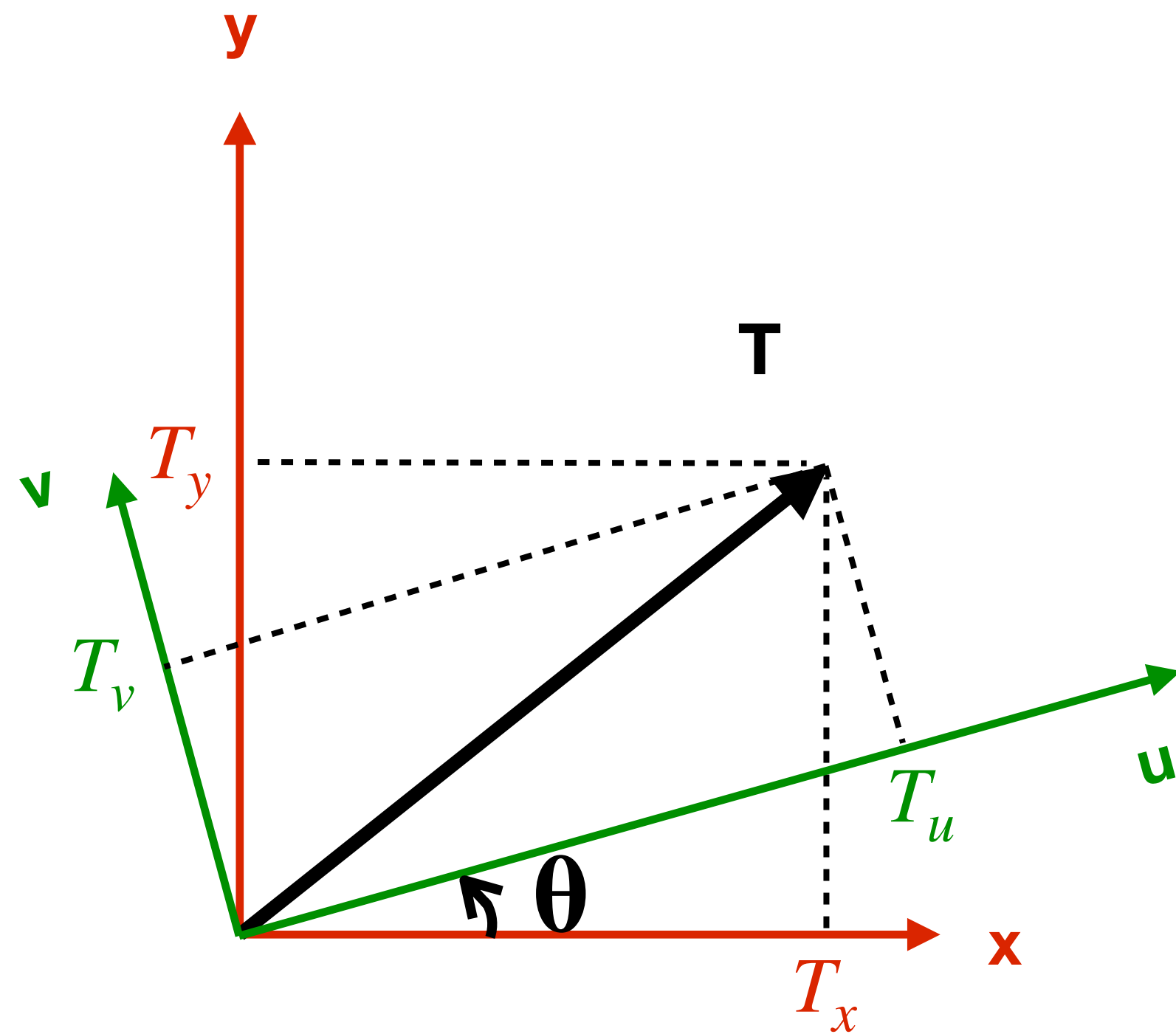
- What's happened here? We can express an arbitrary vector \mathbf{T} using two orthogonal "basis" vectors by projecting the vector to the two basis vectors.

Change of Coordinates

$$T_x = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|}, \quad T_y = \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|}, \quad \mathbf{T} = [T_x, T_y]$$



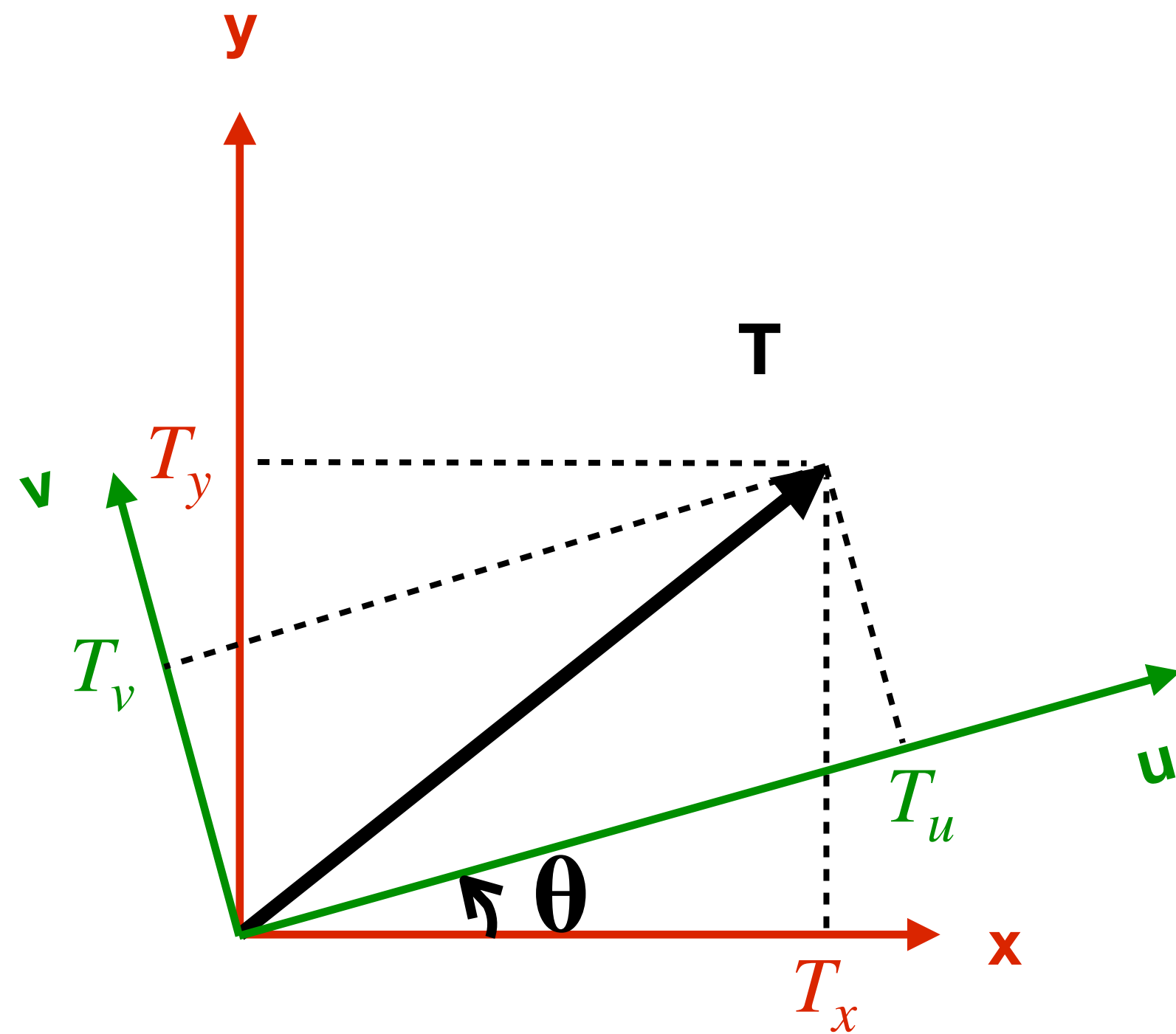
Change of Coordinates



$$T_x = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|}, \quad T_y = \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|}, \quad \mathbf{T} = [T_x, T_y]$$

$$T_u = \frac{\mathbf{T} \cdot \mathbf{u}}{\|\mathbf{u}\|}, \quad T_v = \frac{\mathbf{T} \cdot \mathbf{v}}{\|\mathbf{v}\|}, \quad \mathbf{T} = [T_u, T_v]$$

Change of Coordinates

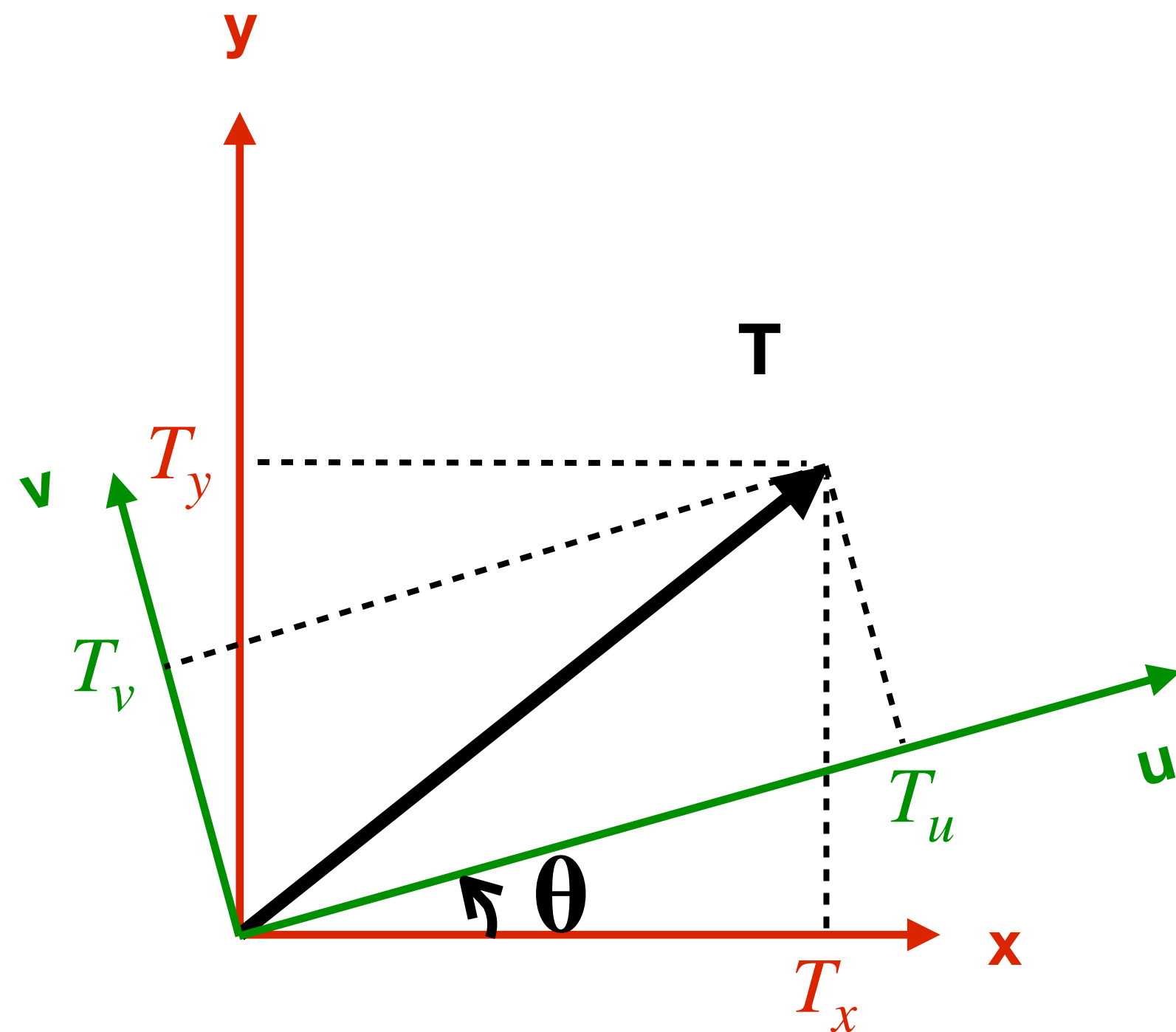


$$T_x = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|}, \quad T_y = \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|}, \quad \mathbf{T} = [T_x, T_y]$$

$$T_u = \frac{\mathbf{T} \cdot \mathbf{u}}{\|\mathbf{u}\|}, \quad T_v = \frac{\mathbf{T} \cdot \mathbf{v}}{\|\mathbf{v}\|}, \quad \mathbf{T} = [T_u, T_v]$$

$$\mathbf{T} = \frac{\mathbf{T} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} + \frac{\mathbf{T} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}$$

Change of Coordinates



$$T_x = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|}, \quad T_y = \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|}, \quad \mathbf{T} = [T_x, T_y]$$

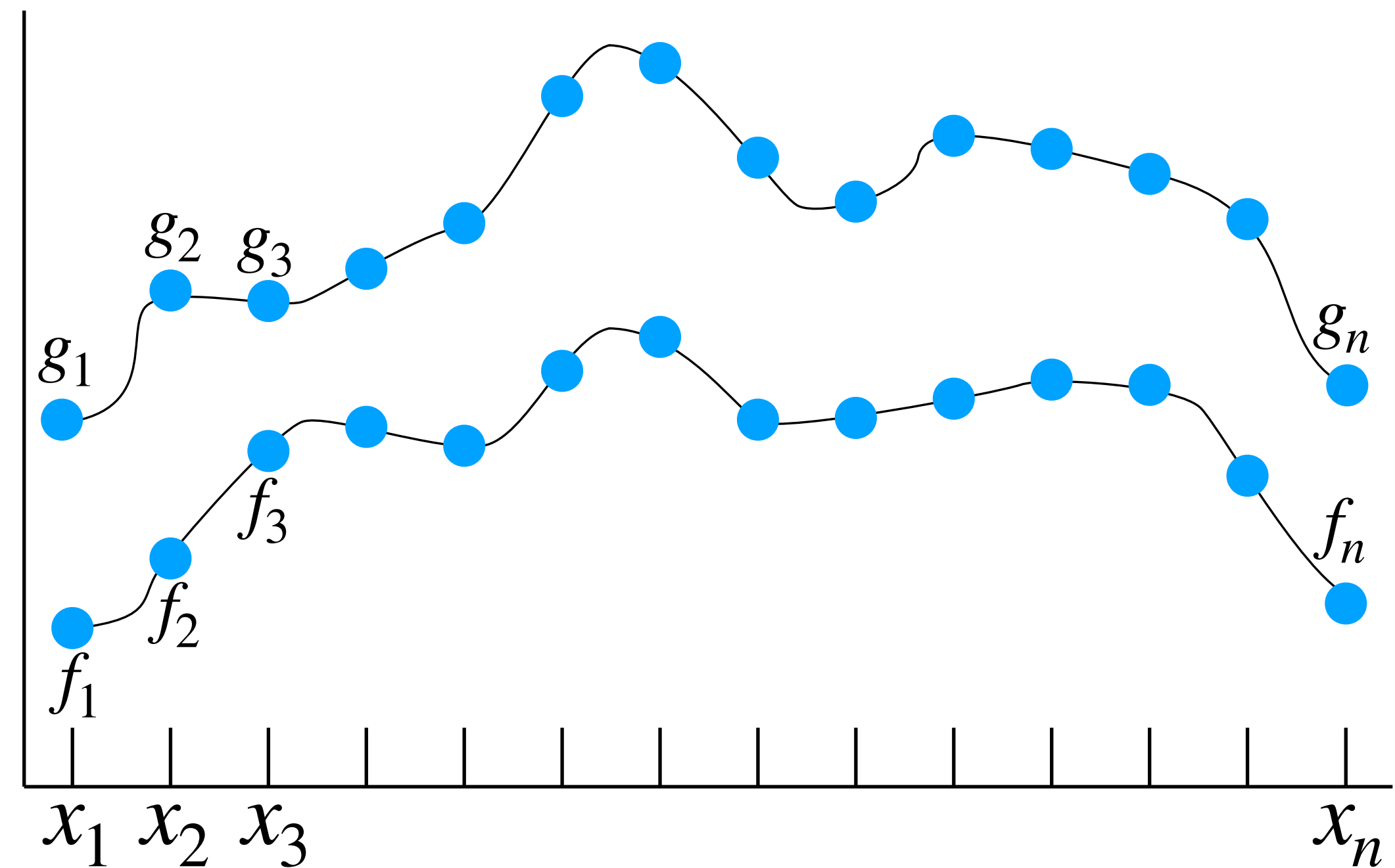
$$T_u = \frac{\mathbf{T} \cdot \mathbf{u}}{\|\mathbf{u}\|}, \quad T_v = \frac{\mathbf{T} \cdot \mathbf{v}}{\|\mathbf{v}\|}, \quad \mathbf{T} = [T_u, T_v]$$

$$\mathbf{T} = \frac{\mathbf{T} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} + \frac{\mathbf{T} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}$$

$[T_x, T_y]$ and $[T_u, T_v]$ are related by a geometric transformation. WA1 asks you to come up with the transformation.

Functions are Vectors

- Assume functions f and g are defined as N discrete points

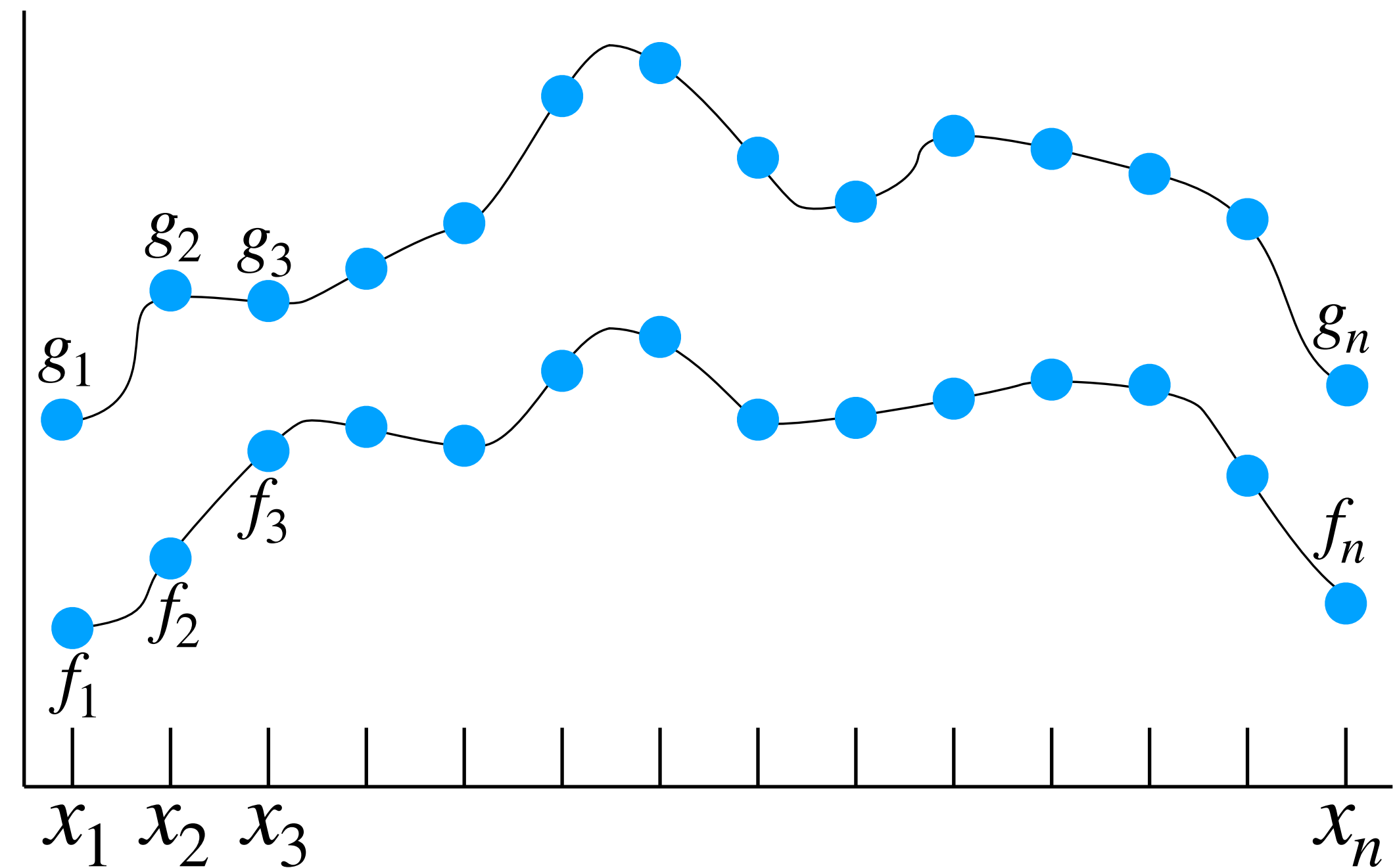


Functions are Vectors

- Assume functions f and g are defined as N discrete points
- f and g are nothing more than two N -dimensional vectors.

$$f = [f_1, f_2, \dots, f_n]$$

$$g = [g_1, g_2, \dots, g_n]$$



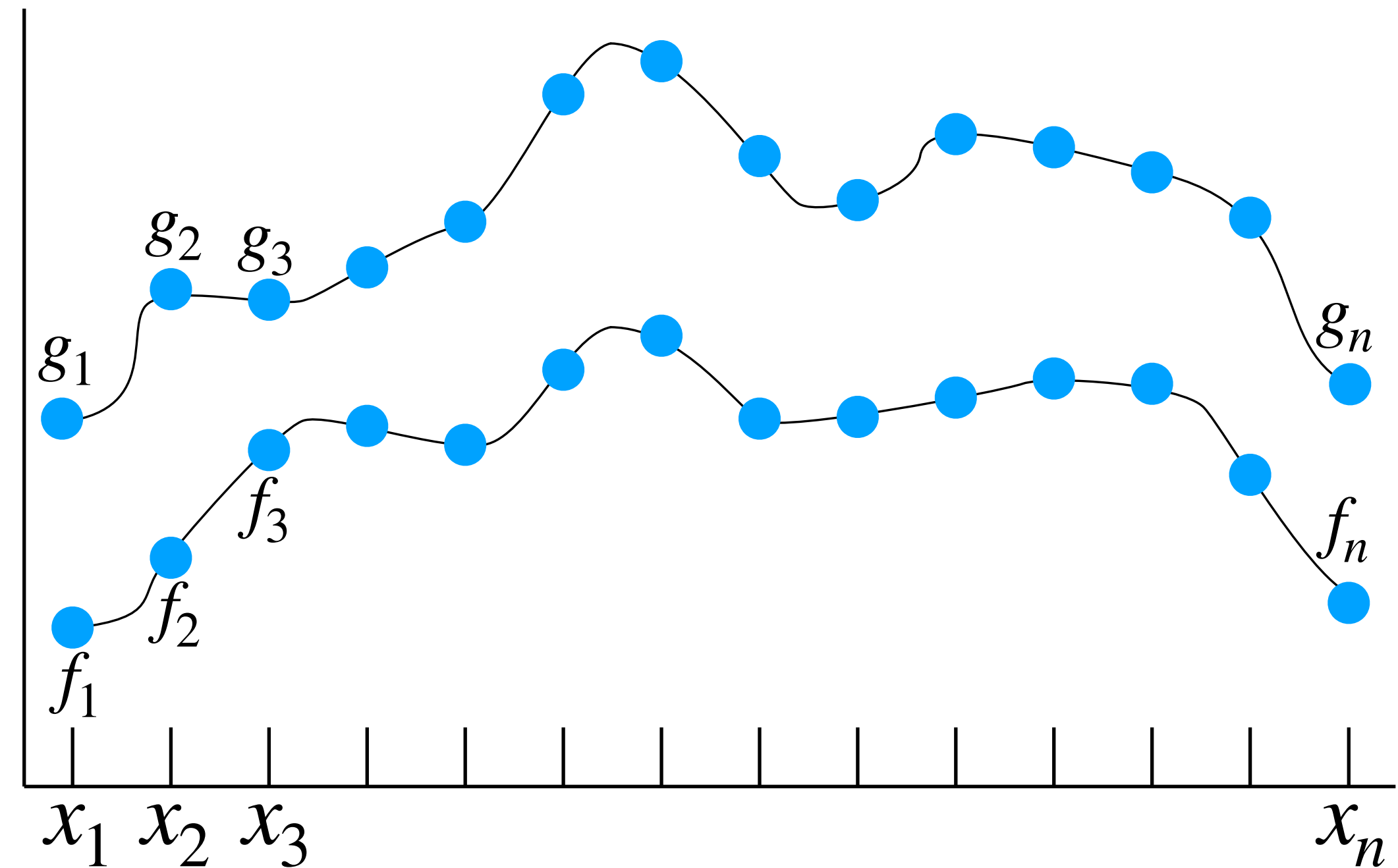
Functions are Vectors

- Assume functions f and g are defined as N discrete points
- f and g are nothing more than two N -dimensional vectors.

$$f = [f_1, f_2, \dots, f_n]$$

$$g = [g_1, g_2, \dots, g_n]$$

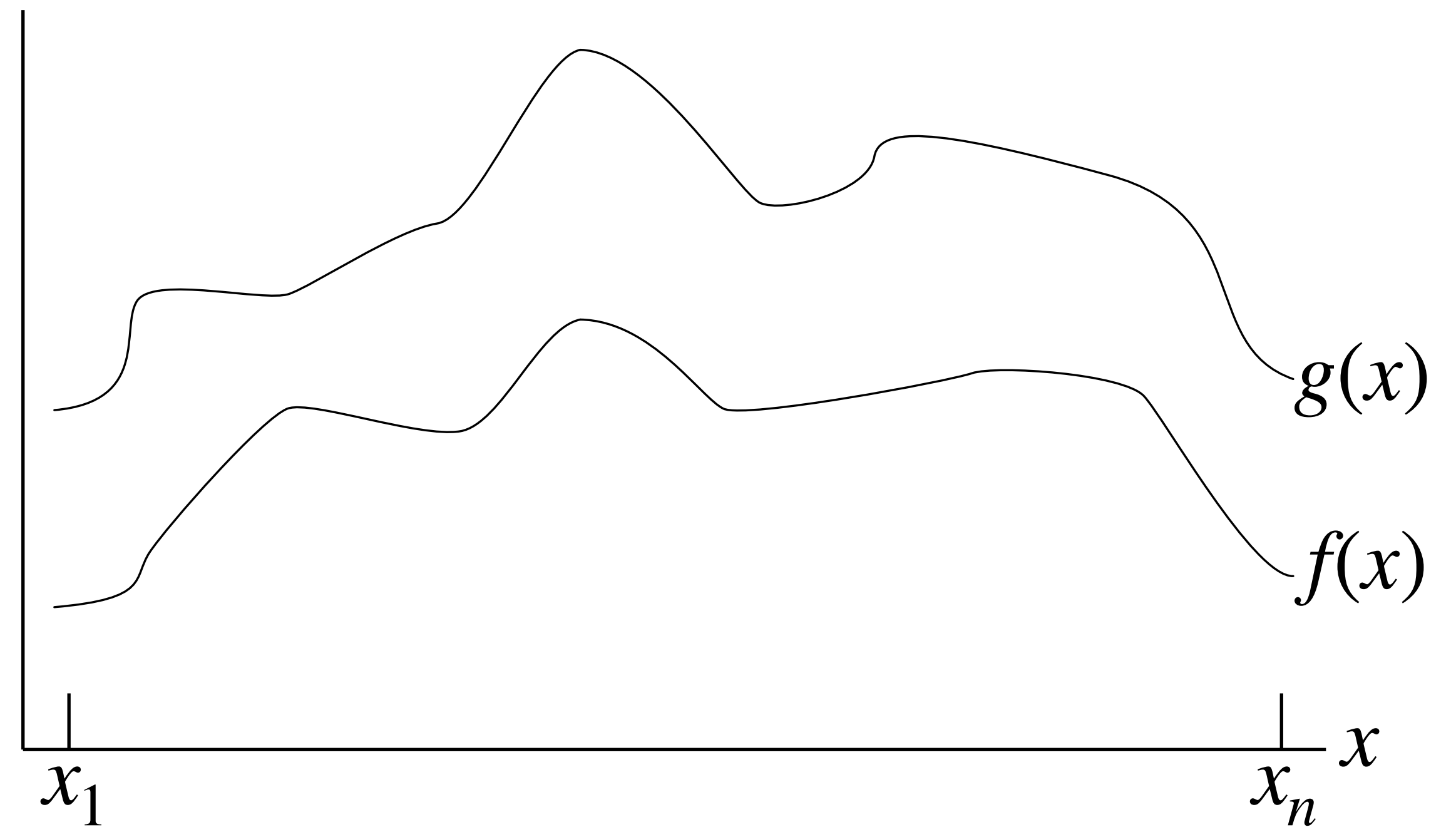
Inner product of two N -dimensional vectors



$$f \cdot g = \sum_{i=1}^n f_i g_i$$

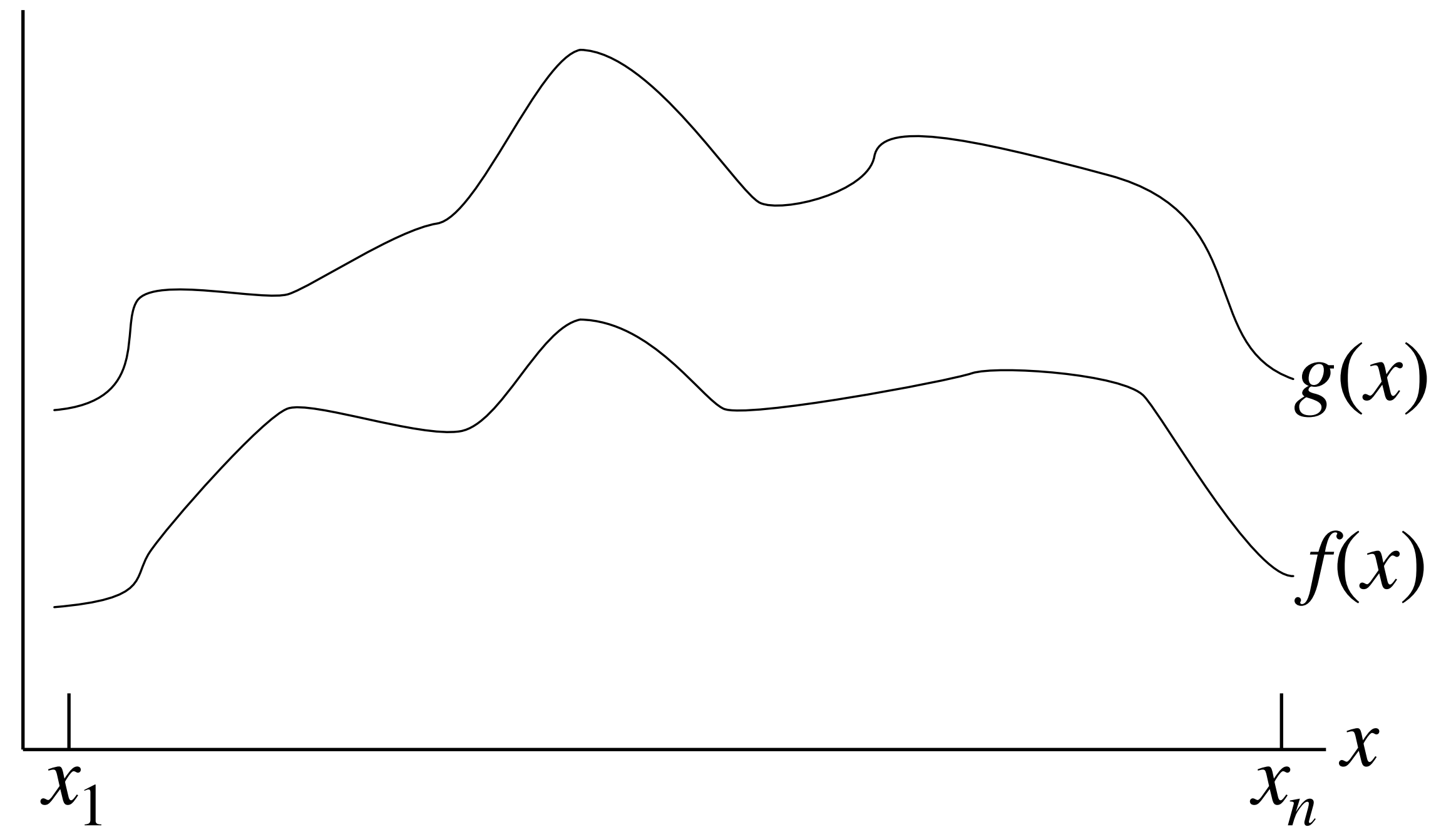
Continuous Function = Infinite-Dimensional Vector

- When f and g are continuous functions, they can be seen as two **infinite-dimensional** vectors.



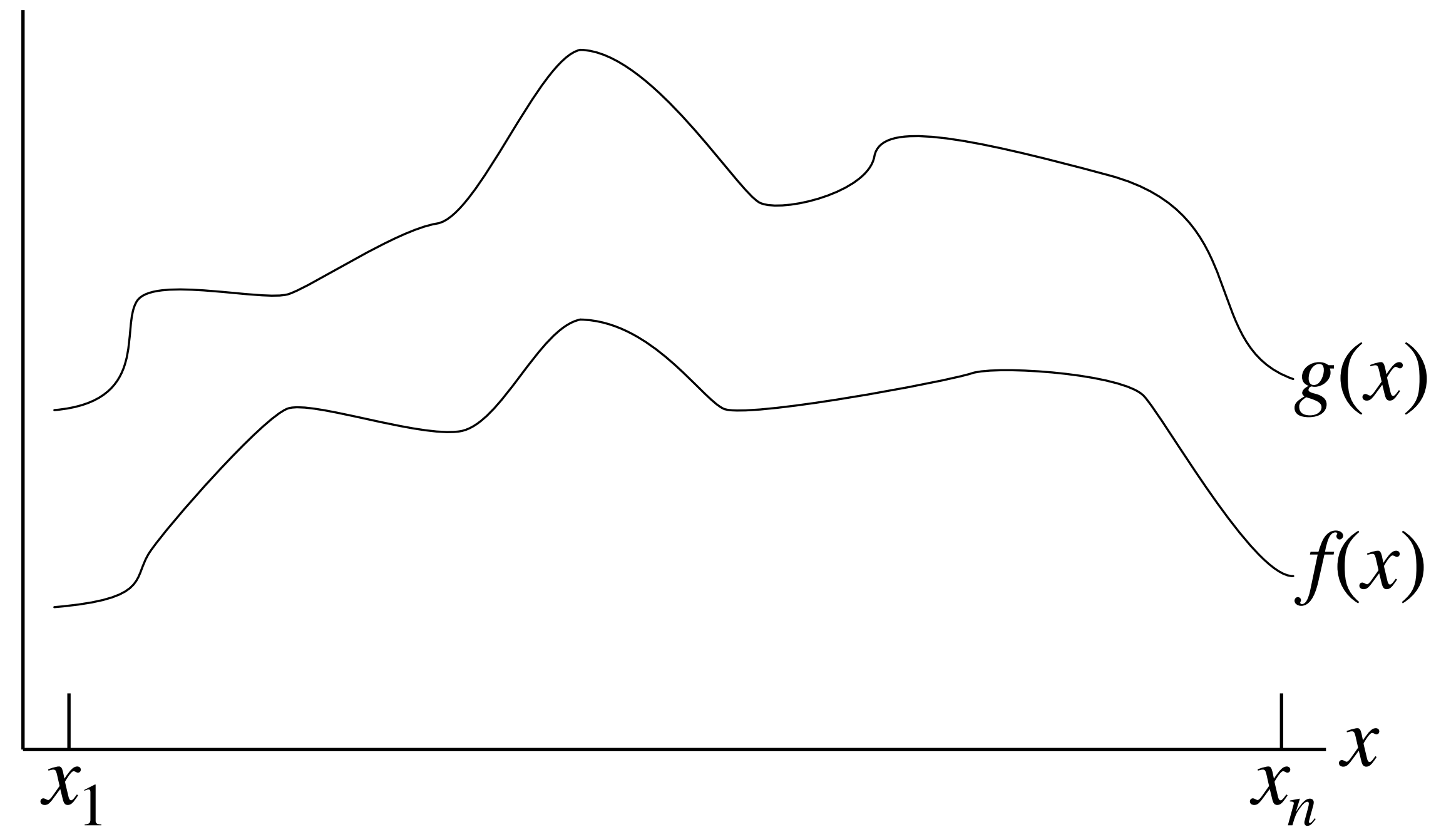
Continuous Function = Infinite-Dimensional Vector

- When f and g are continuous functions, they can be seen as two **infinite-dimensional** vectors.
- The inner product of two continuous functions is the **integration** of the product.



Continuous Function = Infinite-Dimensional Vector

- When f and g are continuous functions, they can be seen as two **infinite-dimensional** vectors.
- The inner product of two continuous functions is the **integration** of the product.

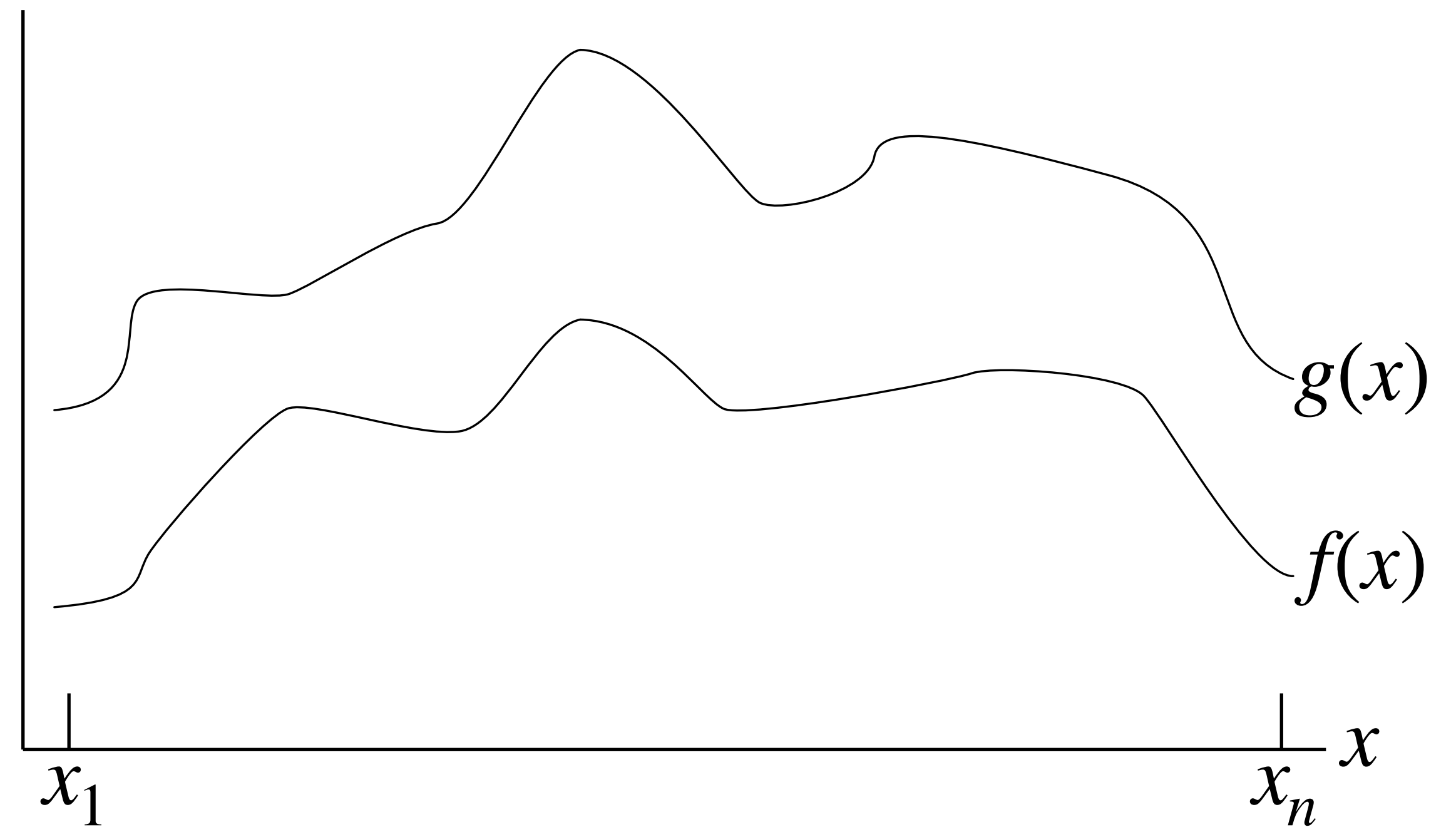


Inner product of two
continuous functions

$$f \cdot g = \int_{x_1}^{x_2} f(x)g(x)dx$$

Continuous Function = Infinite-Dimensional Vector

- When f and g are continuous functions, they can be seen as two **infinite-dimensional** vectors.
- The inner product of two continuous functions is the **integration** of the product.
- Use complex conjugate of g if they are complex-valued.



Inner product of two
continuous functions

$$f \cdot g = \int_{x_1}^{x_2} f(x)g(x)dx$$

Function Inner Product

Inner product of two continuous functions

$$f \cdot g = \int_{x_1}^{x_2} f(x)g(x)dx$$

Function f and g are orthogonal if:

$$f \cdot g = \int_{x_1}^{x_2} f(x)g(x)dx = 0$$

Norm of a function g is:

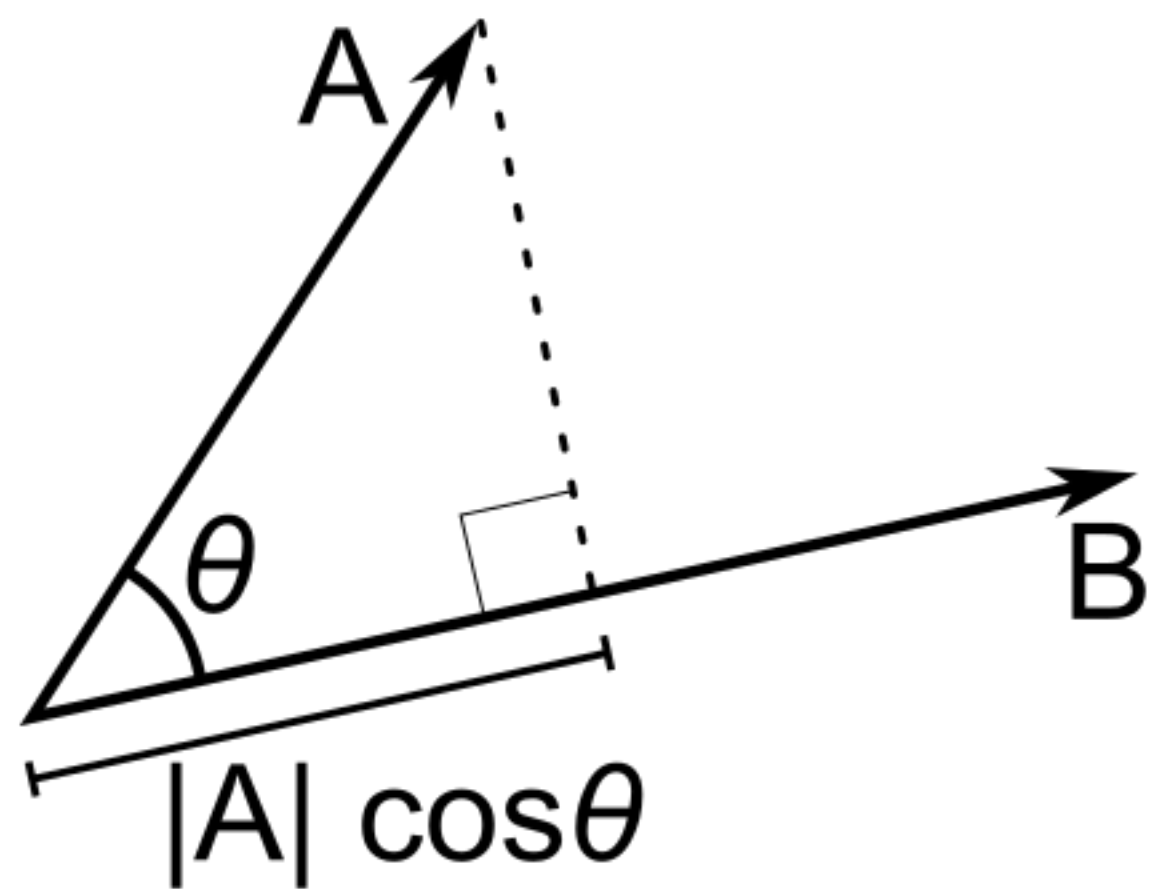
$$\|g\| = \left(\int_{x_1}^{x_2} g(x)^2 dx \right)^{1/2}$$

- An infinite-dimensional space that also defines vector operations (e.g., dot product) is called a *Hilbert space*.

Vector vs. Function Projection

Vector A projected to
vector B

$$A_B \hat{\mathbf{B}} = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|^2} \mathbf{B}$$



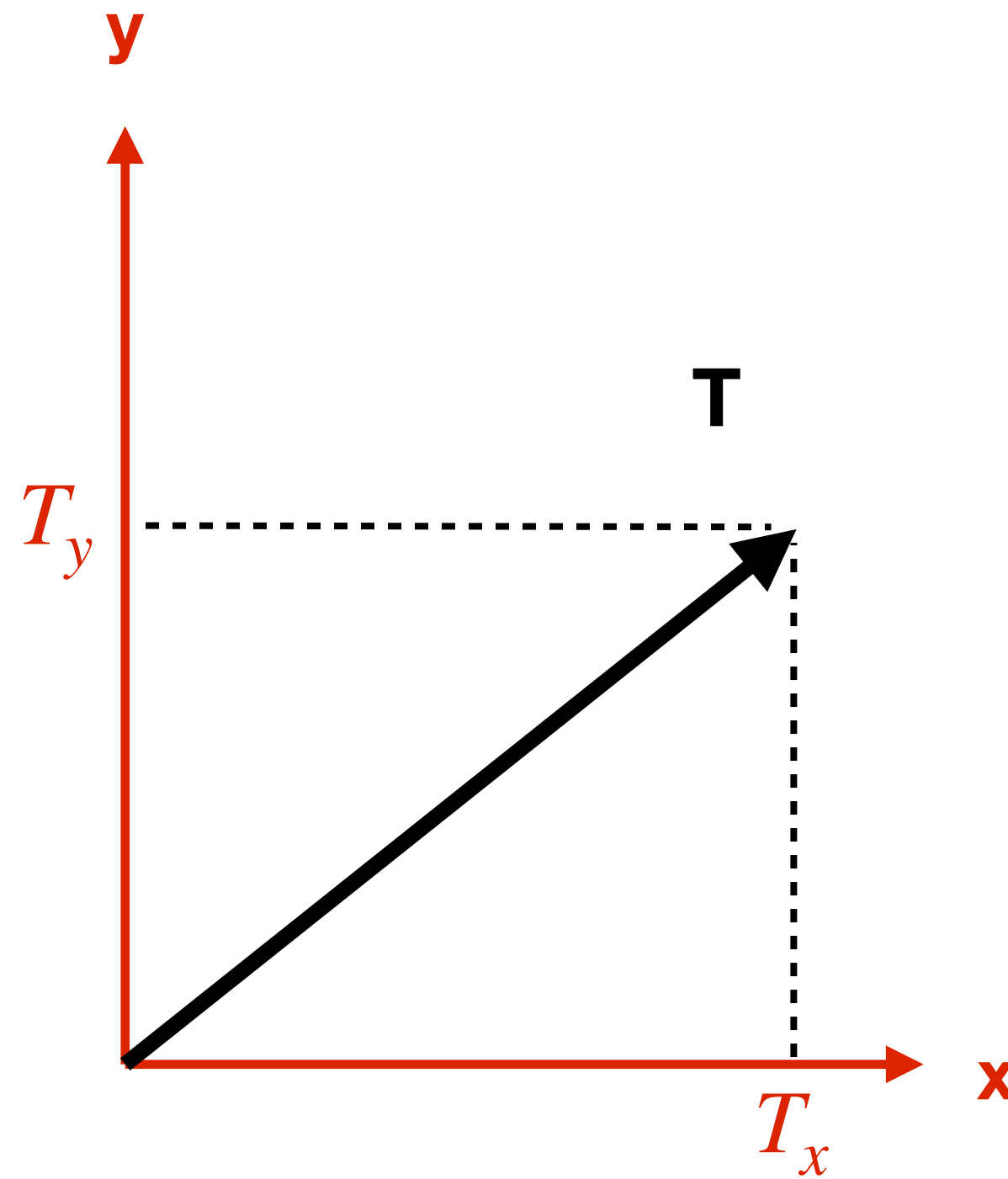
Function f projected to
function g

$$f_g = \frac{f \cdot g}{\|g\|^2} g$$

Recall: Expressing a Vector Using Basis Vectors

- We can express a 2D vector as a sum of two orthogonal basis vectors.
 - By projecting \mathbf{T} to the two basis vectors

$$\mathbf{T} = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} + \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y}$$



Expressing a Function as Basis Functions

- Similarly, we can express a function using “basis functions”.
 - How many basis functions do we need?
- Expressing a continuous function requires infinitely many basis functions.
 - Basis functions must be orthogonal.

$$\mathbf{T} = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} + \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y}$$

$$f = \sum_{i=1}^{\infty} \frac{f \cdot v_i}{\|v_i\|^2} v_i$$

v_1, v_2, \dots are orthogonal functions

Problem Setup

- Now assuming we have a continuous function f defined over $[t, t+L)$
- How do we represent f as a sum of a set of basis functions?
 - The basis functions need to be defined over $[t, t+L)$ too.

$$\mathbf{T} = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} + \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y}$$

$$f = \sum_{i=1}^{\infty} \frac{f \cdot v_i}{\|v_i\|^2} v_i$$

v_1, v_2, \dots are orthogonal functions

A Special Set of Orthogonal Basis Functions

$$\left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \quad \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, \quad k \in \mathbb{Z}$$

mutually orthogonal functions
on any $[t, t+L)$ interval

A Special Set of Orthogonal Basis Functions

$$\left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \quad \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, \quad k \in \mathbb{Z}$$

mutually orthogonal functions
on any $[t, t+L)$ interval

$$\cos\left(\frac{0\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \cos\left(\frac{4\pi x}{L}\right), \cos\left(\frac{6\pi x}{L}\right), \dots, \sin\left(\frac{2\pi x}{L}\right), \sin\left(\frac{4\pi x}{L}\right), \sin\left(\frac{6\pi x}{L}\right), \dots$$

A Special Set of Orthogonal Basis Functions

$$\left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \quad \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, \quad k \in \mathbb{Z}$$

mutually orthogonal functions
on any $[t, t+L)$ interval

$$\cos\left(\frac{0\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \cos\left(\frac{4\pi x}{L}\right), \cos\left(\frac{6\pi x}{L}\right), \dots, \sin\left(\frac{2\pi x}{L}\right), \sin\left(\frac{4\pi x}{L}\right), \sin\left(\frac{6\pi x}{L}\right), \dots$$

For instance: $\int_t^{t+L} \cos\left(\frac{2\pi x}{L}\right) \sin\left(\frac{6\pi x}{L}\right) dx = 0$

A Special Set of Orthogonal Basis Functions

$$\left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \quad \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, \quad k \in \mathbb{Z}$$

mutually orthogonal functions
on any $[t, t+L)$ interval

$$\cos\left(\frac{0\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \cos\left(\frac{4\pi x}{L}\right), \cos\left(\frac{6\pi x}{L}\right), \dots, \sin\left(\frac{2\pi x}{L}\right), \sin\left(\frac{4\pi x}{L}\right), \sin\left(\frac{6\pi x}{L}\right), \dots$$

For instance:

$$\int_t^{t+L} \cos\left(\frac{2\pi x}{L}\right) \sin\left(\frac{6\pi x}{L}\right) dx = 0$$
$$\int_t^{t+L} \cos\left(\frac{2\pi x}{L}\right) \cos\left(\frac{6\pi x}{L}\right) dx = 0$$

A Special Set of Orthogonal Basis Functions

$$\left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \quad \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, \quad k \in \mathbb{Z}$$

mutually orthogonal functions
on any $[t, t+L)$ interval

$$\cos\left(\frac{0\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \cos\left(\frac{4\pi x}{L}\right), \cos\left(\frac{6\pi x}{L}\right), \dots, \sin\left(\frac{2\pi x}{L}\right), \sin\left(\frac{4\pi x}{L}\right), \sin\left(\frac{6\pi x}{L}\right), \dots$$

For instance:

$$\int_t^{t+L} \cos\left(\frac{2\pi x}{L}\right) \sin\left(\frac{6\pi x}{L}\right) dx = 0$$
$$\int_t^{t+L} \cos\left(\frac{2\pi x}{L}\right) \cos\left(\frac{6\pi x}{L}\right) dx = 0$$

The sine series starts from $k=1$, since $k=0$ leads to a function that's 0 everywhere, i.e. a 0-norm vector, which isn't great. Will see why soon.

Fourier Series

$$f = \sum_{i=1}^{\infty} \frac{f \cdot v_i}{\|v_i\|^2} v_i \quad v_i \in \left\{ \left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, x \in [t, t+L) \right\}$$

Fourier Series

$$f = \sum_{i=1}^{\infty} \frac{f \cdot v_i}{\|v_i\|^2} v_i \quad v_i \in \left\{ \left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, x \in [t, t+L) \right\}$$

$$f(x) = \sum_{k=0}^{\infty} \left(a_k \cos\left(\frac{2\pi kx}{L}\right) \right) + \sum_{k=1}^{\infty} \left(b_k \sin\left(\frac{2\pi kx}{L}\right) \right), x \in [t, t+L)$$

Fourier Series

$$f = \sum_{i=1}^{\infty} \frac{f \cdot v_i}{\|v_i\|^2} v_i \quad v_i \in \left\{ \left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, x \in [t, t+L) \right\}$$

$$f(x) = \sum_{k=0}^{\infty} \left(a_k \cos\left(\frac{2\pi kx}{L}\right) \right) + \sum_{k=1}^{\infty} \left(b_k \sin\left(\frac{2\pi kx}{L}\right) \right), x \in [t, t+L)$$

Fourier Series

$$f = \sum_{i=1}^{\infty} \frac{f \cdot v_i}{\|v_i\|^2} v_i \quad v_i \in \left\{ \left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, x \in [t, t+L) \right\}$$

$$f(x) = \sum_{k=0}^{\infty} \left(a_k \cos\left(\frac{2\pi kx}{L}\right) \right) + \sum_{k=1}^{\infty} \left(b_k \sin\left(\frac{2\pi kx}{L}\right) \right), \quad x \in [t, t+L)$$

$$a_k = \frac{f(x) \cdot \left(\cos\left(\frac{2\pi kx}{L}\right) \right)}{\left\| \cos\left(\frac{2\pi kx}{L}\right) \right\|^2} = \frac{2}{L} \int_t^{t+L} f(x) \cos\left(\frac{2\pi kx}{L}\right) dx, \quad \text{if } k \neq 0$$

Fourier Series

$$f = \sum_{i=1}^{\infty} \frac{f \cdot v_i}{\|v_i\|^2} v_i \quad v_i \in \left\{ \left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, x \in [t, t+L) \right\}$$

$$f(x) = \sum_{k=0}^{\infty} \left(a_k \cos\left(\frac{2\pi kx}{L}\right) \right) + \sum_{k=1}^{\infty} \left(b_k \sin\left(\frac{2\pi kx}{L}\right) \right), \quad x \in [t, t+L)$$

$$\begin{aligned} a_k &= \frac{f(x) \cdot \left(\cos\left(\frac{2\pi kx}{L}\right) \right)}{\left\| \cos\left(\frac{2\pi kx}{L}\right) \right\|^2} = \frac{2}{L} \int_t^{t+L} f(x) \cos\left(\frac{2\pi kx}{L}\right) dx, \quad \text{if } k \neq 0 \\ &= \frac{1}{L} \int_t^{t+L} f(x) \cos\left(\frac{2\pi kx}{L}\right) dx, \quad \text{if } k = 0 \end{aligned}$$

Fourier Series

$$f = \sum_{i=1}^{\infty} \frac{f \cdot v_i}{\|v_i\|^2} v_i \quad v_i \in \left\{ \left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, x \in [t, t+L) \right\}$$

$$f(x) = \sum_{k=0}^{\infty} \left(a_k \cos\left(\frac{2\pi kx}{L}\right) \right) + \sum_{k=1}^{\infty} \left(b_k \sin\left(\frac{2\pi kx}{L}\right) \right), \quad x \in [t, t+L)$$

$$a_k = \frac{f(x) \cdot \left(\cos\left(\frac{2\pi kx}{L}\right) \right)}{\left\| \cos\left(\frac{2\pi kx}{L}\right) \right\|^2} = \frac{2}{L} \int_t^{t+L} f(x) \cos\left(\frac{2\pi kx}{L}\right) dx, \quad \text{if } k \neq 0$$

$$b_k = \frac{f(x) \cdot \left(\sin\left(\frac{2\pi kx}{L}\right) \right)}{\left\| \sin\left(\frac{2\pi kx}{L}\right) \right\|^2} = \frac{2}{L} \int_t^{t+L} f(x) \sin\left(\frac{2\pi kx}{L}\right) dx$$

$$= \frac{1}{L} \int_t^{t+L} f(x) \cos\left(\frac{2\pi kx}{L}\right) dx, \quad \text{if } k = 0$$

Fourier Coefficients

- Rearrange terms to get a more compact equation.
- a_k and b_k are called the **Fourier coefficients** of $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2\pi kx}{L}\right) + b_k \sin\left(\frac{2\pi kx}{L}\right) \right), x \in [t, t+L), k \in \mathbb{Z}^+$$

$$a_k = \frac{2}{L} \int_t^{t+L} f(x) \cos\left(\frac{2\pi kx}{L}\right) dx$$

$$b_k = \frac{2}{L} \int_t^{t+L} f(x) \sin\left(\frac{2\pi kx}{L}\right) dx$$

Fourier Coefficients

- Rearrange terms to get a more compact equation.
- a_k and b_k are called the **Fourier coefficients** of $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(\omega_k x) + b_k \sin(\omega_k x))$$

$$\omega_k = \frac{2k\pi}{L}$$

$$a_k = \frac{2}{L} \int_0^L f(x) \cos(\omega_k x) dx \quad b_k = \frac{2}{L} \int_0^L f(x) \sin(\omega_k x) dx$$

Frequency

- As k increases, the frequency of the basis functions increase.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(\omega_k x) + b_k \sin(\omega_k x)) \quad \omega_k = \frac{2k\pi}{L}$$

$$a_k = \frac{2}{L} \int_0^L f(x) \cos(\omega_k x) dx \quad b_k = \frac{2}{L} \int_0^L f(x) \sin(\omega_k x) dx$$

Frequency-Domain Representation of a Function

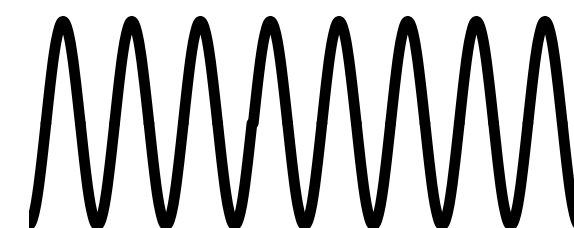
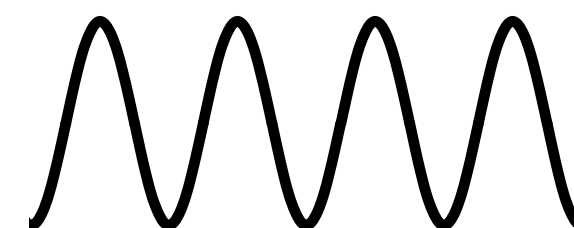
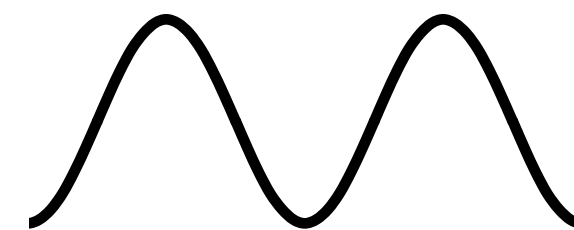
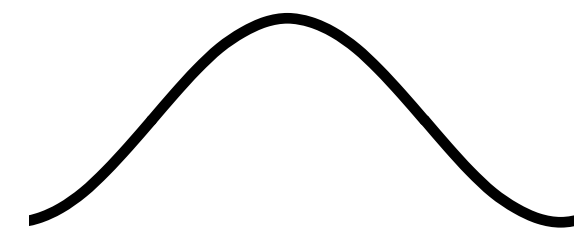
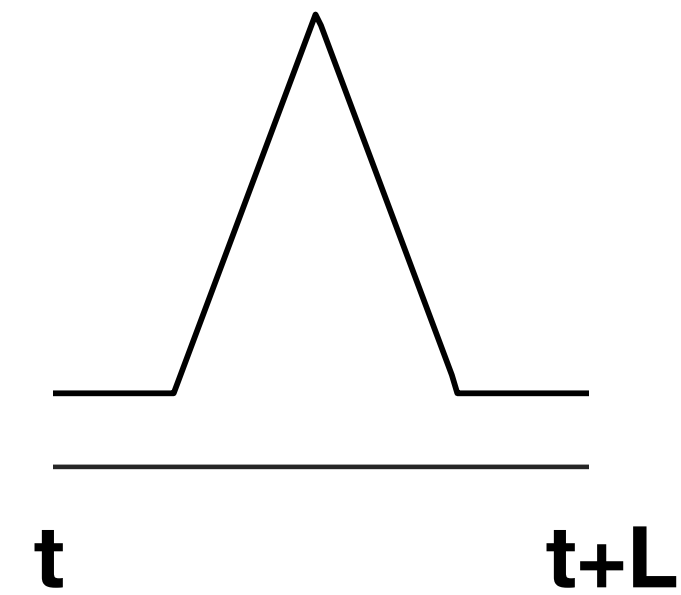
- We have converted a function $f(x)$ into an infinite sequence of a_k and b_k .
- The (infinite) coefficient sequence is called the frequency-domain representation, or the **spectrum**, of the function f .

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(\omega_k x) + b_k \sin(\omega_k x)) \quad \omega_k = \frac{2k\pi}{L}$$

$$a_k = \frac{2}{L} \int_0^L f(x) \cos(\omega_k x) dx \quad b_k = \frac{2}{L} \int_0^L f(x) \sin(\omega_k x) dx$$

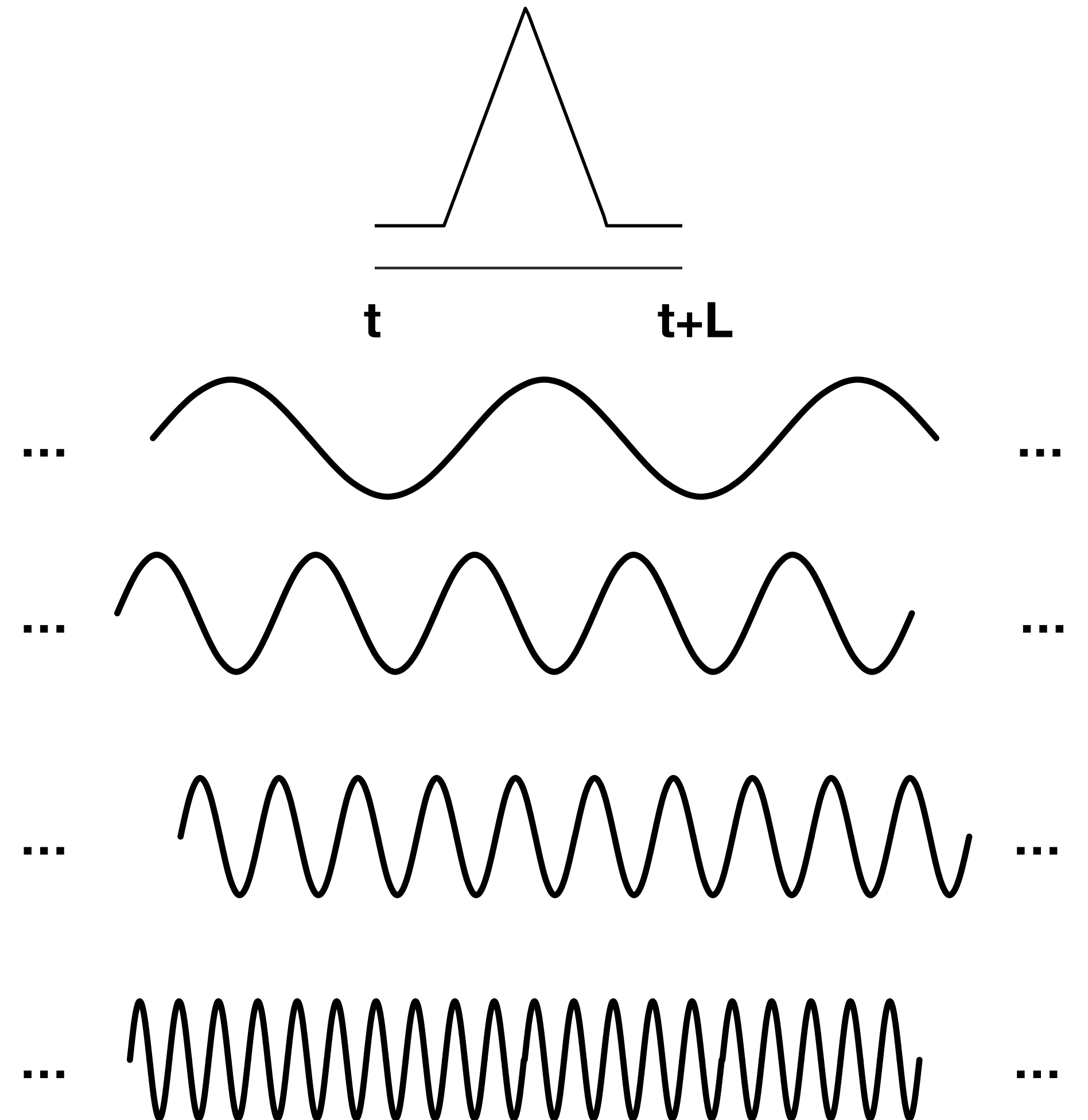
Applies to Only Periodic Functions

- We assumed f and the basis functions are defined only over a fixed interval $[t, t+L)$



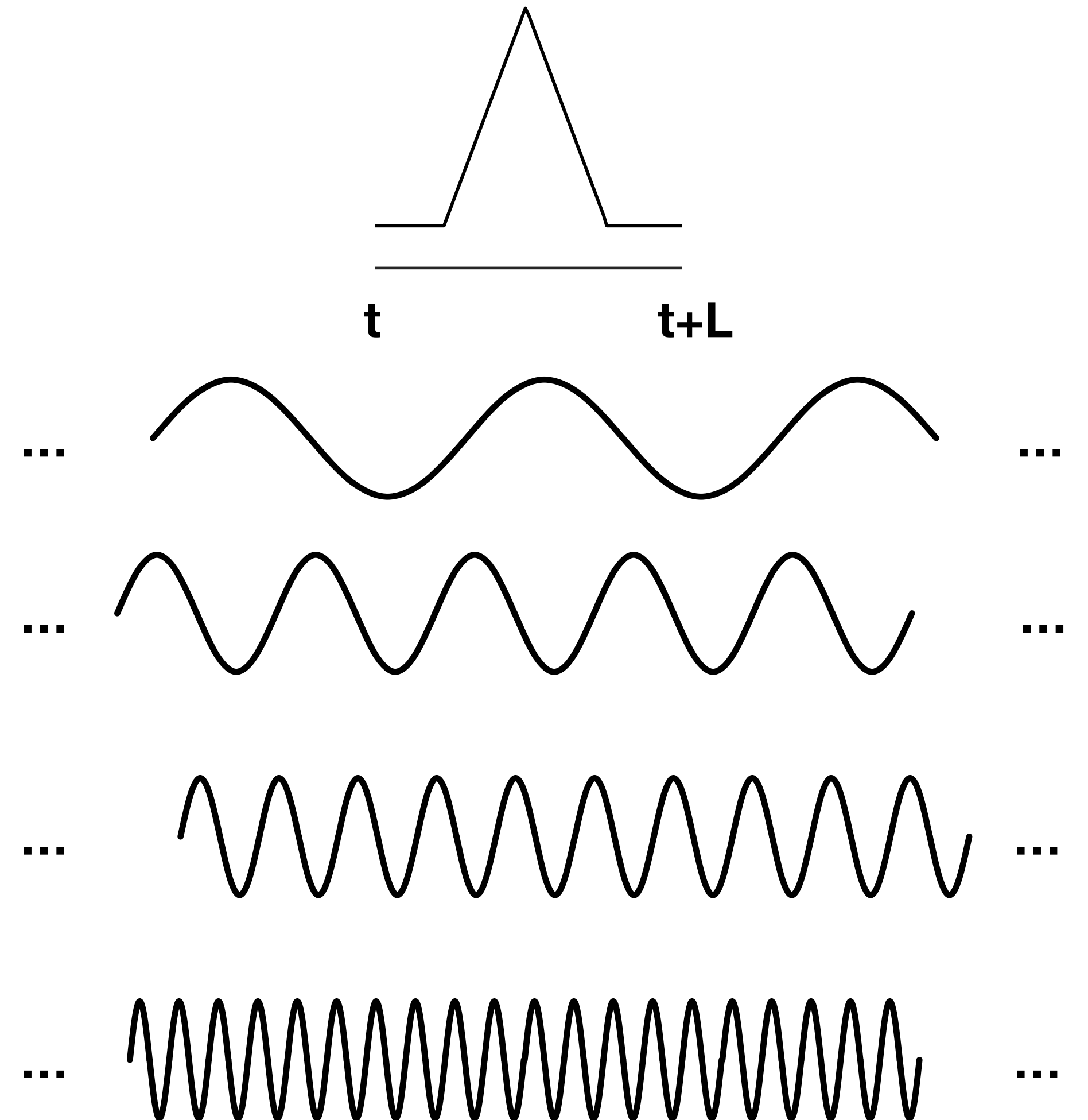
Applies to Only Periodic Functions

- We assumed f and the basis functions are defined only over a fixed interval $[t, t+L)$



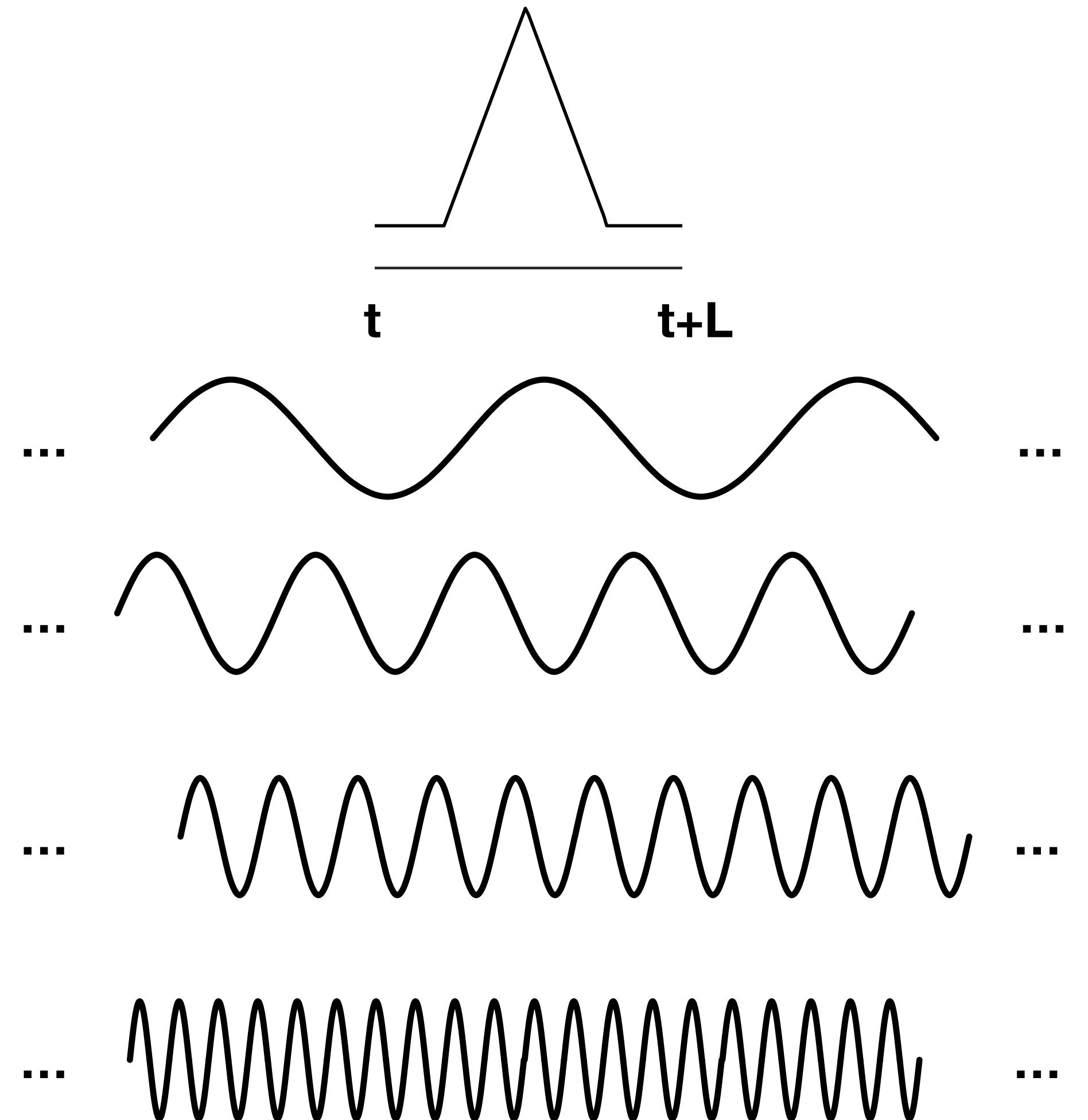
Applies to Only Periodic Functions

- We assumed f and the basis functions are defined only over a fixed interval $[t, t+L)$
- But sinusoidal functions have infinite span; they are not limited to a fixed interval



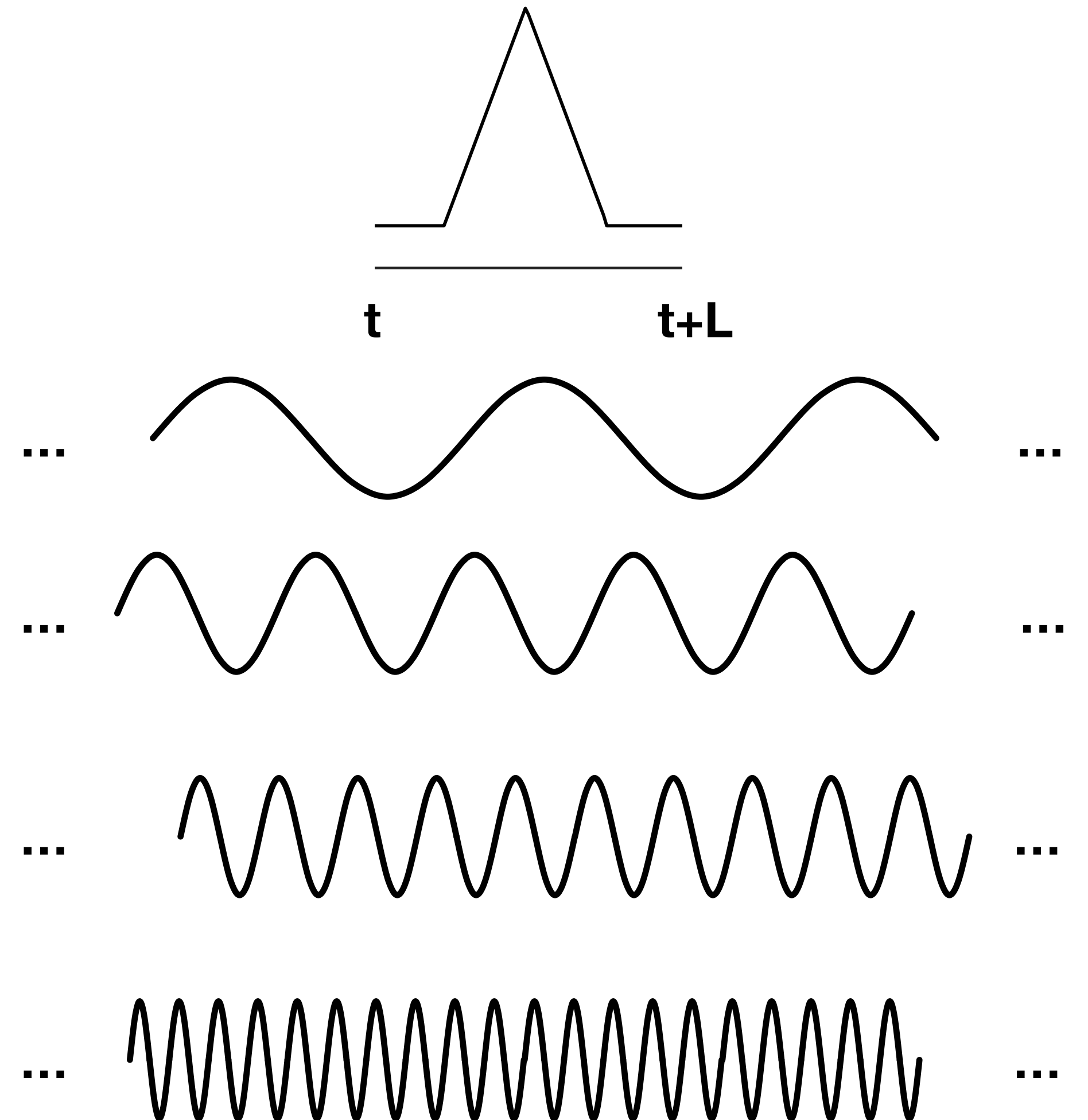
Applies to Only Periodic Functions

- We assumed f and the basis functions are defined only over a fixed interval $[t, t+L)$
- But sinusoidal functions have infinite span; they are not limited to a fixed interval
- What if we just add the Fourier series over the entire span?



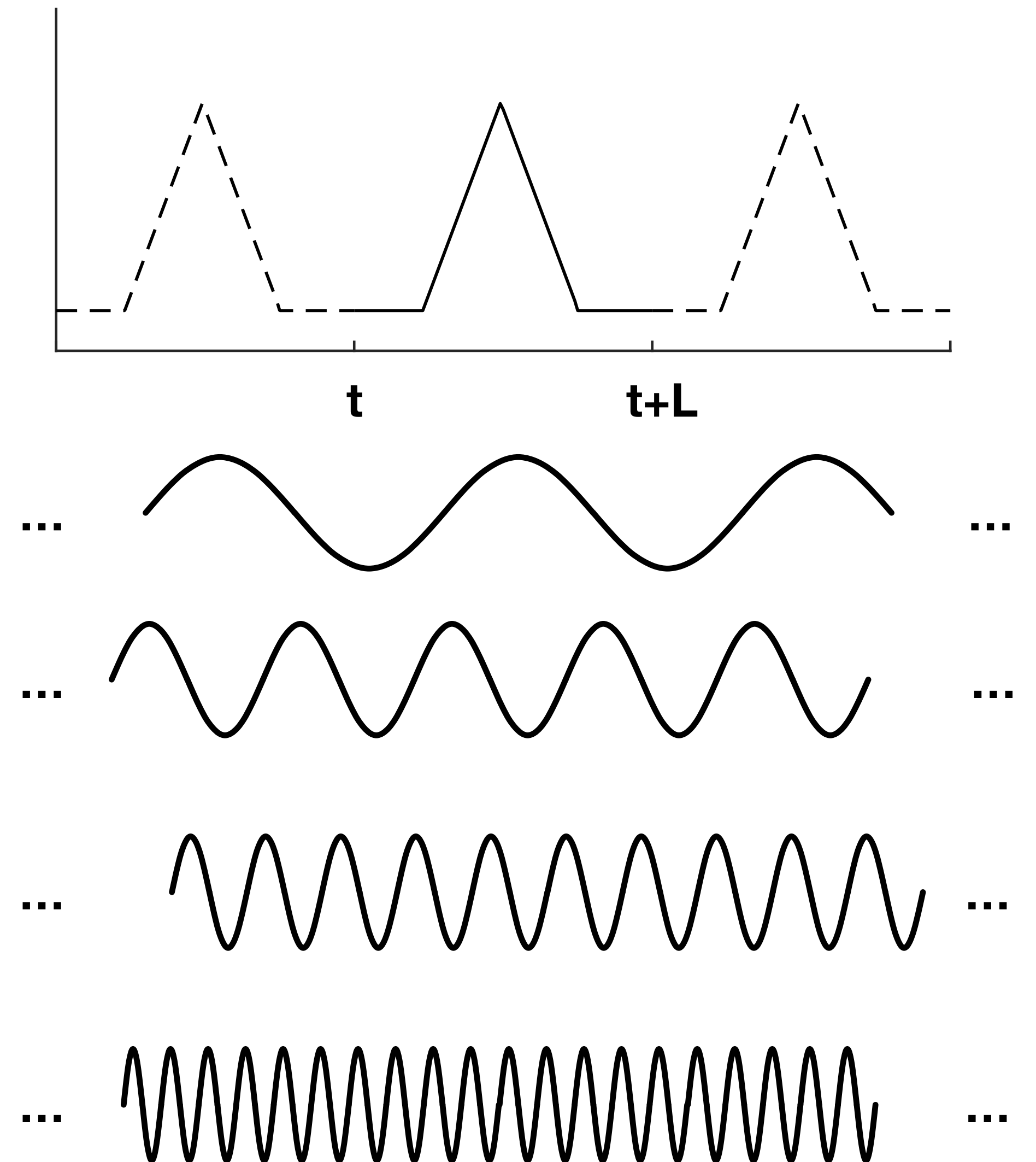
Applies to Only Periodic Functions

- We assumed f and the basis functions are defined only over a fixed interval $[t, t+L)$
- But sinusoidal functions have infinite span; they are not limited to a fixed interval
- What if we just add the Fourier series over the entire span?
- We will generate a **L-periodic** function!



Applies to Only Periodic Functions

- We assumed f and the basis functions are defined only over a fixed interval $[t, t+L)$
- But sinusoidal functions have infinite span; they are not limited to a fixed interval
- What if we just add the Fourier series over the entire span?
- We will generate a **L-periodic** function!



Fourier Series: What It Says

Fourier coefficients

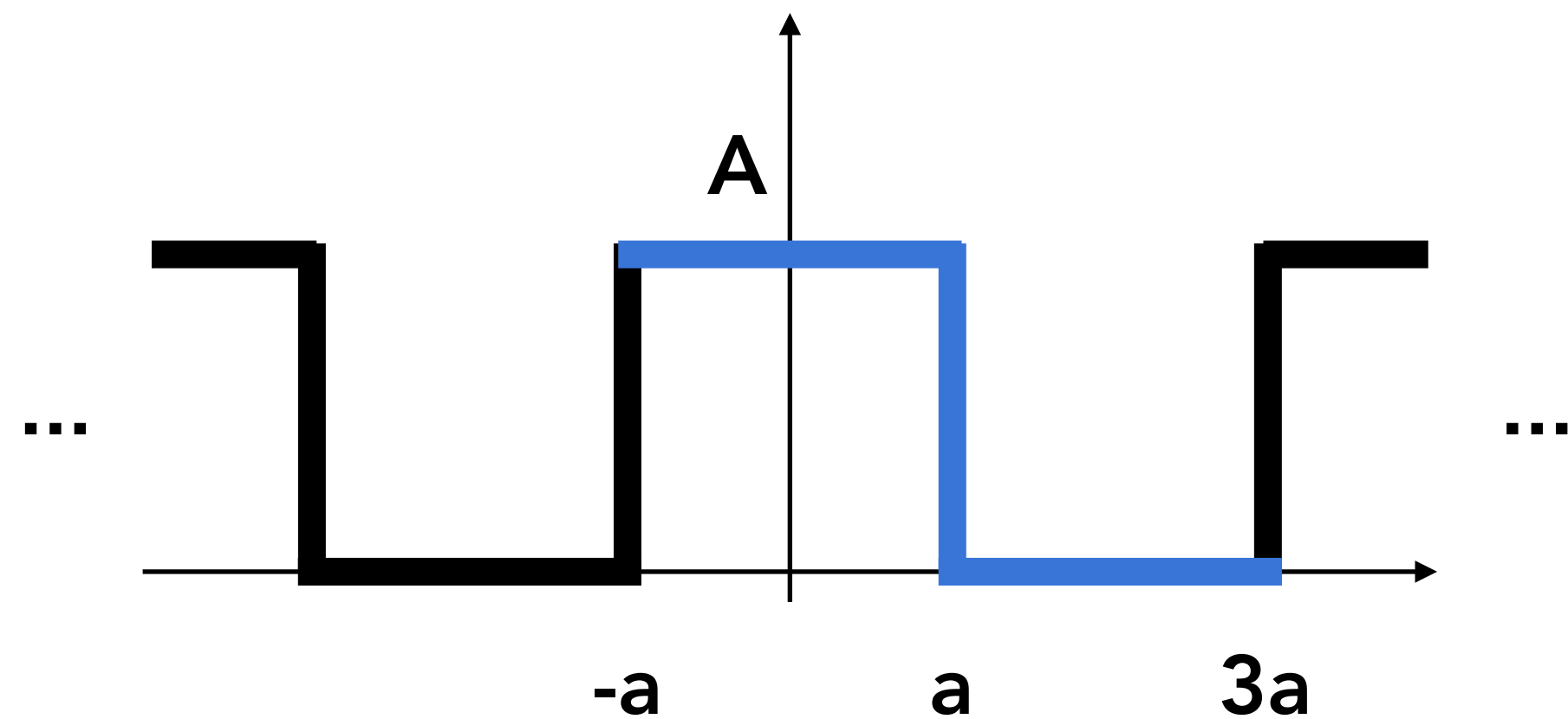


Joseph Fourier (1768 – 1830)

Any **periodic** function can be represented as an infinite **weighted** sum of **sines and cosines** with increasing frequencies.

Basis functions

Back To the First Example



A square function

Period $L = 4a$

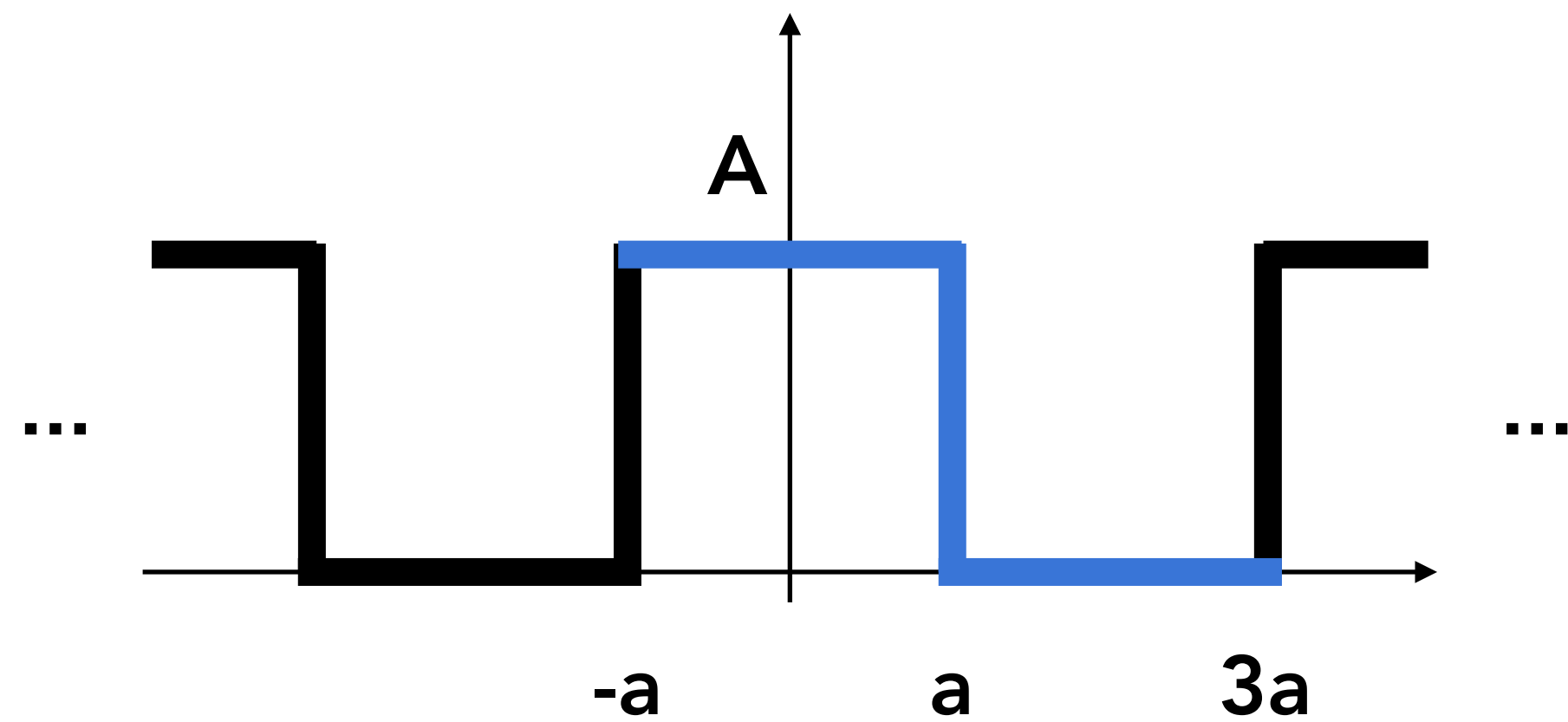
Frequency is $1/4a$

Amplitude is A

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

Back To the First Example

$$\omega_k = \frac{2k\pi}{L}$$



A square function

Period $L = 4a$

Frequency is $1/4a$

Amplitude is A

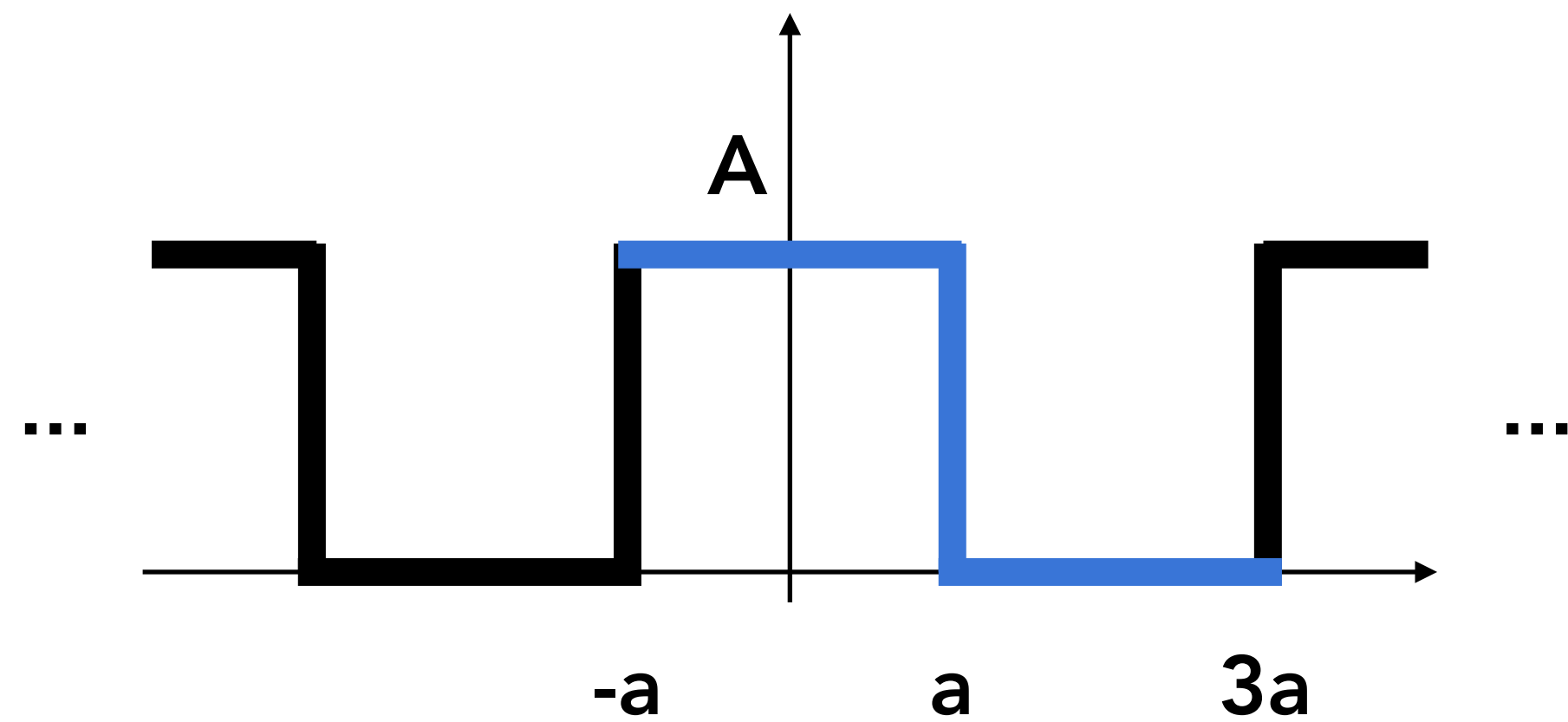
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(\omega_k x) + b_k \sin(\omega_k x))$$

$$a_k = \frac{2}{L} \int_t^{t+L} f(x) \cos(\omega_k x) dx$$

$$b_k = \frac{2}{L} \int_t^{t+L} f(x) \sin(\omega_k x) dx$$

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

Back To the First Example



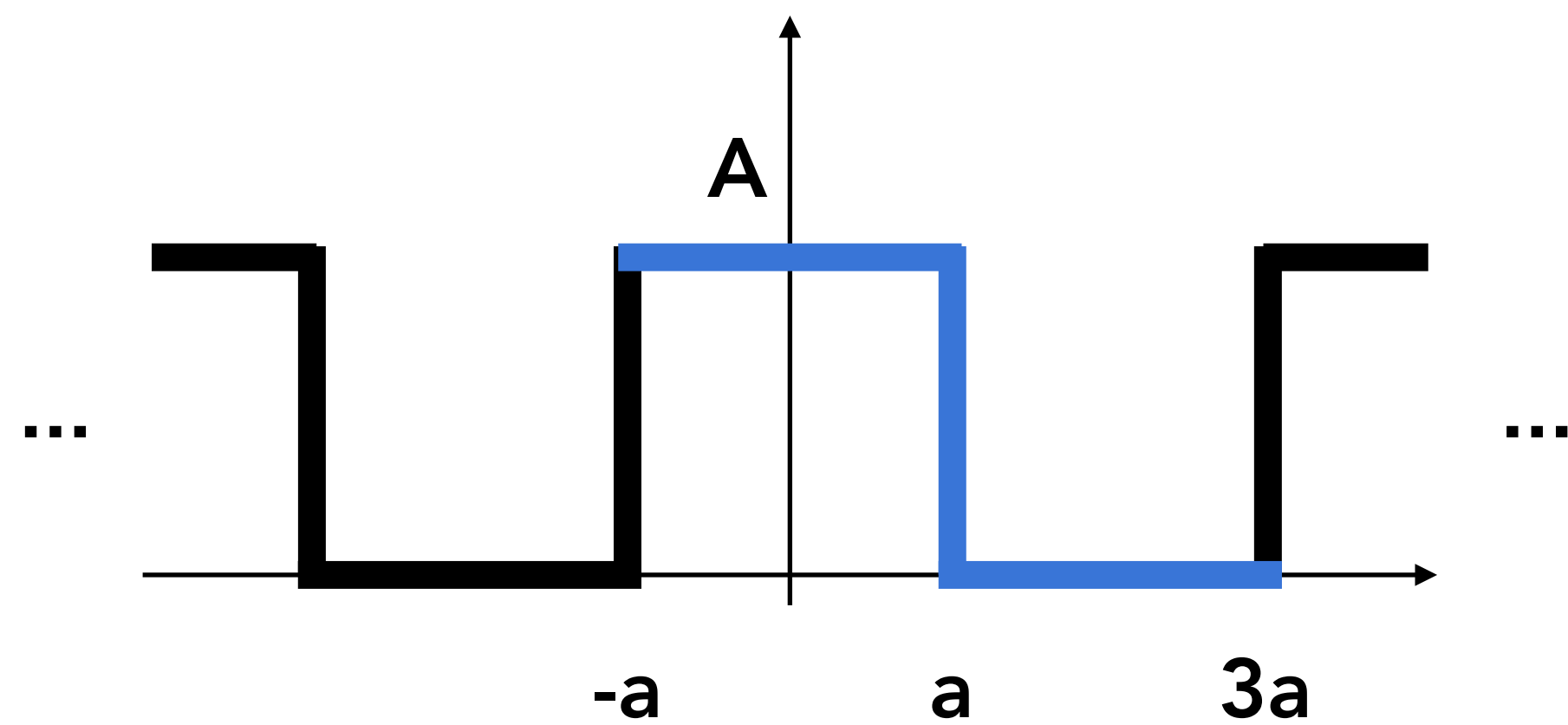
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{2a}x\right) + b_k \sin\left(\frac{k\pi}{2a}x\right) \right)$$

$$a_k = \frac{1}{2a} \int_{-a}^{3a} f(x) \cos\left(\frac{k\pi}{2a}x\right) dx$$

$$b_k = \frac{1}{2a} \int_{-a}^{3a} f(x) \sin\left(\frac{k\pi}{2a}x\right) dx$$

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

Back To the First Example



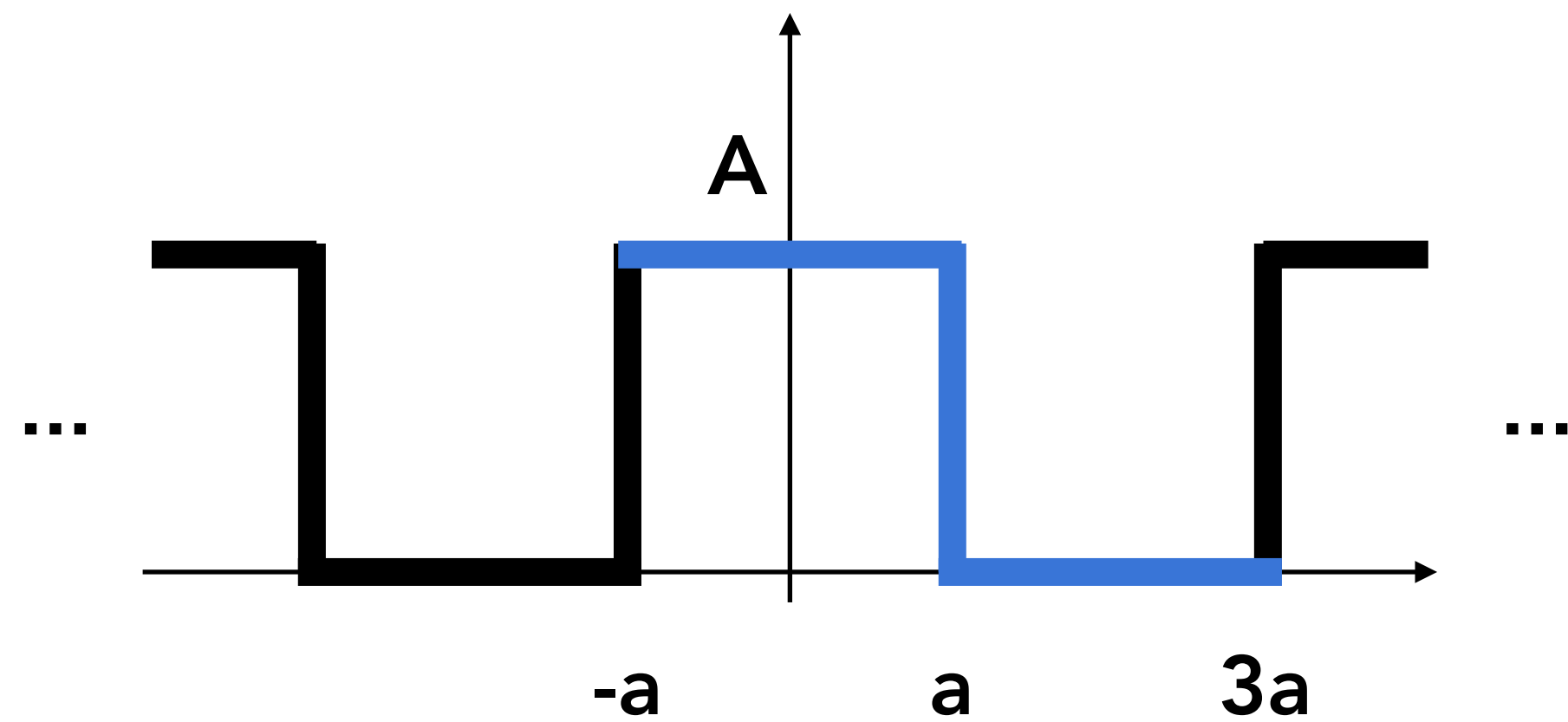
$$f(x) = \boxed{\frac{a_0}{2}} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{2a}x\right) + b_k \sin\left(\frac{k\pi}{2a}x\right) \right)$$

$$a_k = \frac{1}{2a} \int_{-a}^{3a} f(x) \cos\left(\frac{k\pi}{2a}x\right) dx$$

$$a_0 = \frac{1}{2a} \int_{-a}^{3a} f(x) \cos\left(\frac{0\pi}{2a}x\right) dx = \frac{1}{2a} \int_{-a}^{3a} f(x) dx = \frac{1}{2a} 2aA = A$$

$$f(x) = \boxed{\frac{A}{2}} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

Why No *Sine* Term?

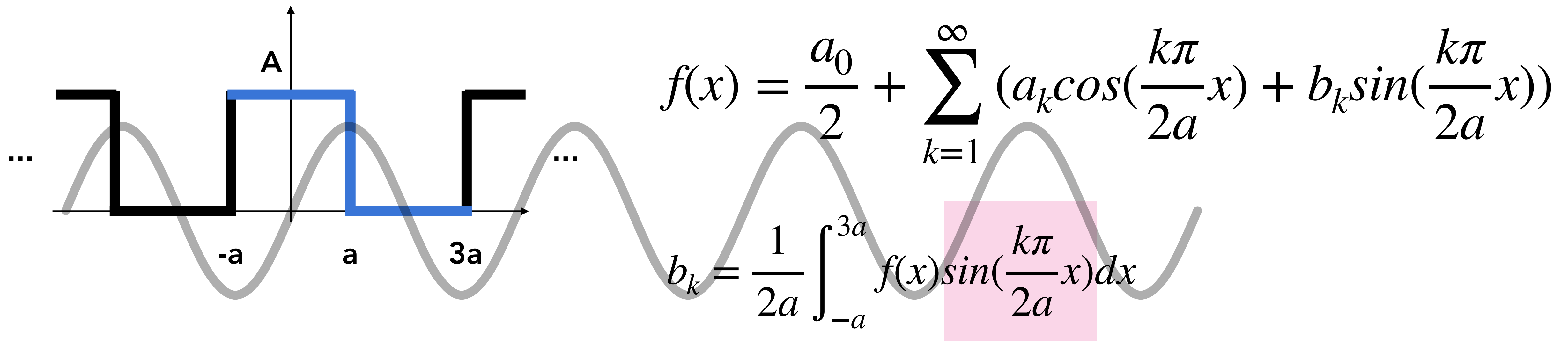


$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{2a}x\right) + b_k \sin\left(\frac{k\pi}{2a}x\right) \right)$$

$$b_k = \frac{1}{2a} \int_{-a}^{3a} f(x) \sin\left(\frac{k\pi}{2a}x\right) dx$$

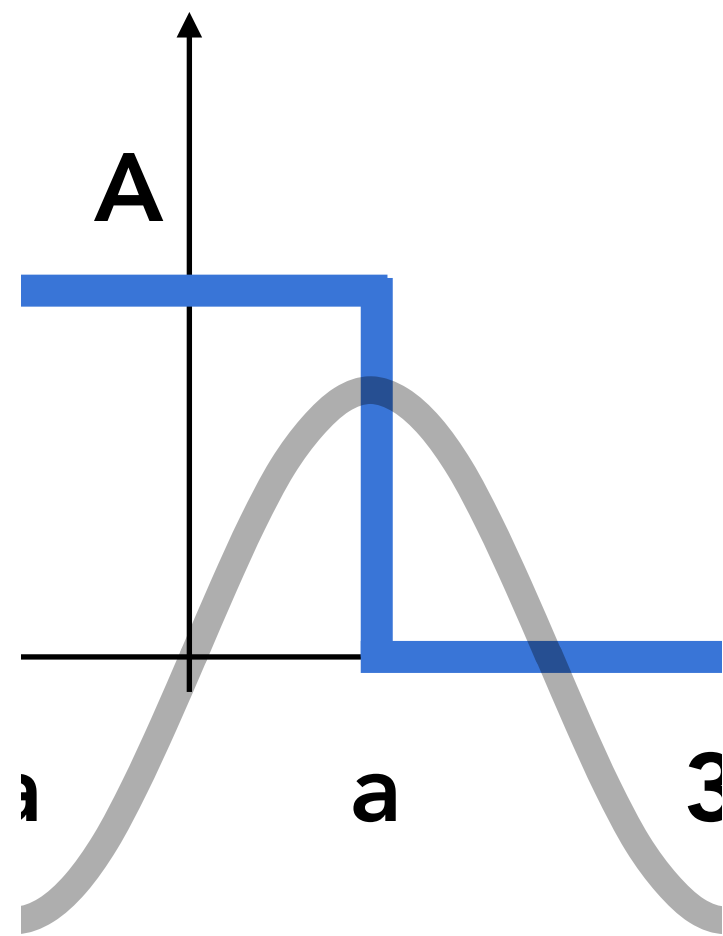
- The period of the *sine* functions are $4a/k$, i.e., $4a, 2a, a, a/2, \dots$

Why No *Sine* Term?



- The period of the *sine* functions are $4a/k$, i.e., $4a, 2a, a, a/2, \dots$

Why No *Sine* Term?

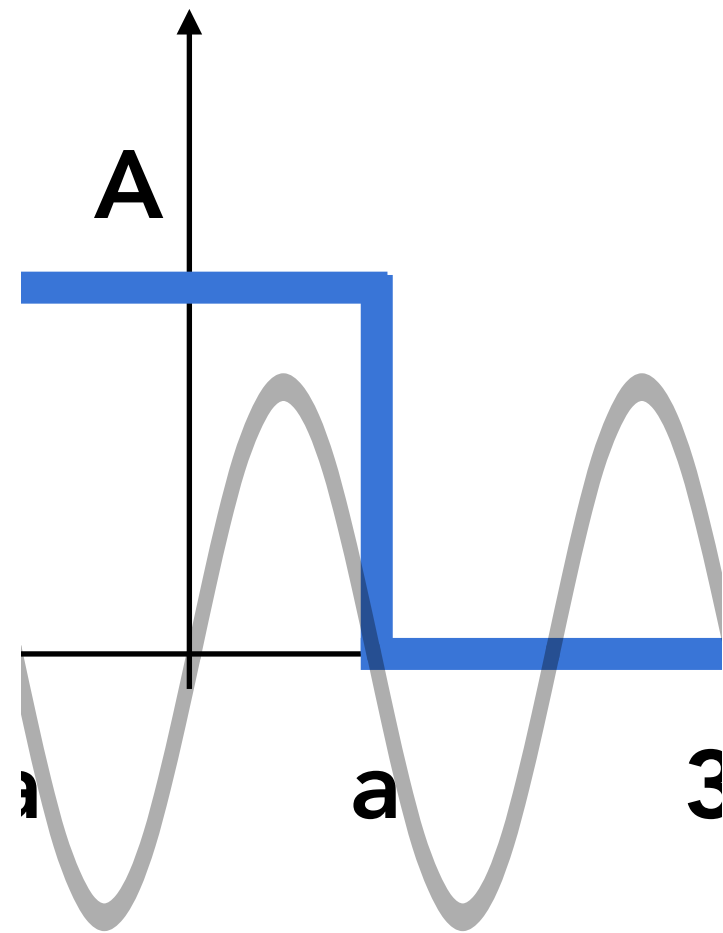


$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{2a}x\right) + b_k \sin\left(\frac{k\pi}{2a}x\right) \right)$$

$$b_k = \frac{1}{2a} \int_{-a}^{3a} f(x) \sin\left(\frac{k\pi}{2a}x\right) dx$$

- The period of the *sine* functions are $4a/k$, i.e., $4a, 2a, a, a/2, \dots$

Why No *Sine* Term?

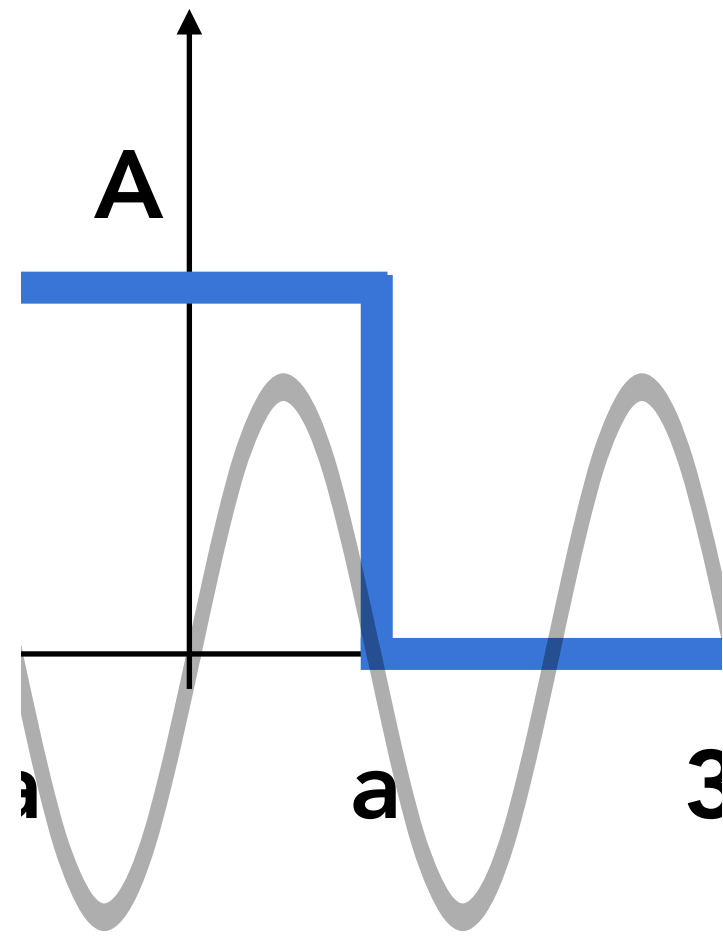


$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{2a}x\right) + b_k \sin\left(\frac{k\pi}{2a}x\right) \right)$$

$$= \frac{1}{2a} \int_{-a}^{3a} f(x) \sin\left(\frac{k\pi}{2a}x\right) dx$$

- The period of the *sine* functions are $4a/k$, i.e., $4a, 2a, a, a/2, \dots$

Why No *Sine* Term?

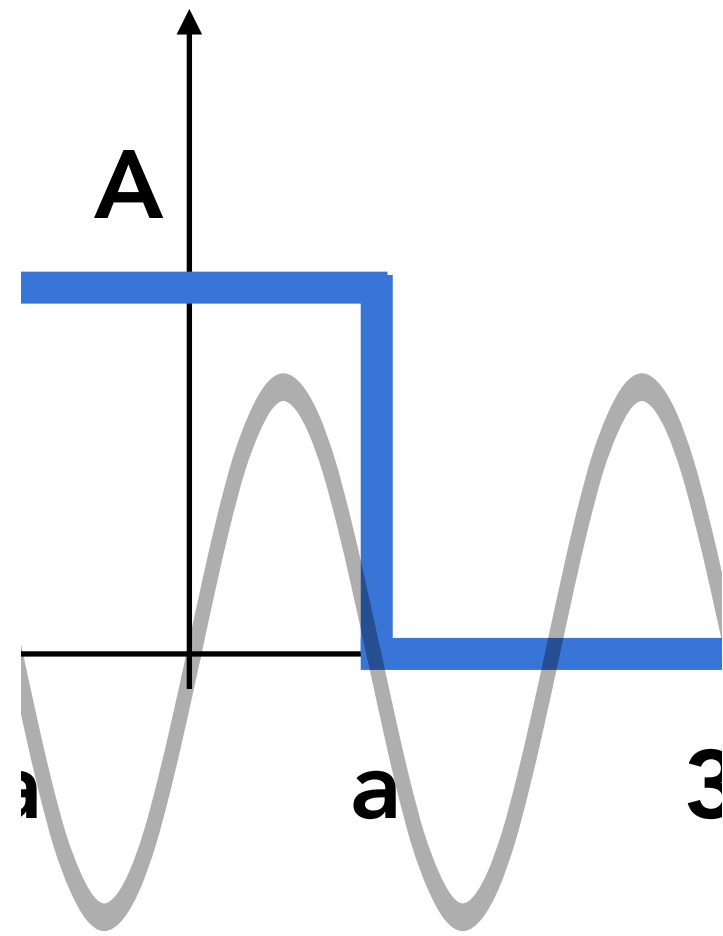


$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{2a}x\right) + b_k \sin\left(\frac{k\pi}{2a}x\right) \right)$$

$$= \frac{1}{2a} \int_{-a}^{3a} f(x) \sin\left(\frac{k\pi}{2a}x\right) dx$$

- The period of the *sine* functions are $4a/k$, i.e., $4a, 2a, a, a/2, \dots$
- Since *sine* is an odd function, b_k always becomes 0

Why No *Sine* Term?



$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{2a}x\right) + b_k \sin\left(\frac{k\pi}{2a}x\right) \right)$$

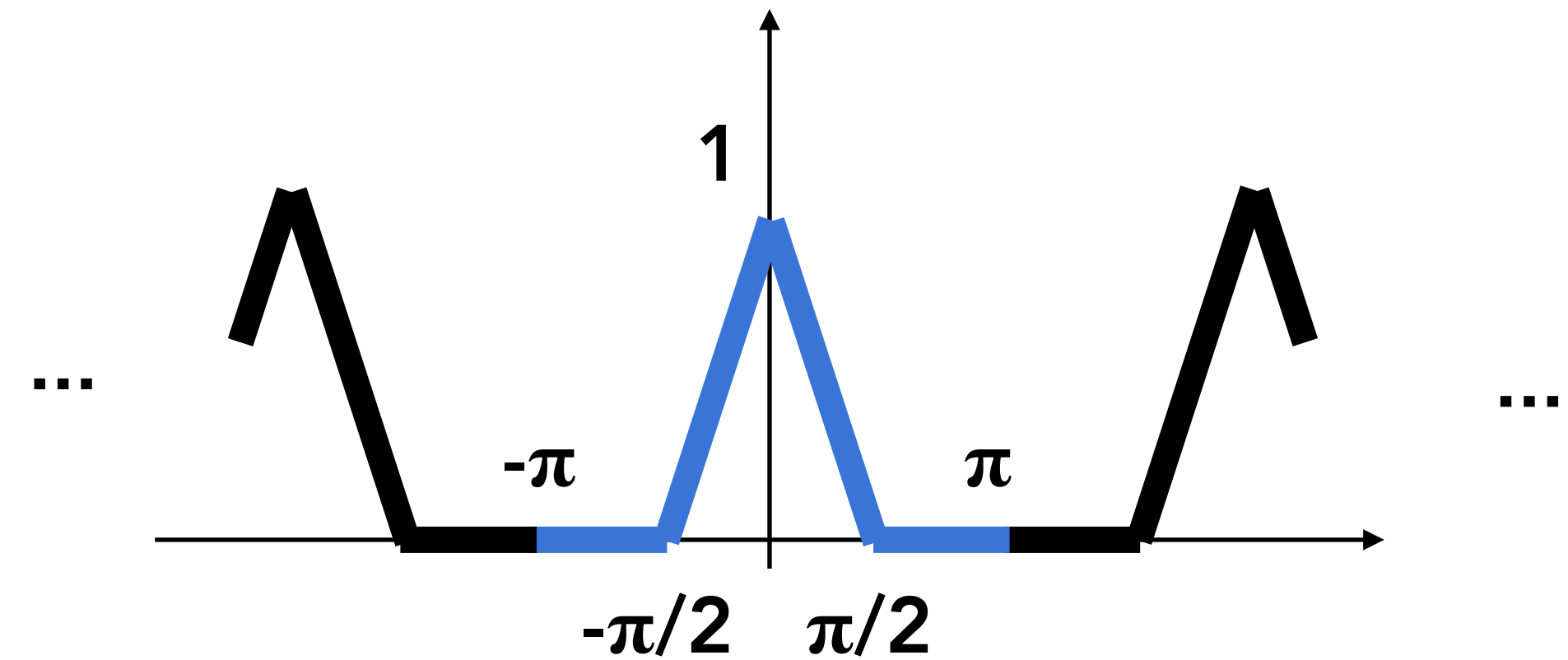
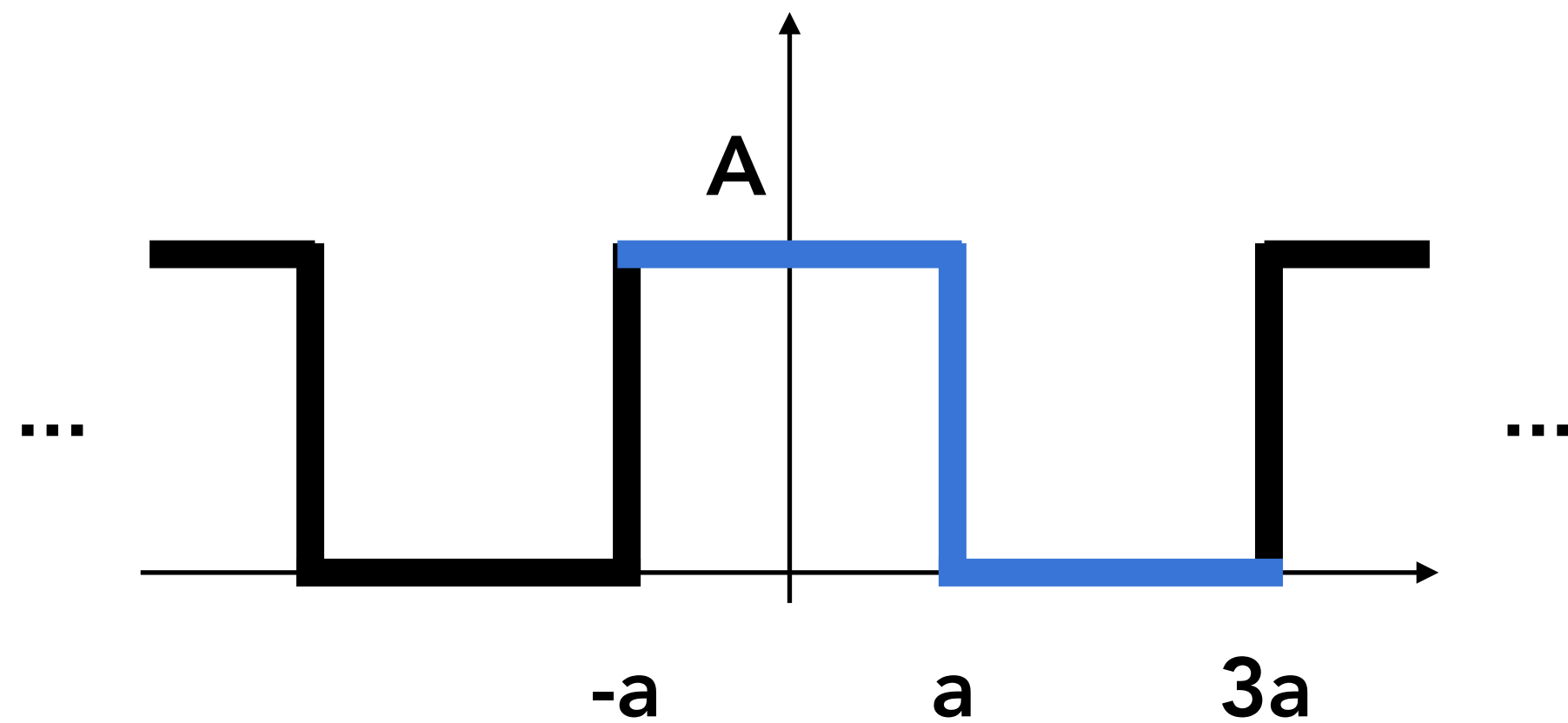
$$= \frac{1}{2a} \int_{-a}^{3a} f(x) \sin\left(\frac{k\pi}{2a}x\right) dx$$

- The period of the *sine* functions are $4a/k$, i.e., $4a, 2a, a, a/2, \dots$
- Since *sine* is an odd function, b_k always becomes 0

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

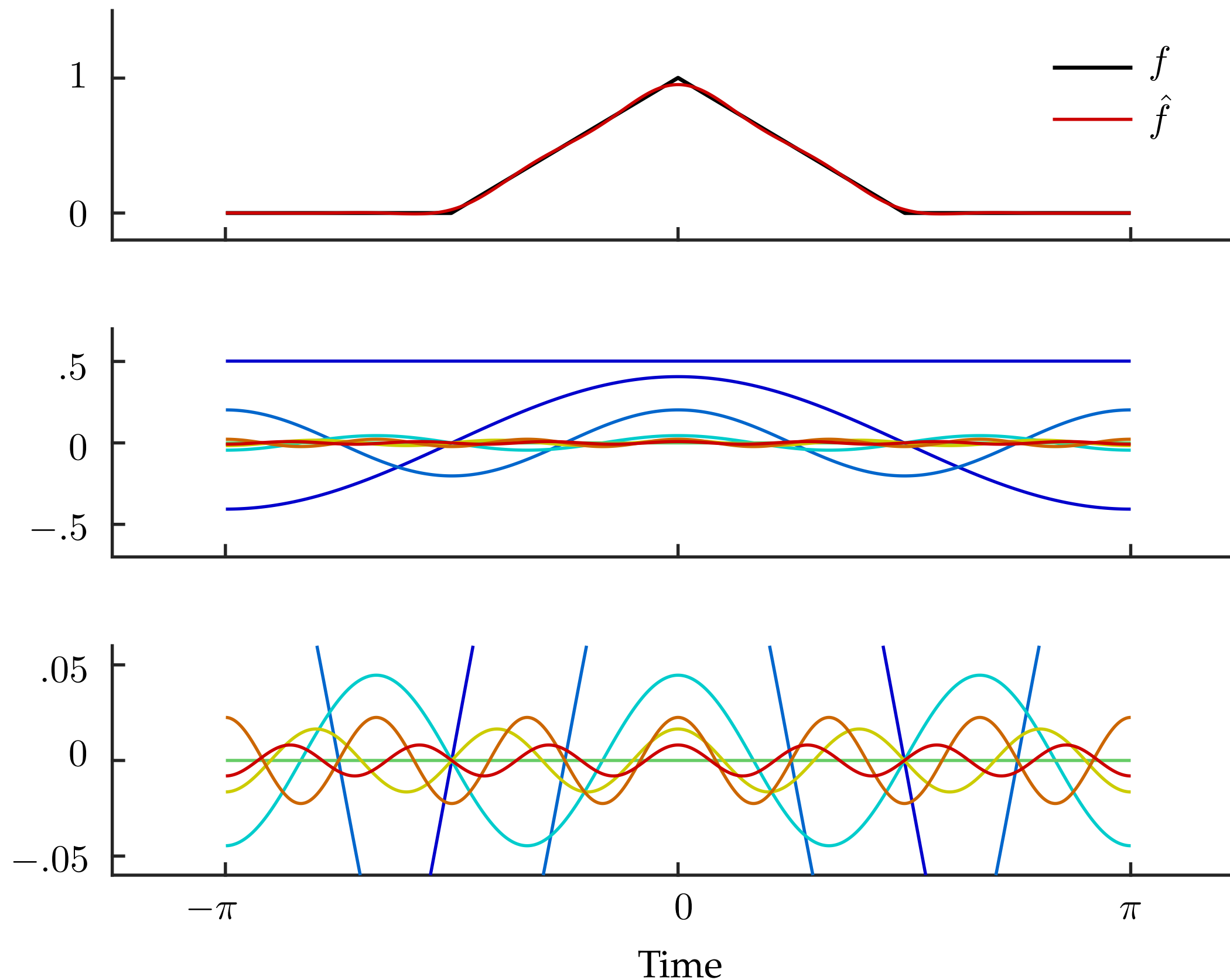
Even Functions Need only Cosine Terms

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2} \cos(x) + \frac{4}{(3\pi)^2} \cos(3x) + \frac{4}{(5\pi)^2} \cos(5x) + \frac{4}{(7\pi)^2} \cos(7x) + \dots$$



$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

Function Approximation



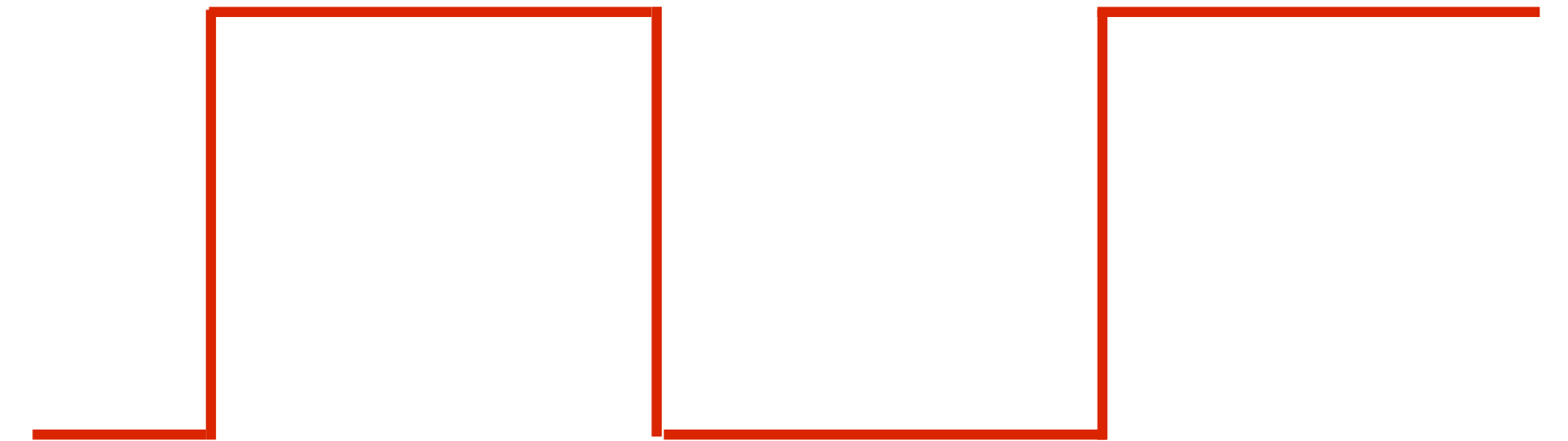
The periodic hat function and the sum of the first seven terms

The first seven terms

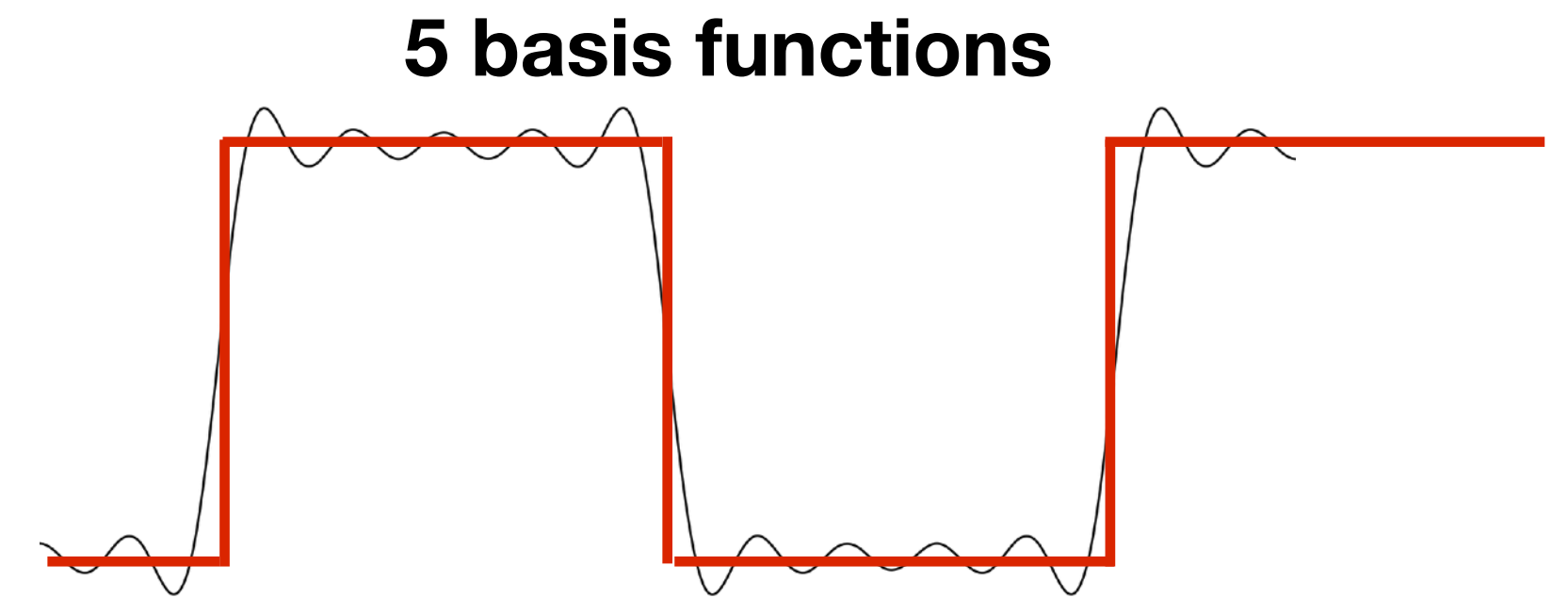
Zoom in so you see the high-frequency part

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2} \cos(x) + \frac{4}{(3\pi)^2} \cos(3x) + \frac{4}{(5\pi)^2} \cos(5x) + \frac{4}{(7\pi)^2} \cos(7x) + \dots$$

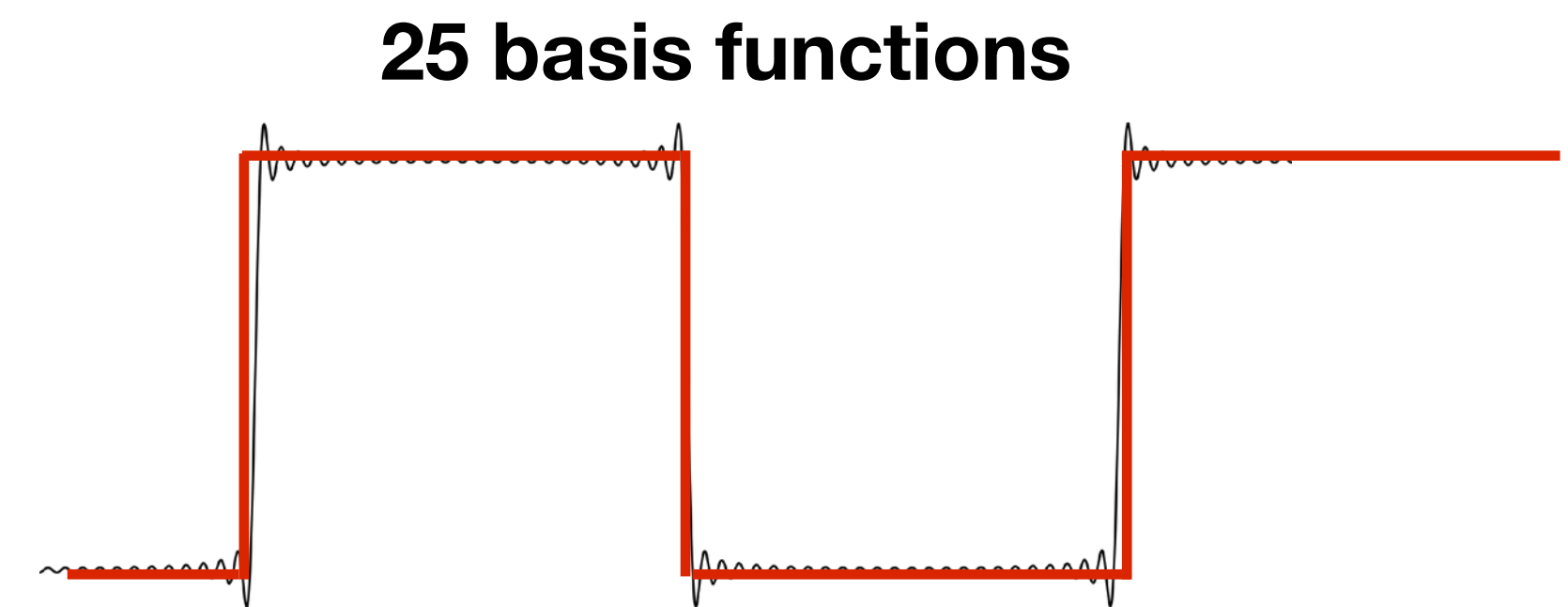
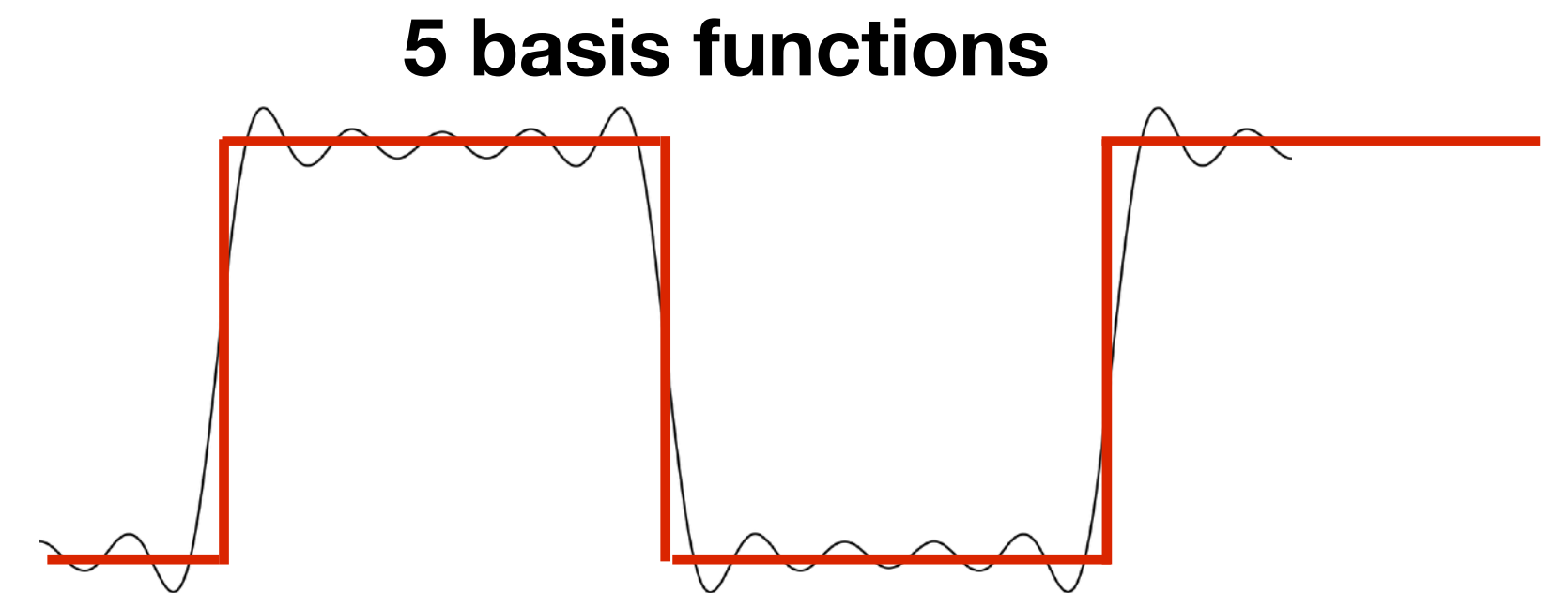
Gibbs Phenomena (Ringing)



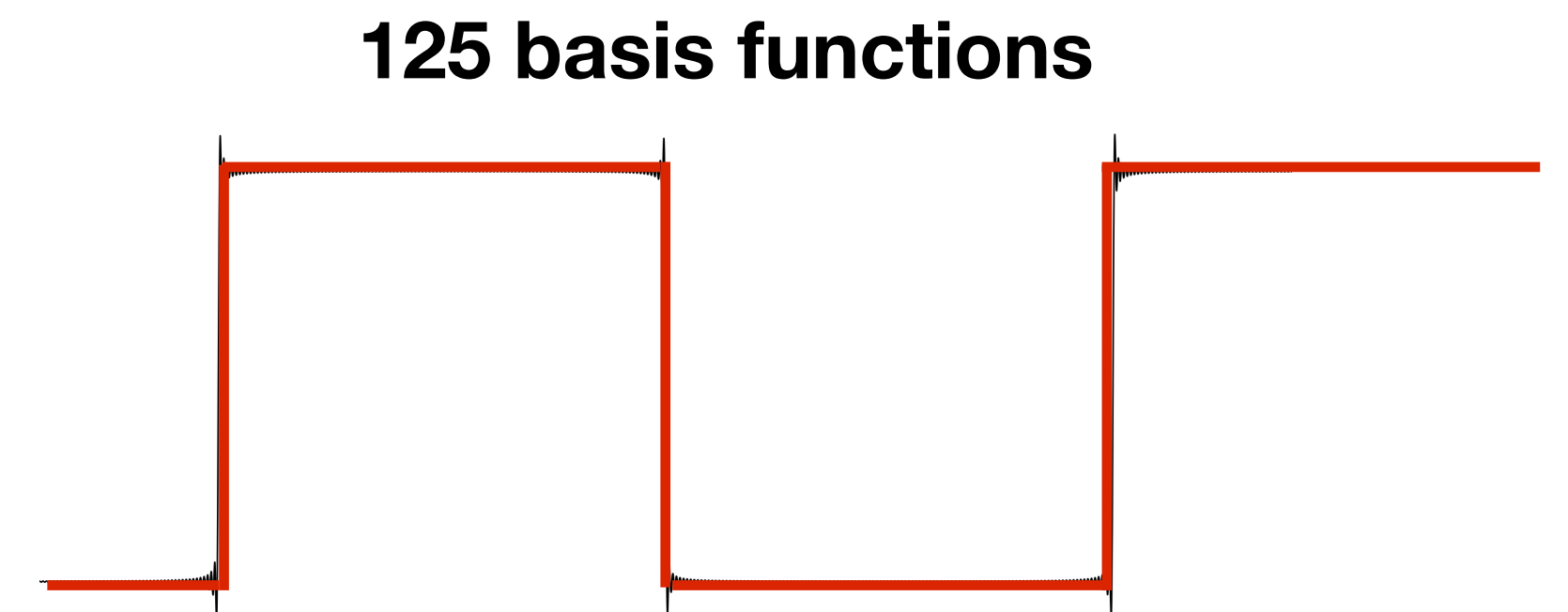
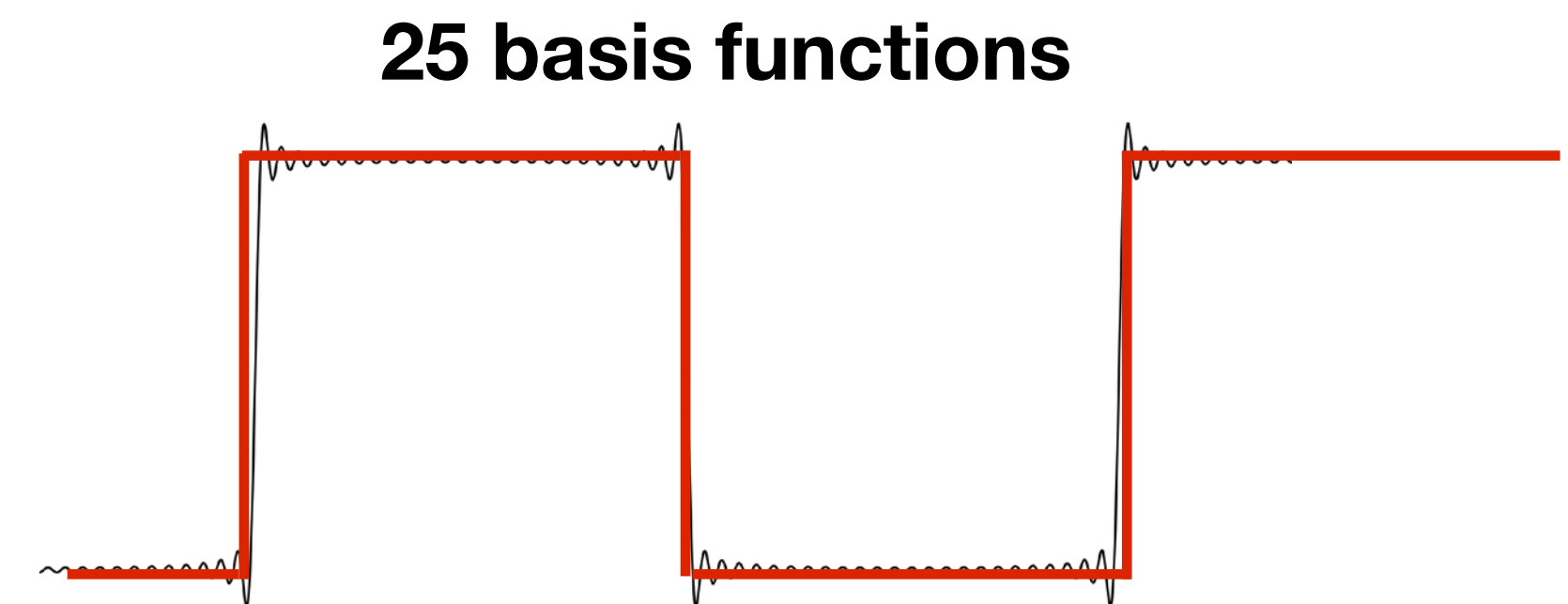
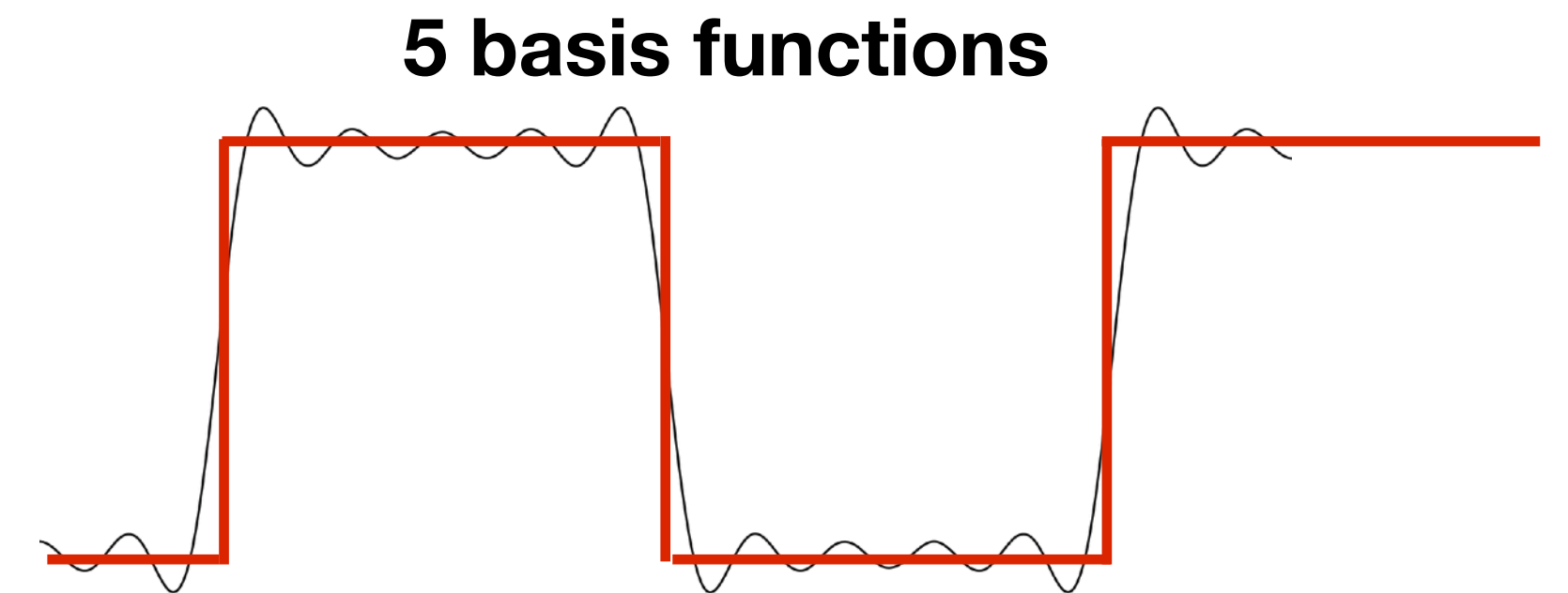
Gibbs Phenomena (Ringing)



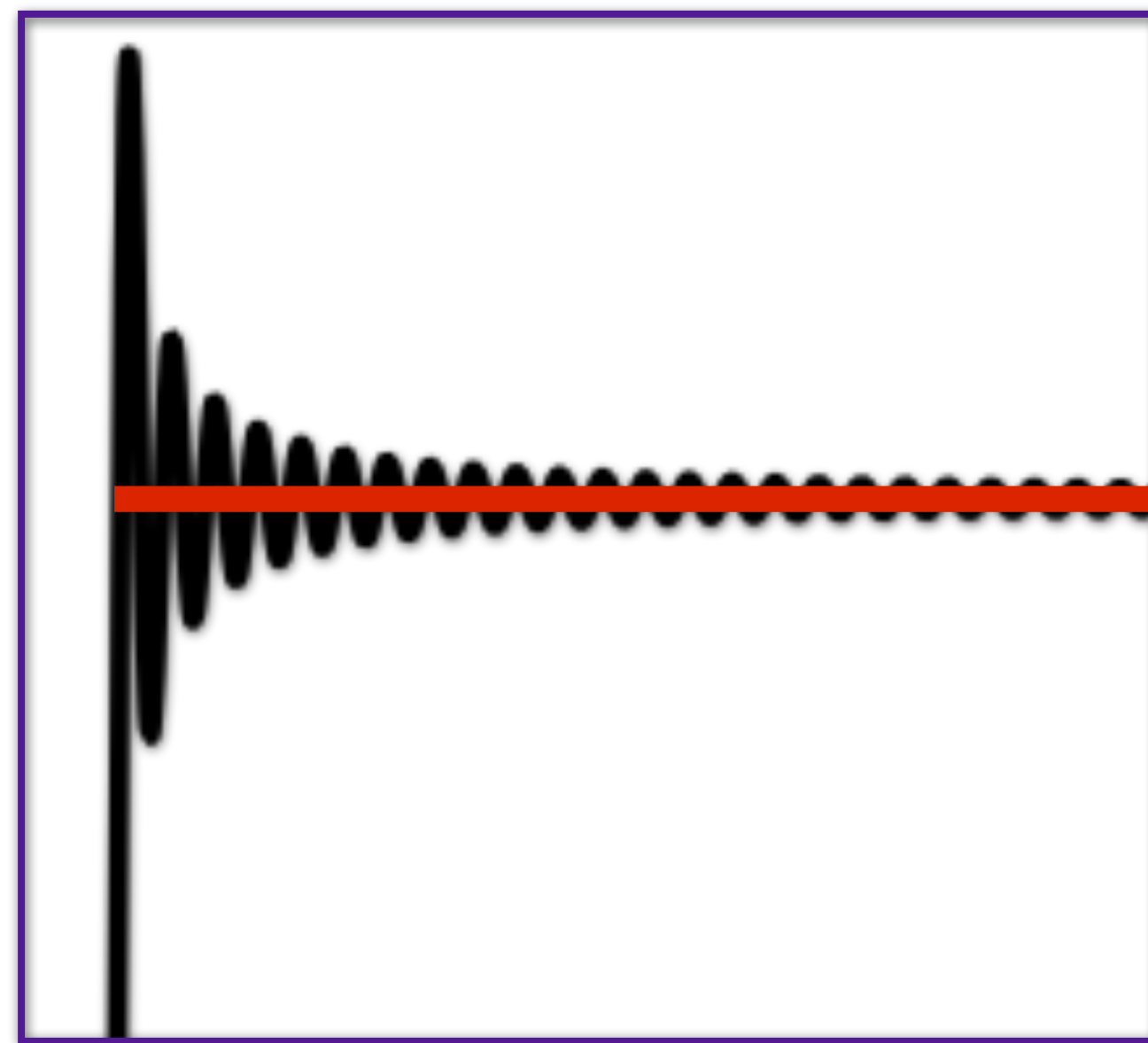
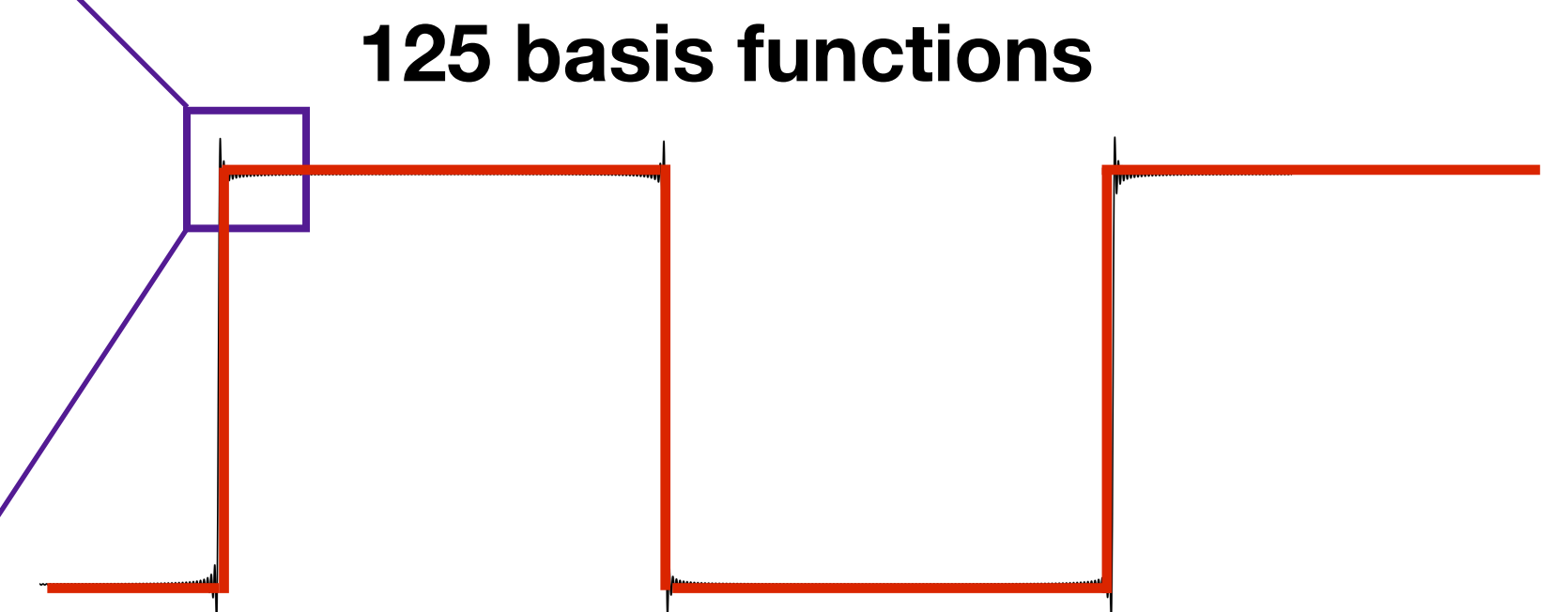
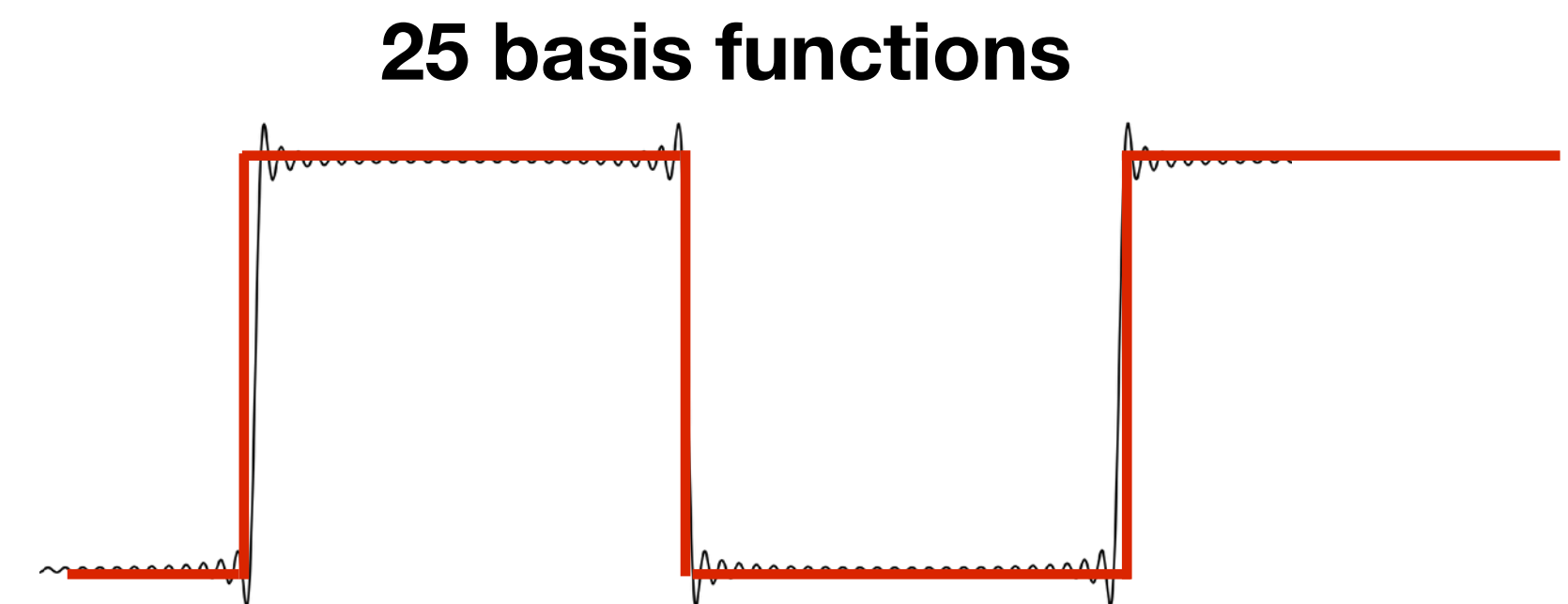
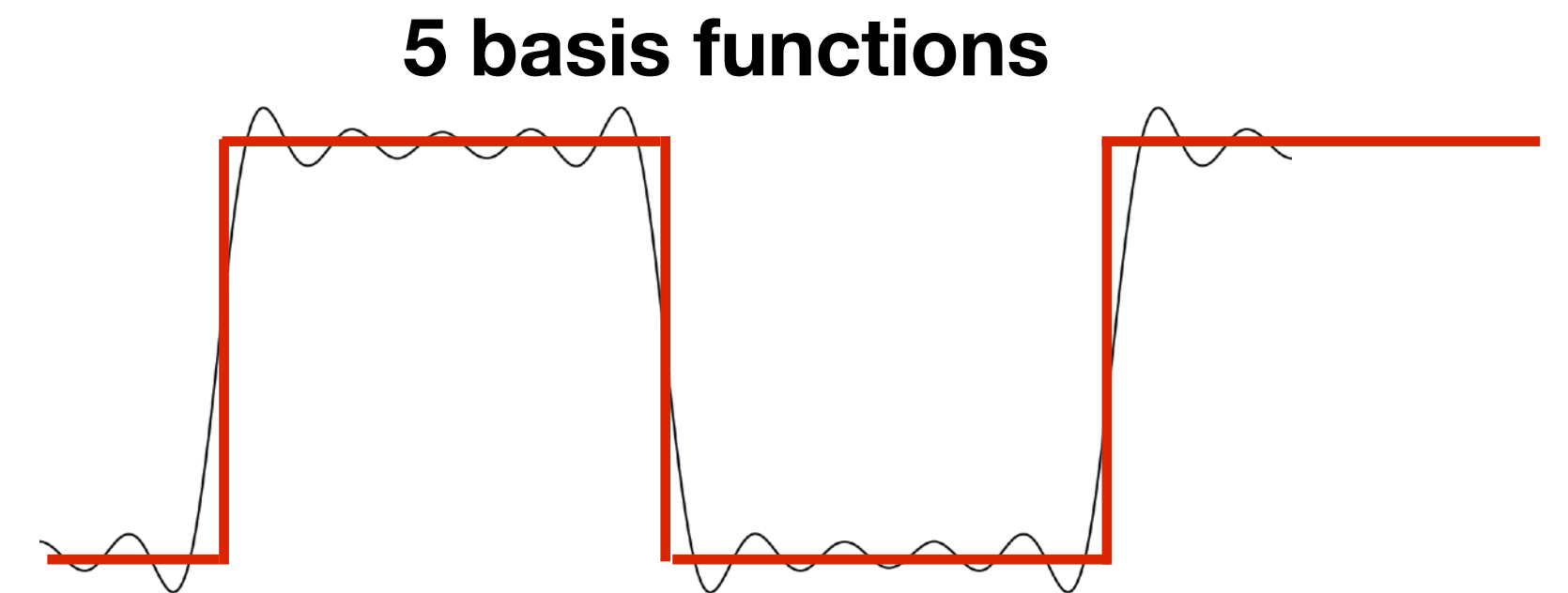
Gibbs Phenomena (Ringing)



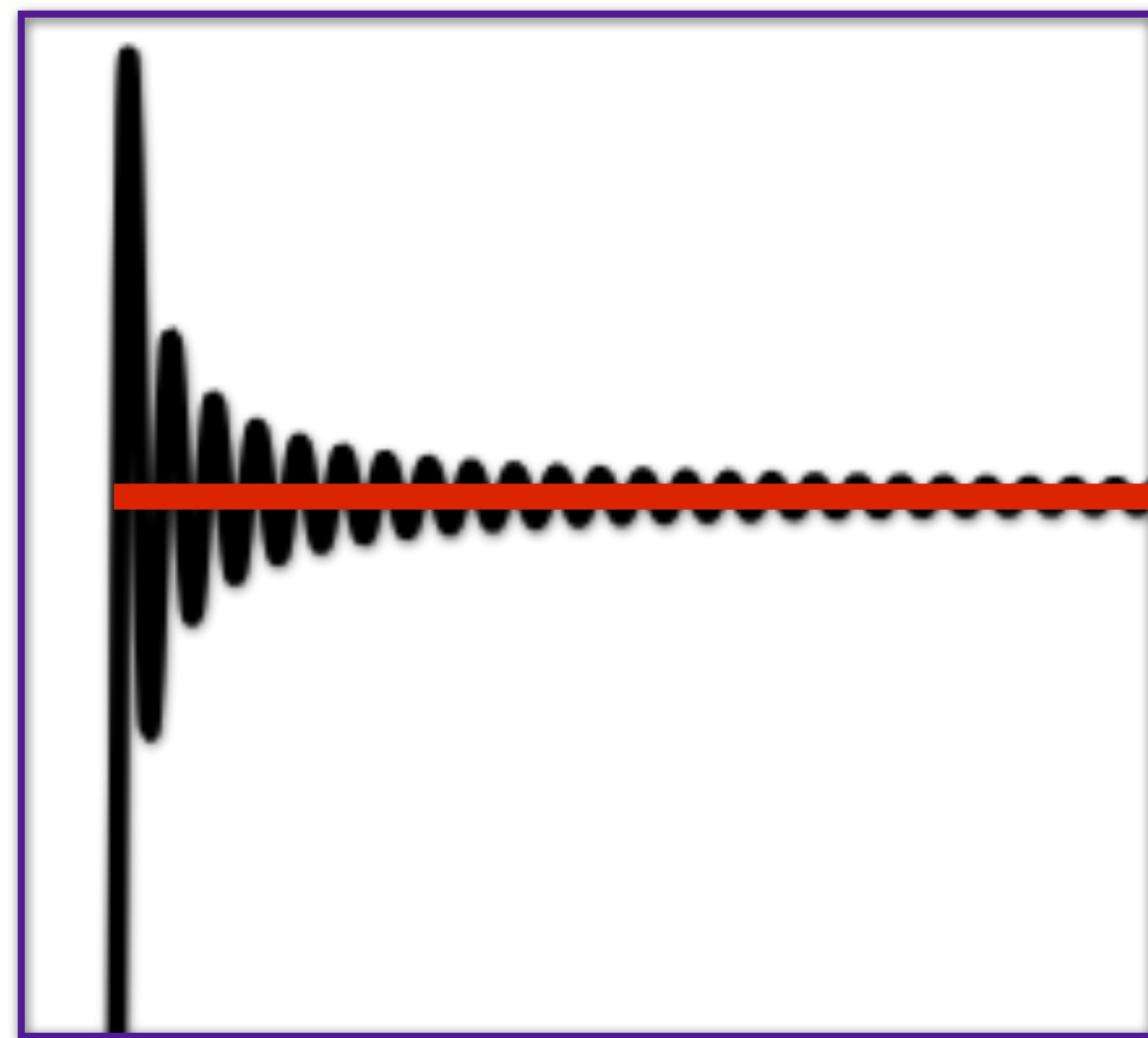
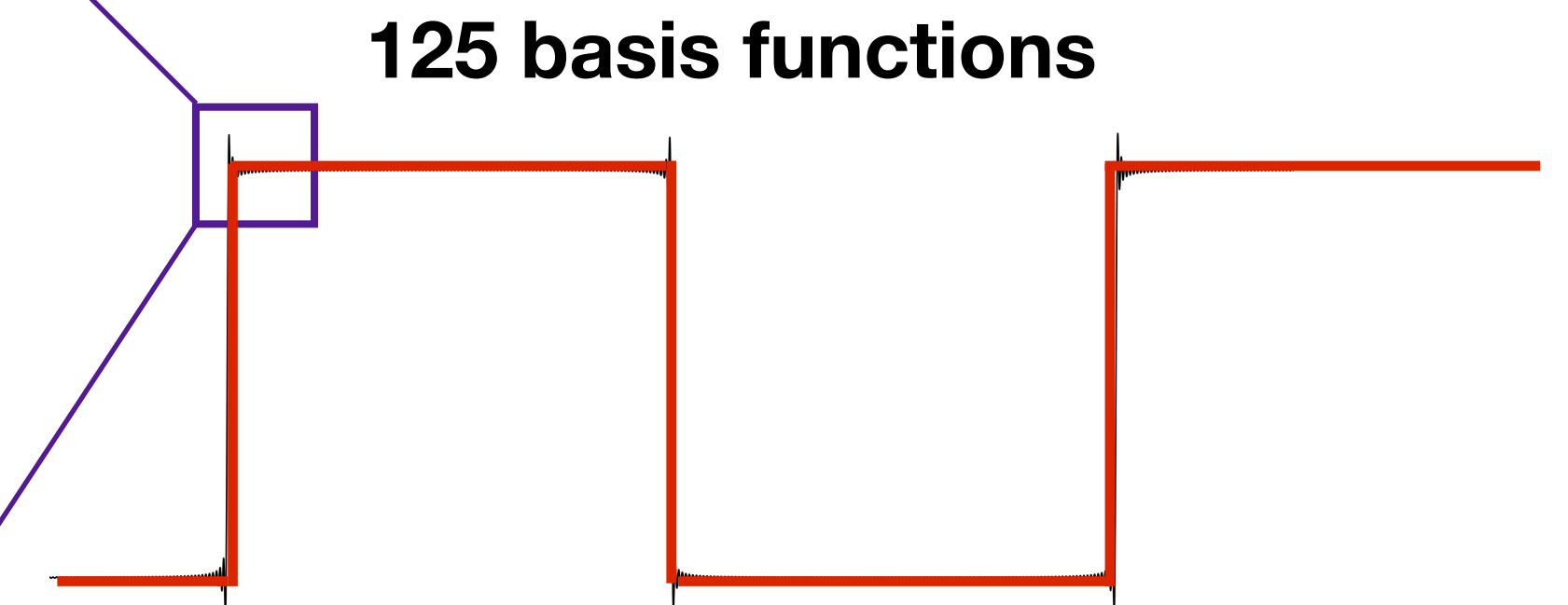
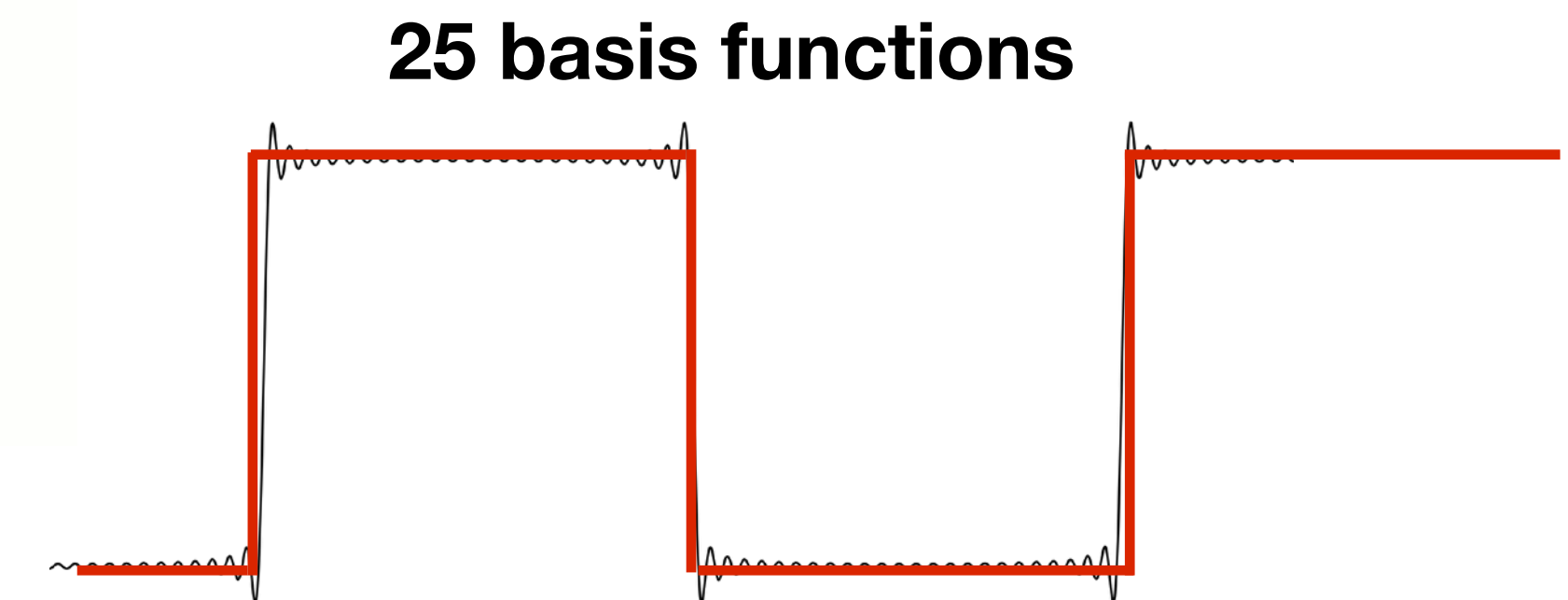
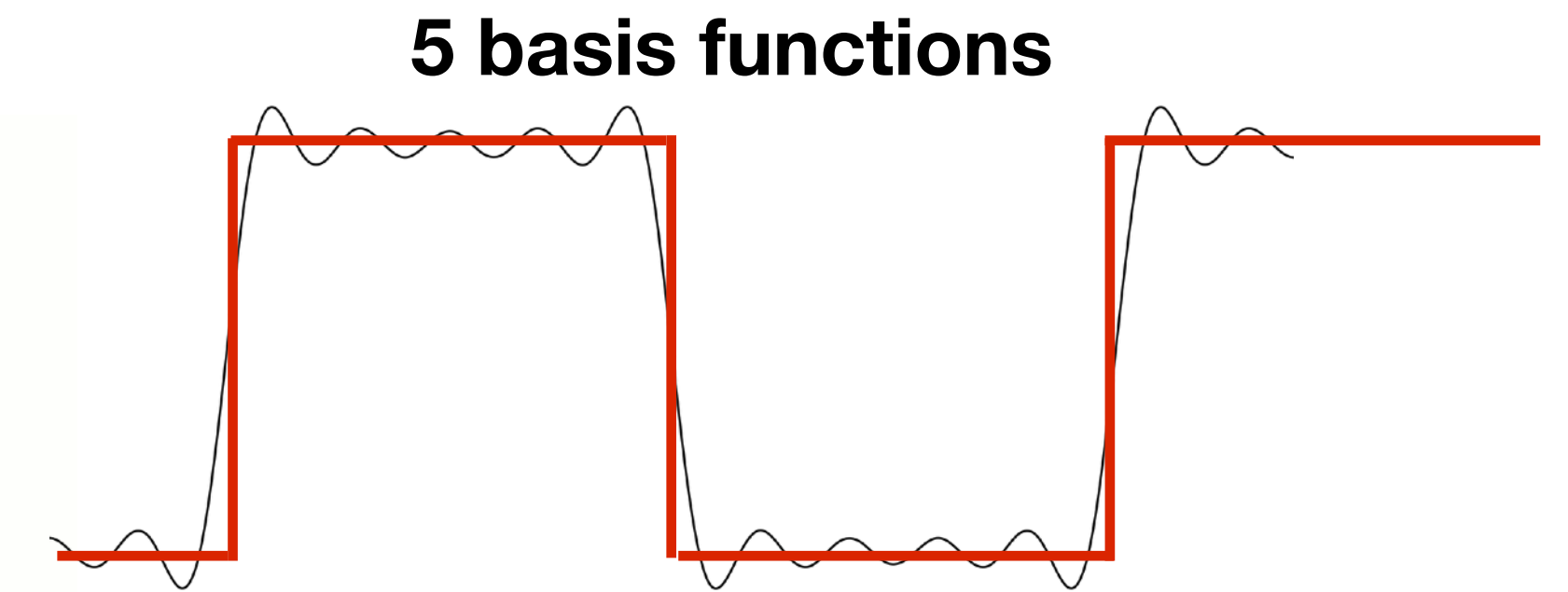
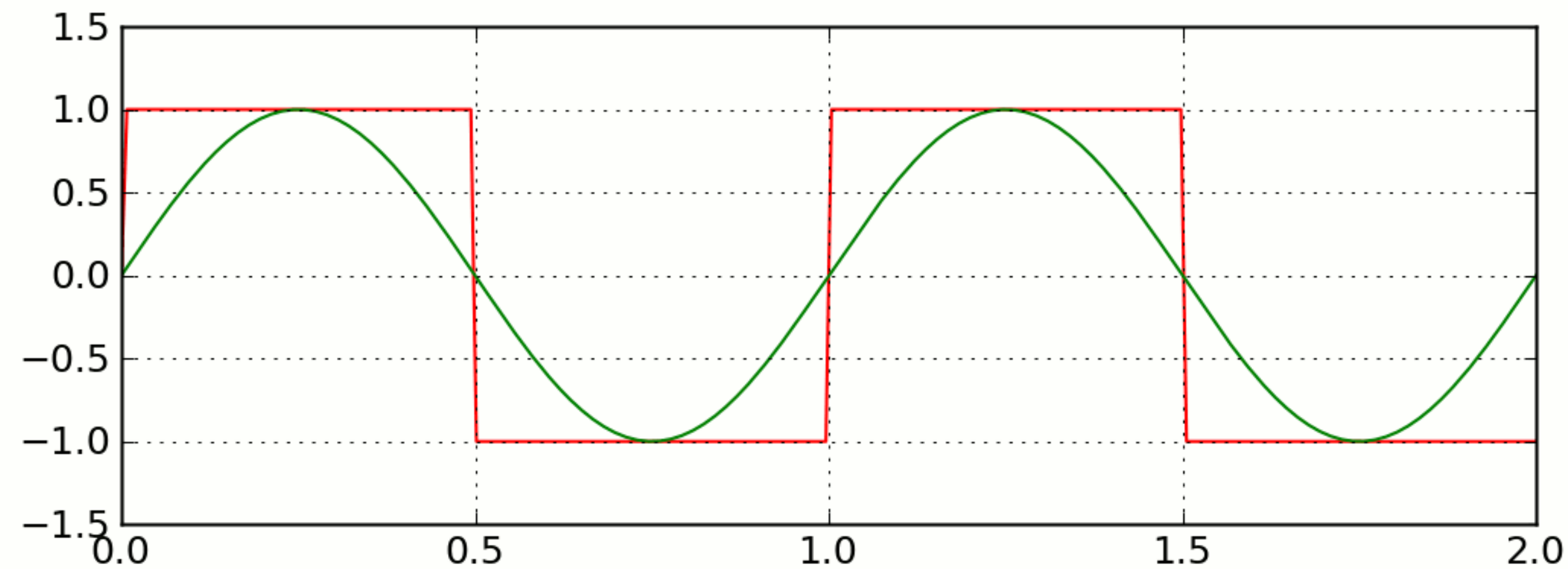
Gibbs Phenomena (Ringing)



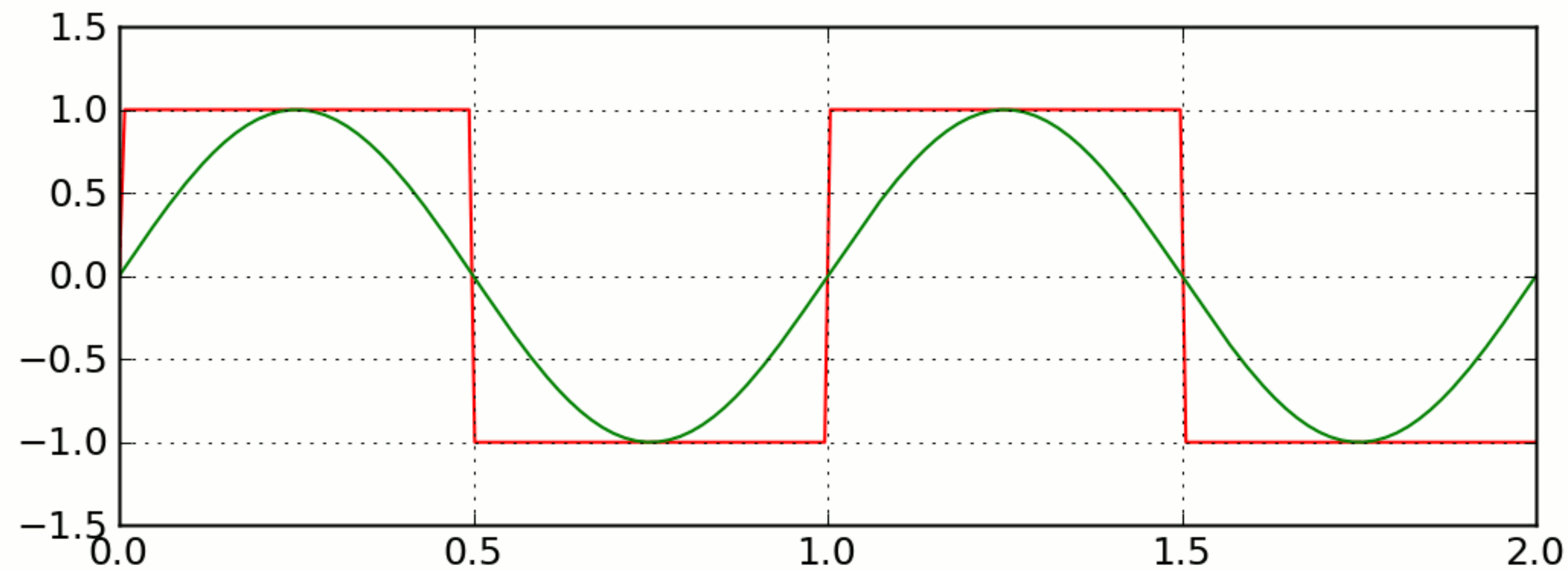
Gibbs Phenomena (Ringing)



Gibbs Phenomena (Ringing)

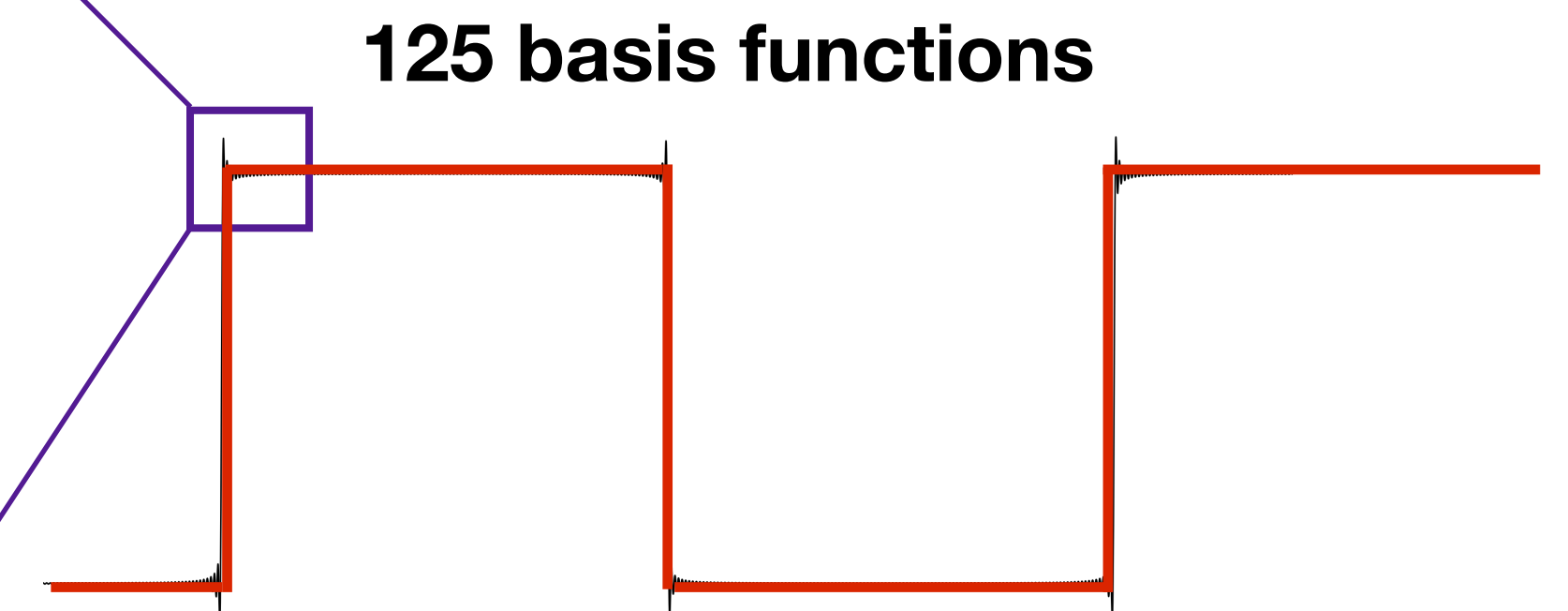
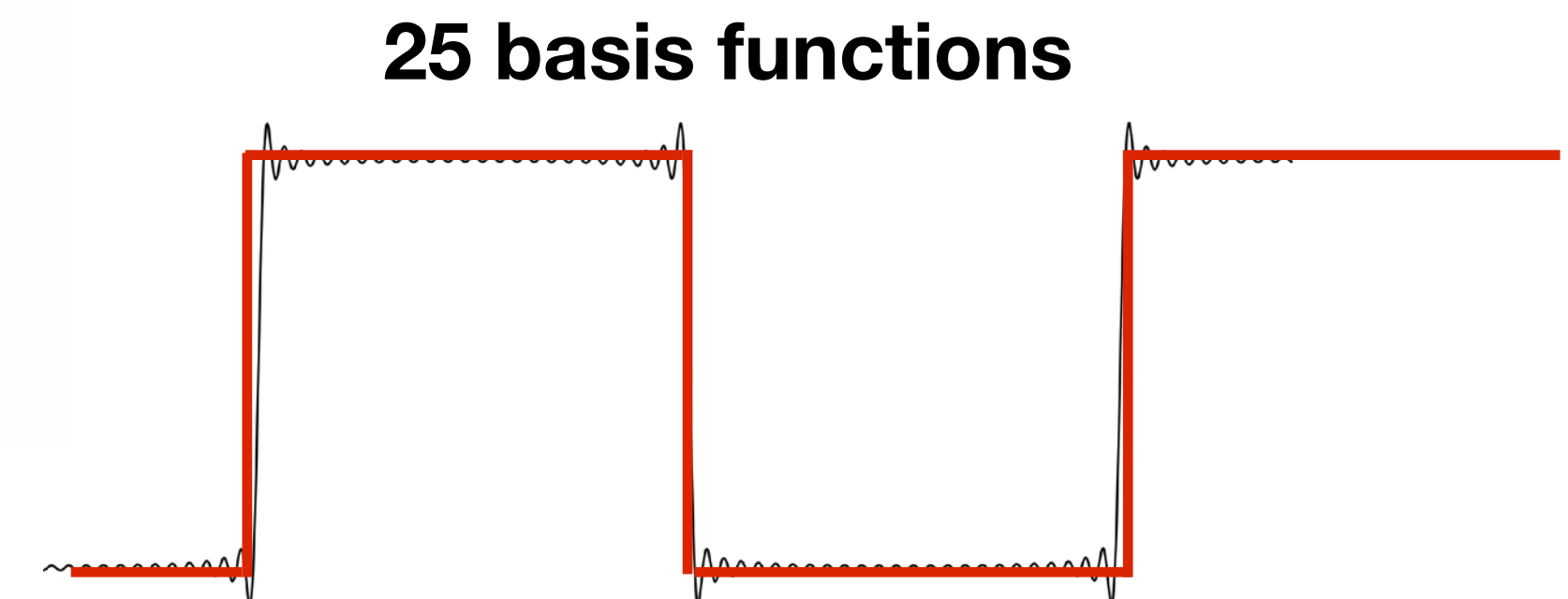
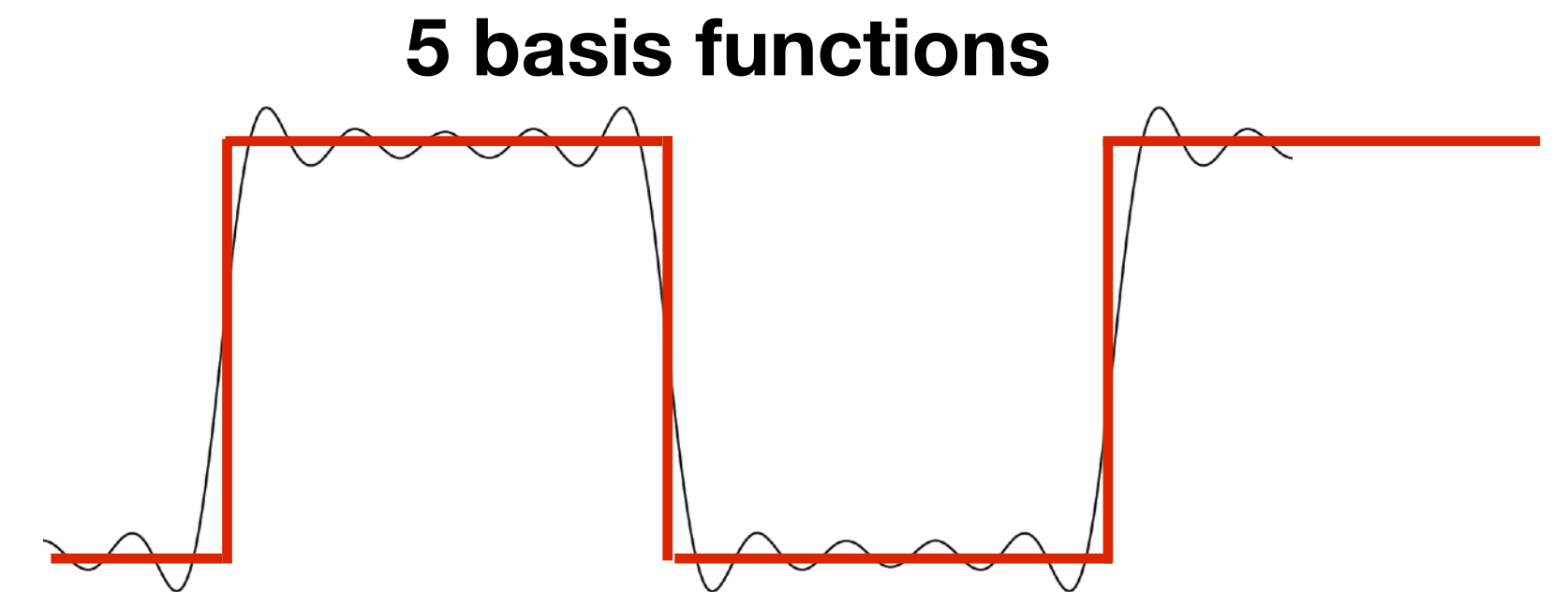
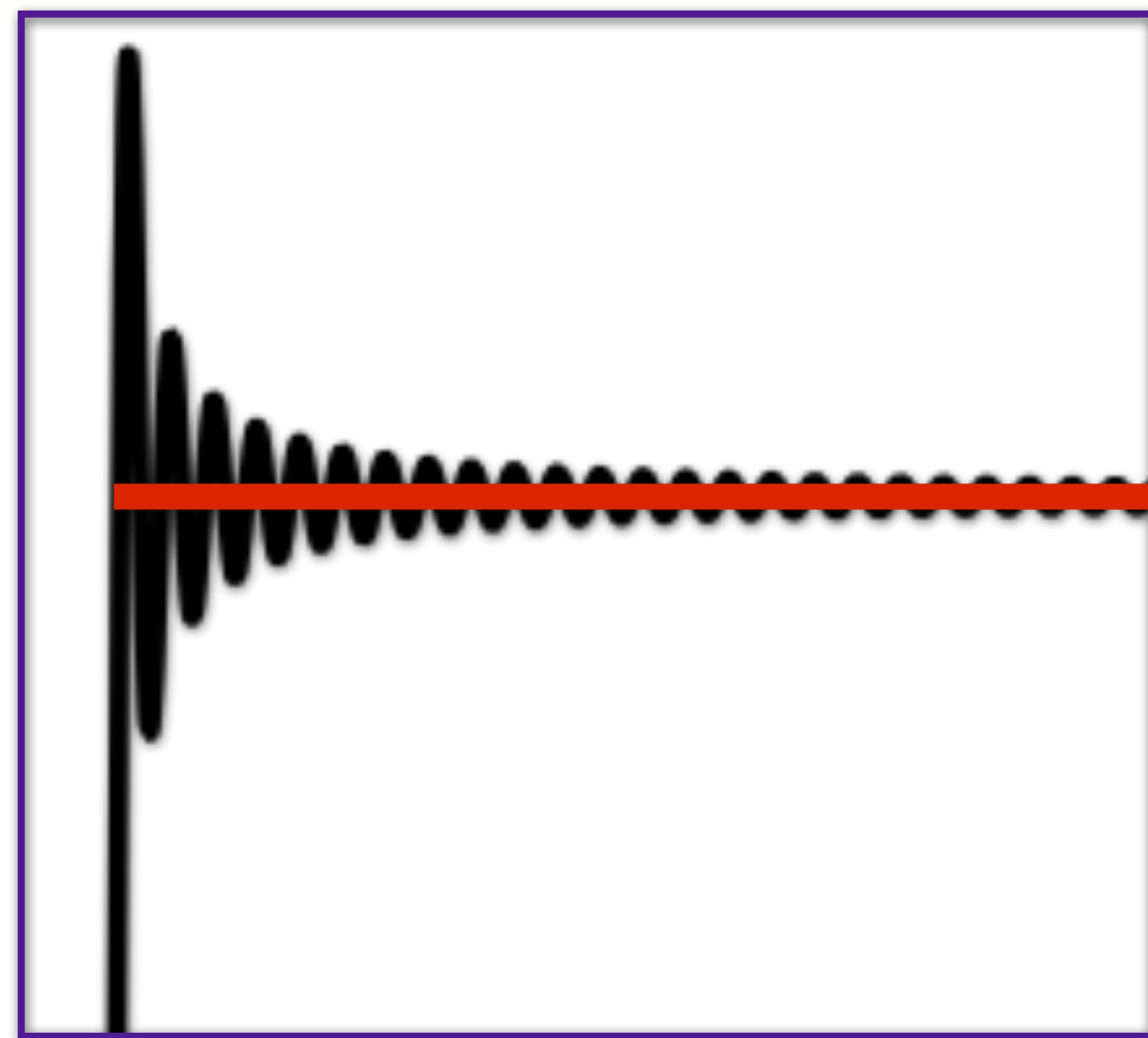


Gibbs Phenomena (Ringing)



Sudden change of value in a signal requires extremely high-frequency signals to represent.

We say the signal “contains **high-frequency** components.”



Complex Basis Functions

Complex Basis Functions

$$\left\{ e^{\frac{ik\pi x}{L}} \right\}_{k=-\infty}^{k=\infty}, \quad x \in [-L, L), \quad k \in \mathbb{Z}$$

$$\left\{ \dots, e^{\frac{-2i\pi x}{L}}, e^{\frac{-i\pi x}{L}}, e^{\frac{i0\pi x}{L}}, e^{\frac{i1\pi x}{L}}, \dots \right\}$$

Complex Basis Functions

$$\left\{ e^{\frac{ik\pi x}{L}} \right\}_{k=-\infty}^{k=\infty}, \quad x \in [-L, L), \quad k \in \mathbb{Z}$$

$$\left\{ \dots, e^{\frac{-2i\pi x}{L}}, e^{\frac{-i\pi x}{L}}, e^{\frac{i0\pi x}{L}}, e^{\frac{i1\pi x}{L}}, \dots \right\}$$

$$\phi_k = e^{\frac{ik\pi x}{L}}$$

Complex Basis Functions

$$\left\{ e^{\frac{ik\pi x}{L}} \right\}_{k=-\infty}^{k=\infty}, \quad x \in [-L, L), \quad k \in \mathbb{Z} \quad \left\{ \dots, e^{\frac{-2i\pi x}{L}}, e^{\frac{-i\pi x}{L}}, e^{\frac{i0\pi x}{L}}, e^{\frac{i1\pi x}{L}}, \dots \right\}$$

$$\phi_k = e^{\frac{ik\pi x}{L}} \quad \phi_i \cdot \phi_j = \begin{cases} 0, & \text{if } i \neq j, \\ 2L, & \text{if } i = j. \end{cases}$$

Mutually orthogonal vectors over $[-L, L)$ with a norm of $\sqrt{2L}$

Complex Fourier Series

$$\phi_k = e^{\frac{ik\pi x}{L}}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k$$

Complex Fourier Series

$$\phi_k = e^{\frac{ik\pi x}{L}}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k$$

$$c_k = \frac{f(x) \cdot \phi_k}{\|\phi_k\|^2} = \frac{1}{2L} \int_{-L}^L f(x) \bar{\phi}_k dx = \frac{1}{2L} \int_{-L}^L f(x) e^{\frac{-ik\pi x}{L}} dx$$

Complex Fourier Series

- Can prove equivalence using Euler's formula: $e^{ikx} = \cos(kx) + i \sin(kx)$

$$\phi_k = e^{\frac{ik\pi x}{L}}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right] \quad x \in [-L, L)$$

$$c_k = \frac{f(x) \cdot \phi_k}{\|\phi_k\|^2} = \frac{1}{2L} \int_{-L}^L f(x) \bar{\phi}_k dx = \frac{1}{2L} \int_{-L}^L f(x) e^{-\frac{ik\pi x}{L}} dx$$

Complex Fourier Series

- Can prove equivalence using Euler's formula: $e^{ikx} = \cos(kx) + i \sin(kx)$
- More general and concise

$$\phi_k = e^{\frac{ik\pi x}{L}}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right] \quad x \in [-L, L)$$

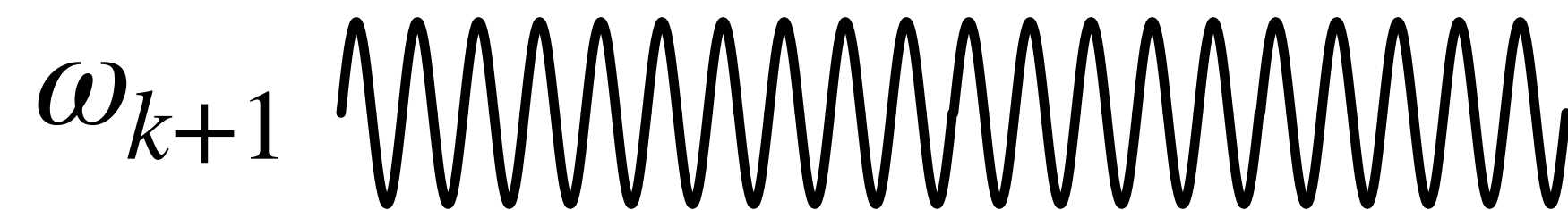
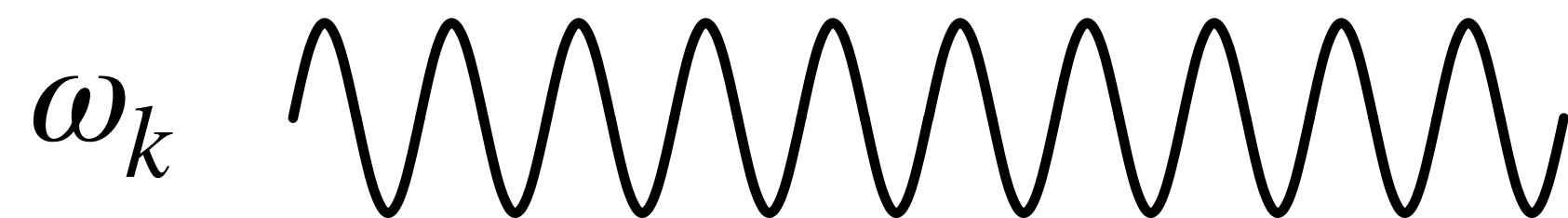
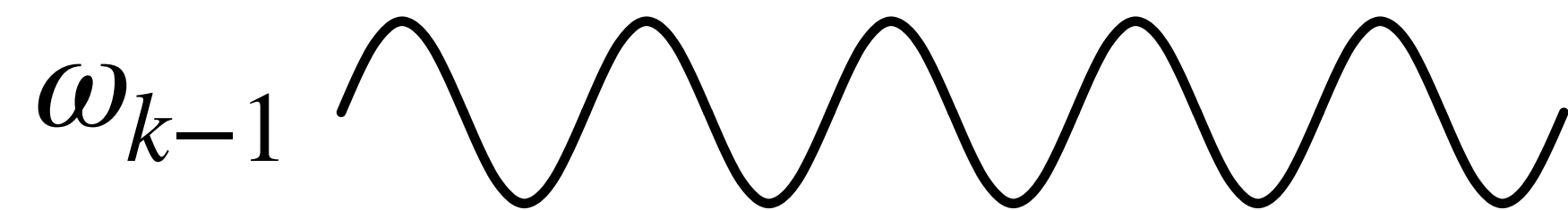
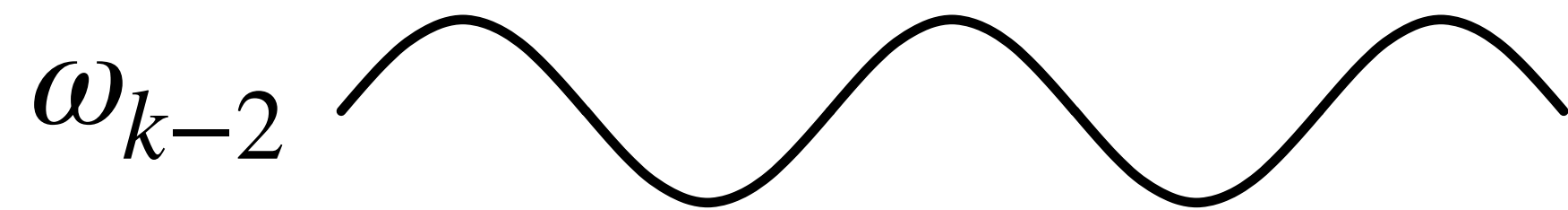
$$c_k = \frac{f(x) \cdot \phi_k}{\|\phi_k\|^2} = \frac{1}{2L} \int_{-L}^L f(x) \bar{\phi}_k dx = \frac{1}{2L} \int_{-L}^L f(x) e^{\frac{-ik\pi x}{L}} dx$$

Basis Functions of Complex Fourier Series

$$\omega_k = \frac{k\pi}{L} \quad \left\{ e^{\frac{ik\pi x}{L}} \right\}_{k=-\infty}^{k=\infty} \Rightarrow \left\{ e^{i\omega_k x} \right\}_{k=-\infty}^{k=\infty}$$

Frequency

...



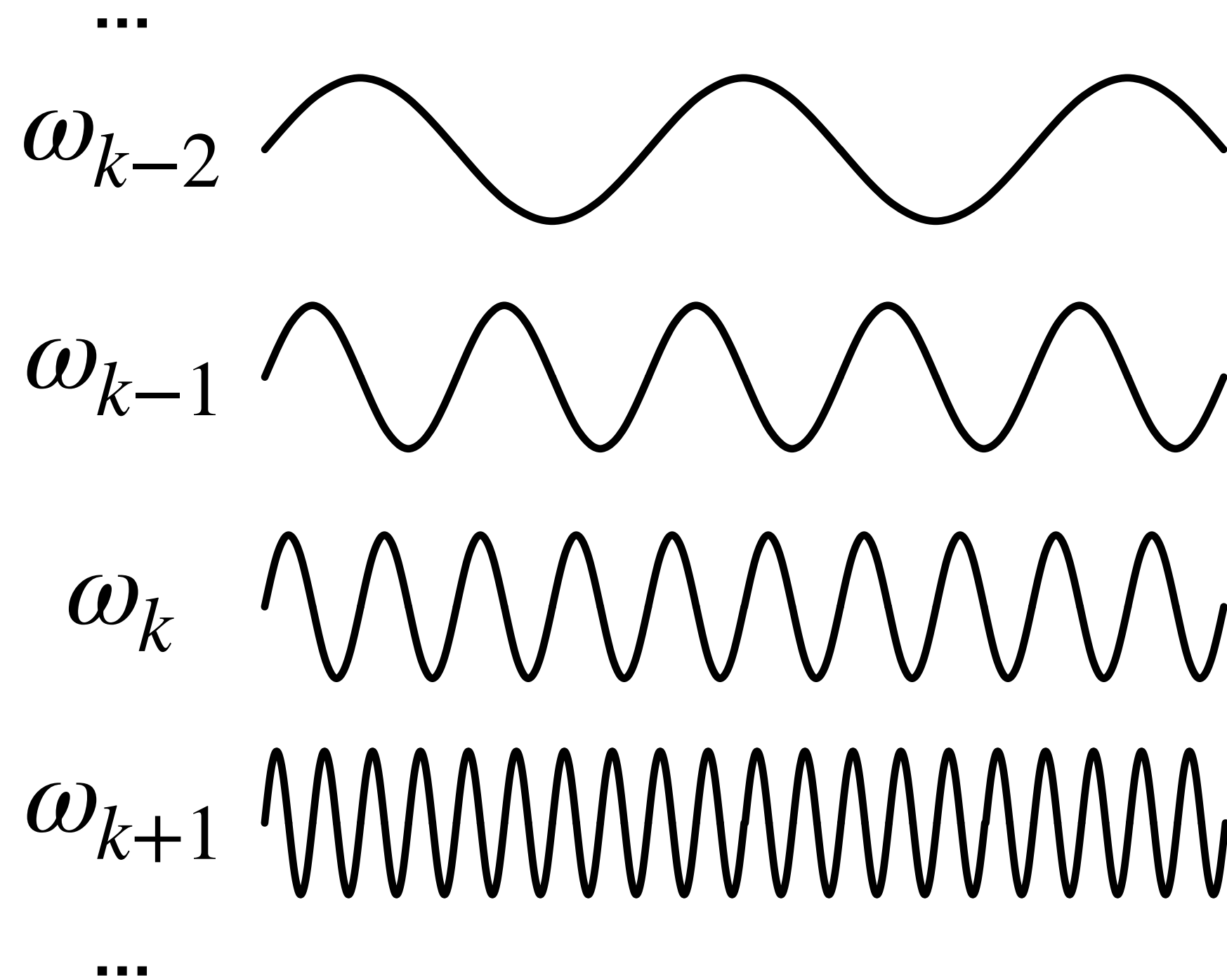
...

Basis Functions of Complex Fourier Series

$$\omega_k = \frac{k\pi}{L} \quad \left\{ e^{\frac{ik\pi x}{L}} \right\}_{k=-\infty}^{k=\infty} \Rightarrow \left\{ e^{i\omega_k x} \right\}_{k=-\infty}^{k=\infty}$$

As **k** increases, **ω_k** increase, i.e., the **frequency** of the basis functions increase, at a uniform interval of **π/L** .

Frequency



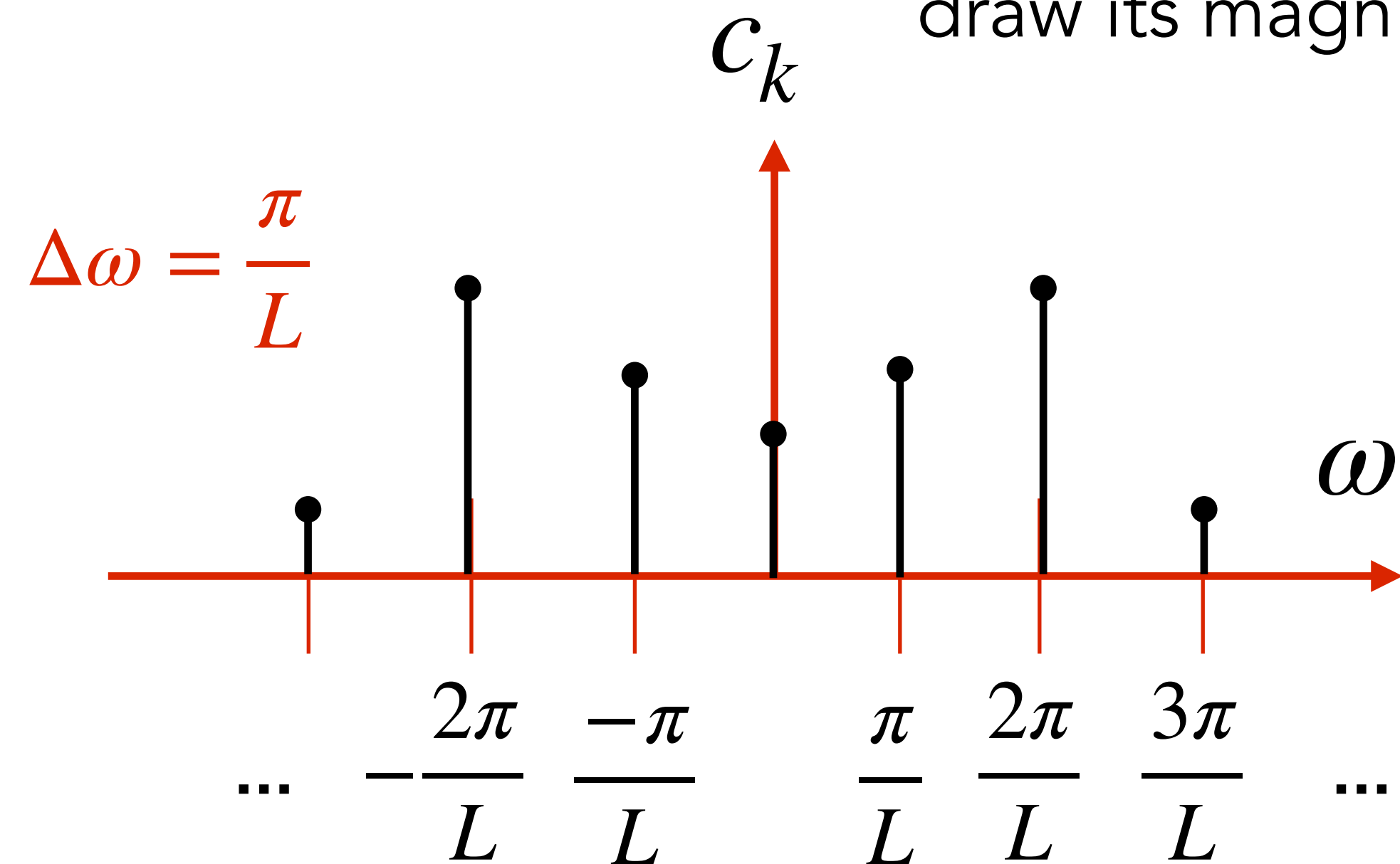
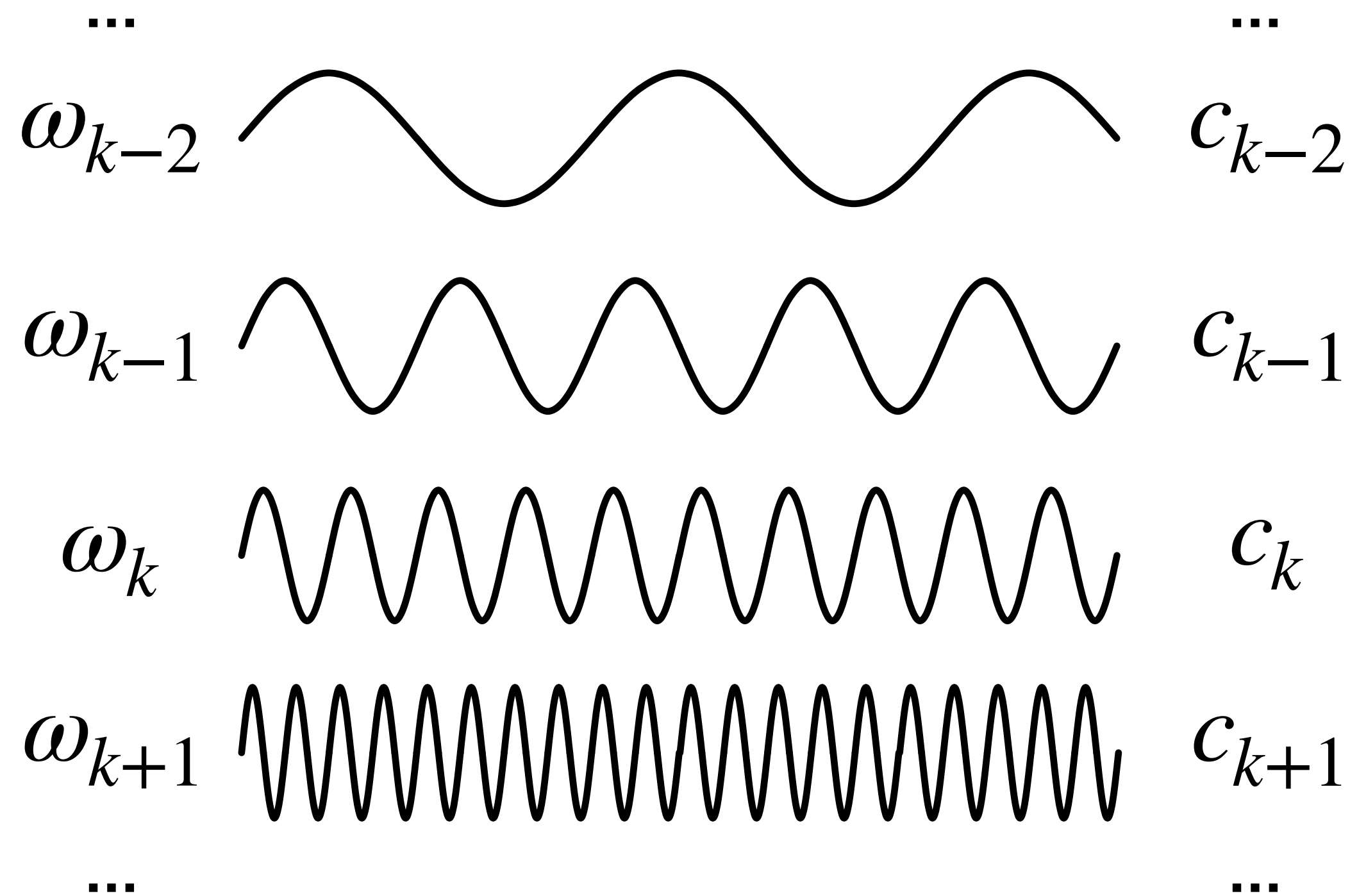
Coefficients of Complex Fourier Series

$$\omega_k = \frac{k\pi}{L} \quad f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x} \quad c_k = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\omega_k x} dx$$

\mathbf{C}_k is complex-valued. Usually we draw its magnitude.

Frequency

Coeff.



Geometric Interpretation

$$\phi_k = e^{\frac{ik\pi x}{L}}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k$$

Geometric Interpretation

$$\phi_k = e^{\frac{ik\pi x}{L}}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k$$

$$\begin{bmatrix} | & | & | & | & | \\ \cdots & \phi_{k-1} & \phi_k & \phi_{k+1} & \cdots \\ | & | & | & | & | \end{bmatrix} \times \begin{bmatrix} | \\ c_{k-1} \\ c_k \\ c_{k+1} \\ | \end{bmatrix} = \begin{bmatrix} | \\ f \\ | \end{bmatrix}$$

Geometric Interpretation

- $\{c_k\}$ is a Hilbert-space vector transformed from f , another Hilbert-space vector.

$$\phi_k = e^{\frac{ik\pi x}{L}}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k$$

$$\begin{bmatrix} | & | & | & | & | \\ \cdots & \phi_{k-1} & \phi_k & \phi_{k+1} & \cdots \\ | & | & | & | & | \end{bmatrix} \times \begin{bmatrix} | \\ c_{k-1} \\ c_k \\ c_{k+1} \\ | \end{bmatrix} = \begin{bmatrix} | \\ f \\ | \end{bmatrix}$$

Geometric Interpretation

- $\{c_k\}$ is a Hilbert-space vector transformed from f , another Hilbert-space vector.
- What kind of transform is this? Homework!

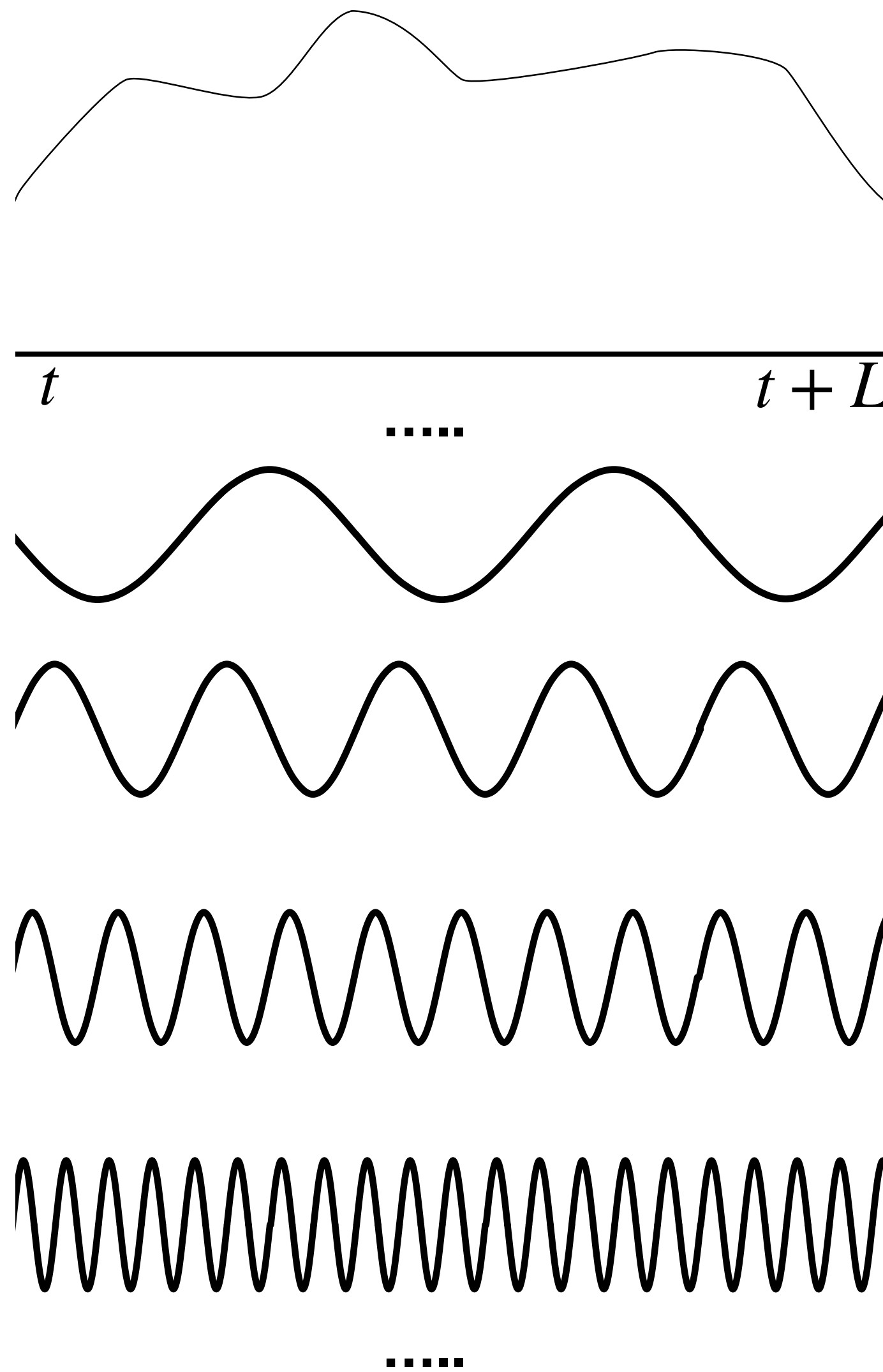
$$\phi_k = e^{\frac{ik\pi x}{L}}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k$$

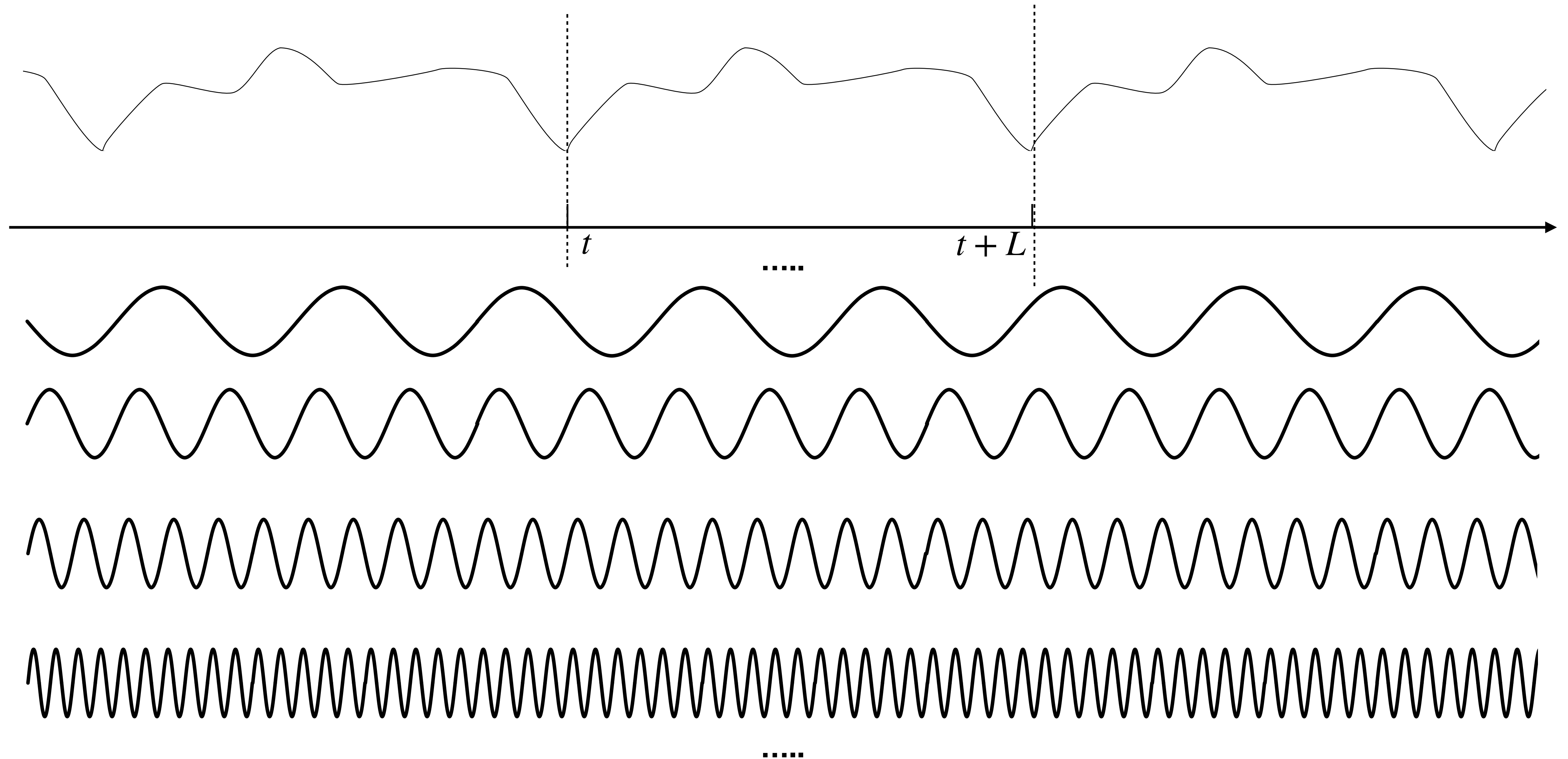
$$\begin{bmatrix} | & | & | & | & | \\ \cdots & \phi_{k-1} & \phi_k & \phi_{k+1} & \cdots \\ | & | & | & | & | \end{bmatrix} \times \begin{bmatrix} | \\ c_{k-1} \\ c_k \\ c_{k+1} \\ | \end{bmatrix} = \begin{bmatrix} | \\ f \\ | \end{bmatrix}$$

Discrete Fourier Transform (DFT)

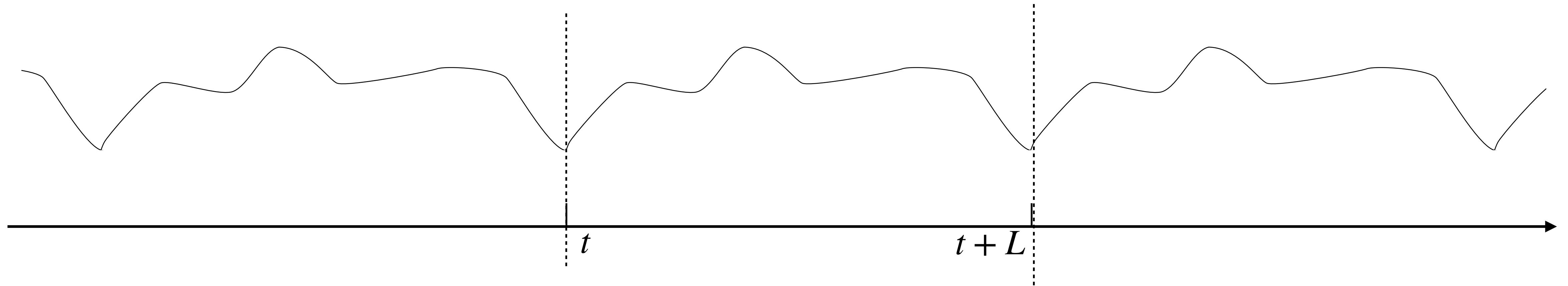
Recall: Fourier Series



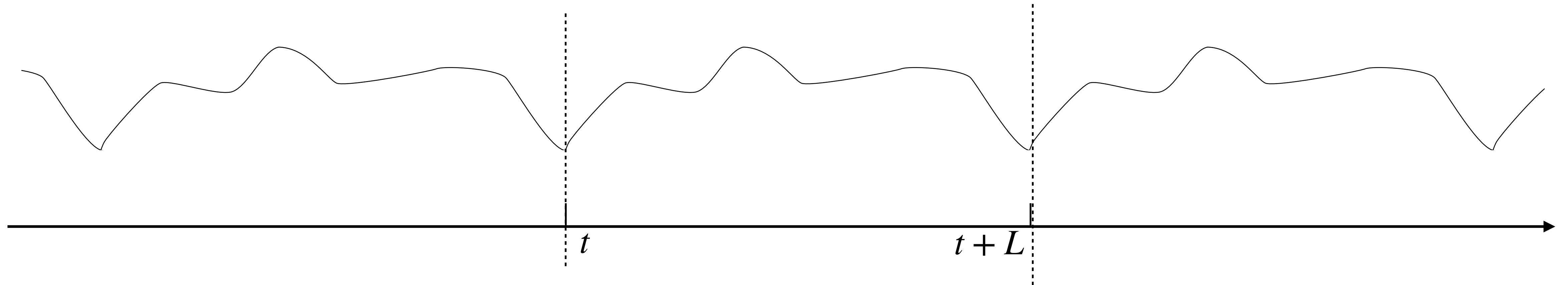
Recall: Fourier Series



Recall: Fourier Series

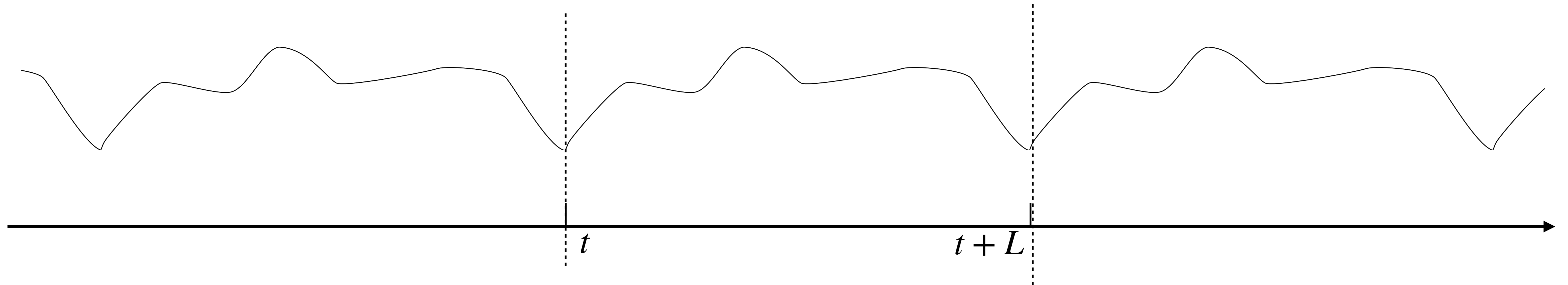


Recall: Fourier Series



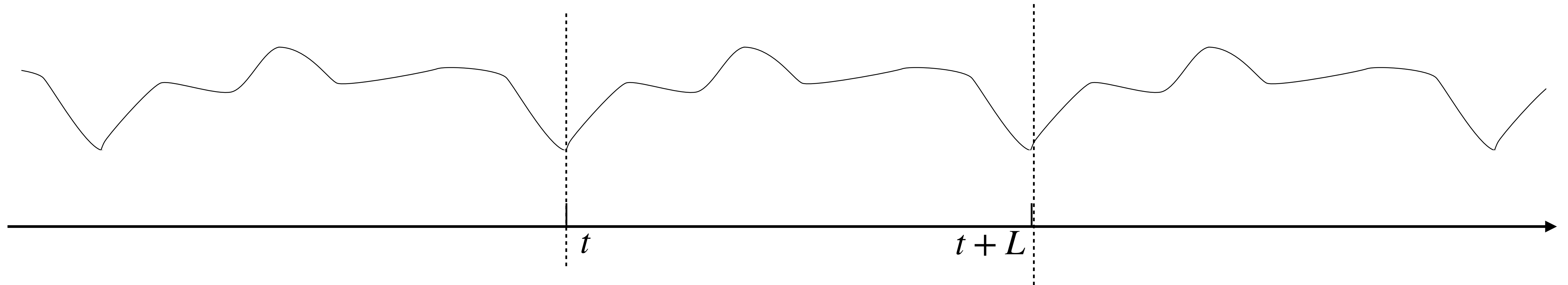
- Fourier series applies to:

Recall: Fourier Series



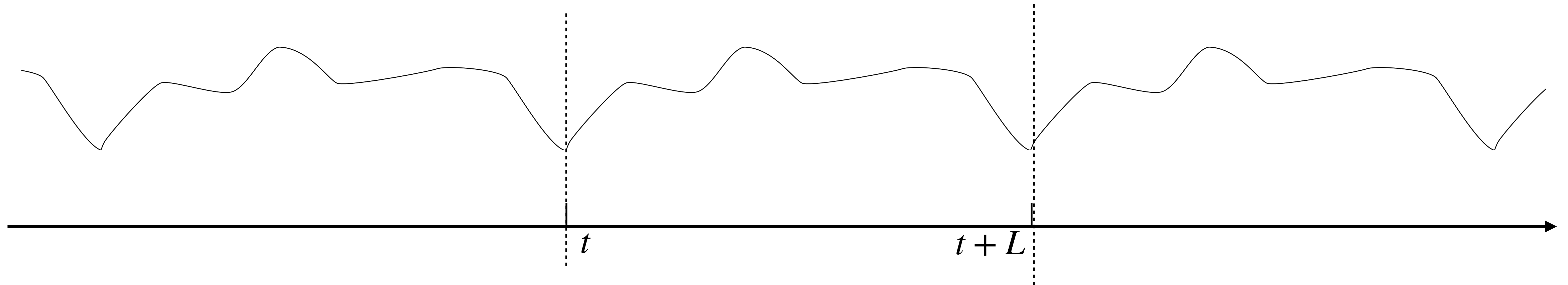
- Fourier series applies to:
 - Finite interval (say L) + Continuous function

Recall: Fourier Series



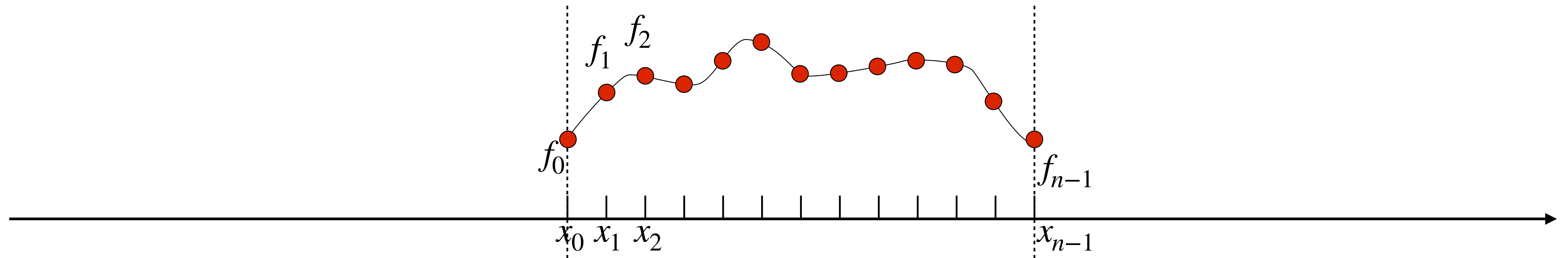
- Fourier series applies to:
 - Finite interval (say L) + Continuous function
 - Or equivalently: an L -periodic function with an infinite span

Recall: Fourier Series



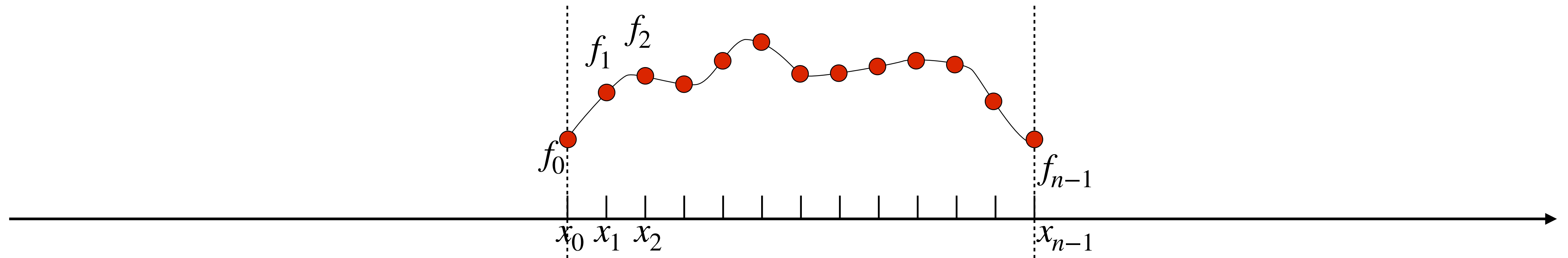
- Fourier series applies to:
 - Finite interval (say L) + Continuous function
 - Or equivalently: an L -periodic function with an infinite span
- We need infinitely many basis functions, since a continuous function is an infinitely-dimensional vector

Real-World Scenarios Impose Restrictions



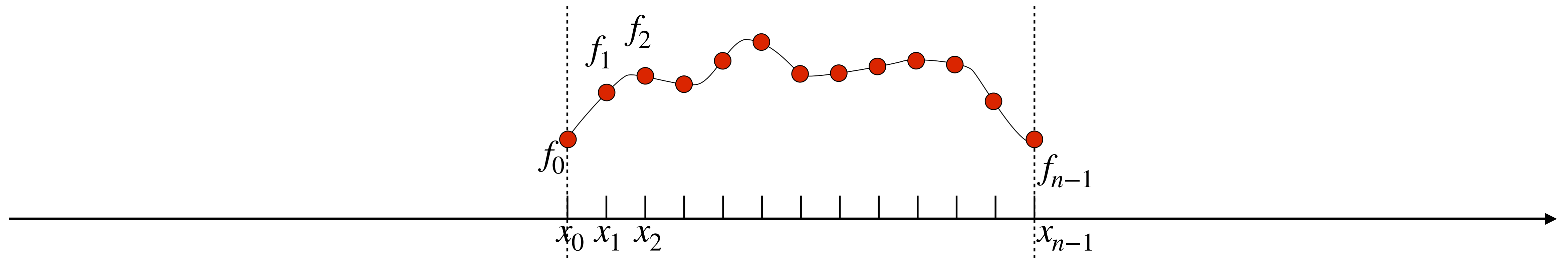
- In engineering/real-world applications, the signals we obtain are discrete samples over a finite interval

Real-World Scenarios Impose Restrictions



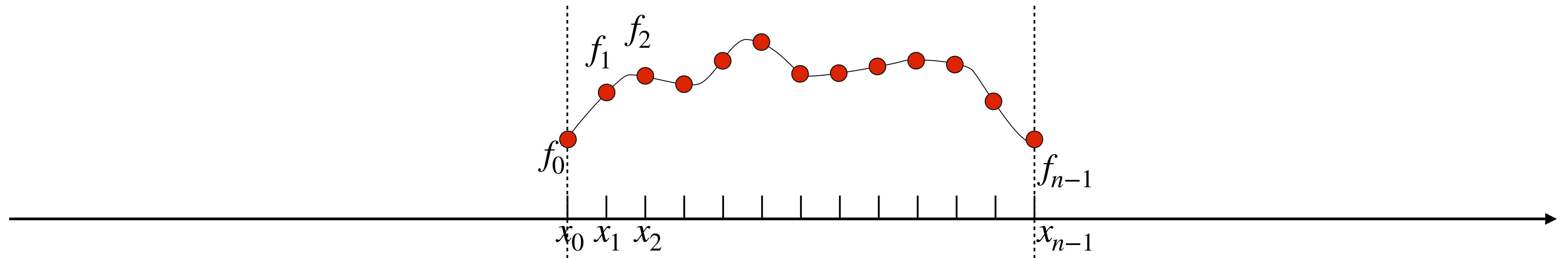
- In engineering/real-world applications, the signals we obtain are discrete samples over a finite interval
 - For instance, audio files are usually 44.1 kHz, i.e., 44,100 samples per second

Real-World Scenarios Impose Restrictions



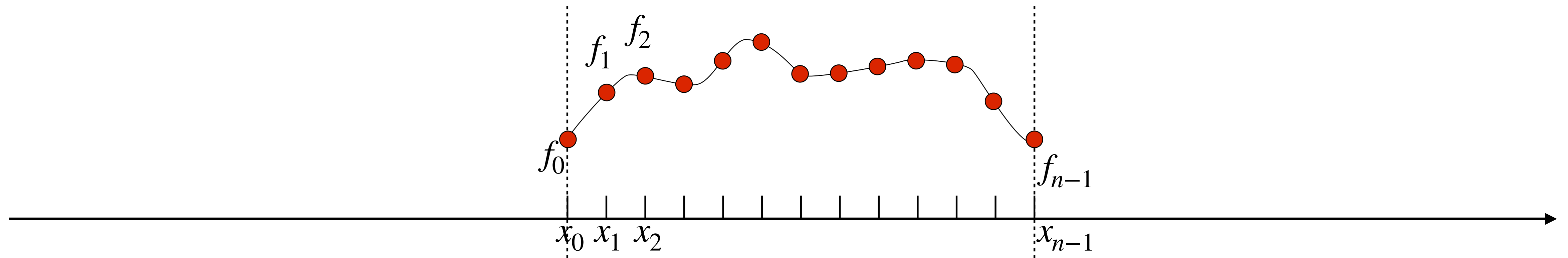
- In engineering/real-world applications, the signals we obtain are discrete samples over a finite interval
 - For instance, audio files are usually 44.1 kHz, i.e., 44,100 samples per second
- We need a discrete form of Fourier series

Real-World Scenarios Impose Restrictions



- In engineering/real-world applications, the signals we obtain are discrete samples over a finite interval
 - For instance, audio files are usually 44.1 kHz, i.e., 44,100 samples per second
- We need a discrete form of Fourier series
- Given a sequence of n points, express it as a sum of other sequences

Real-World Scenarios Impose Restrictions



- In engineering/real-world applications, the signals we obtain are discrete samples over a finite interval
 - For instance, audio files are usually 44.1 kHz, i.e., 44,100 samples per second
- We need a discrete form of Fourier series
- Given a sequence of n points, express it as a sum of other sequences
 - This time, we need n other sequences

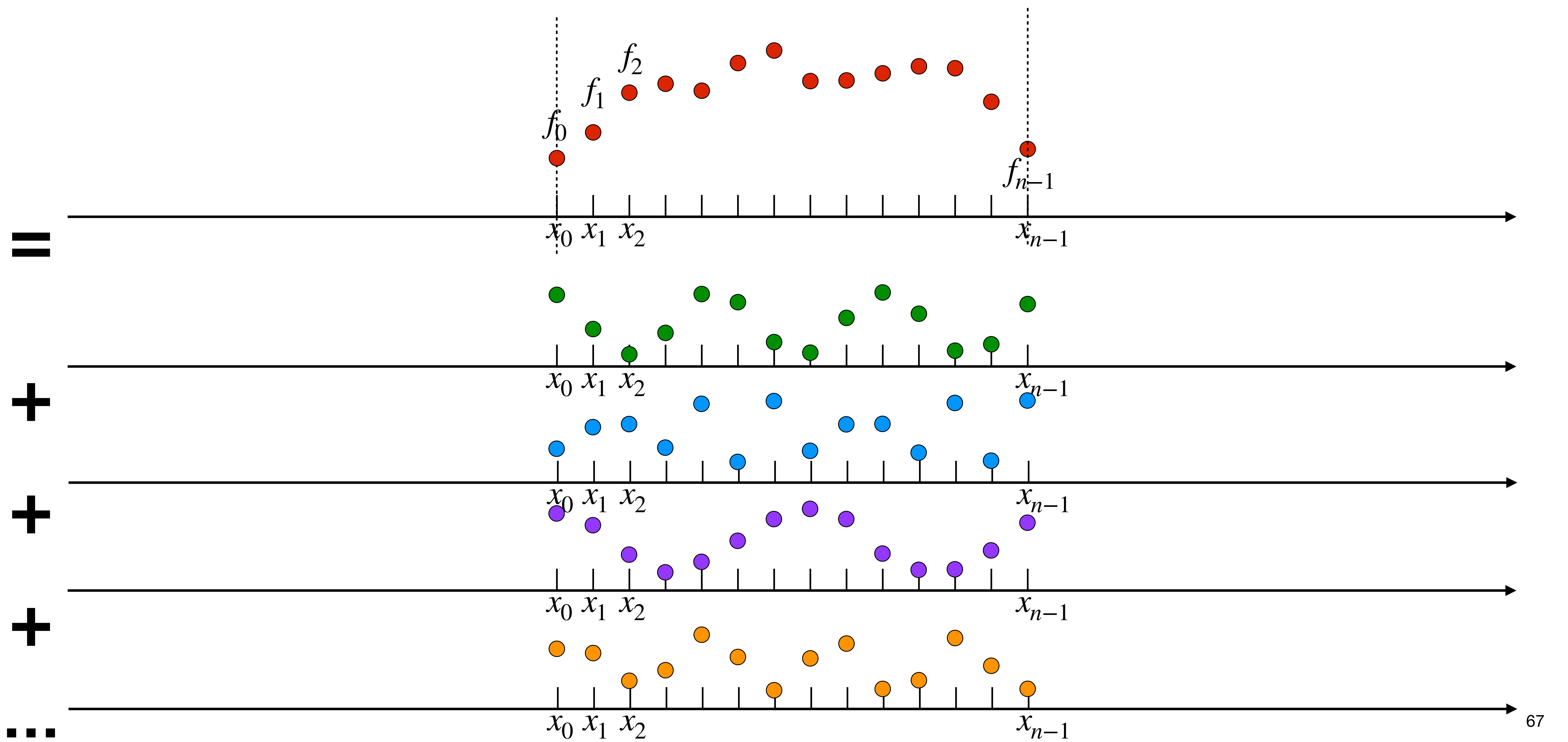
Discrete Complex Basis Functions

$\left\{ e^{\frac{i2\pi jk}{n}} \right\}, \quad j \in [0, 1, \dots, n-1]$ are n mutually orthogonal n -dimensional vectors

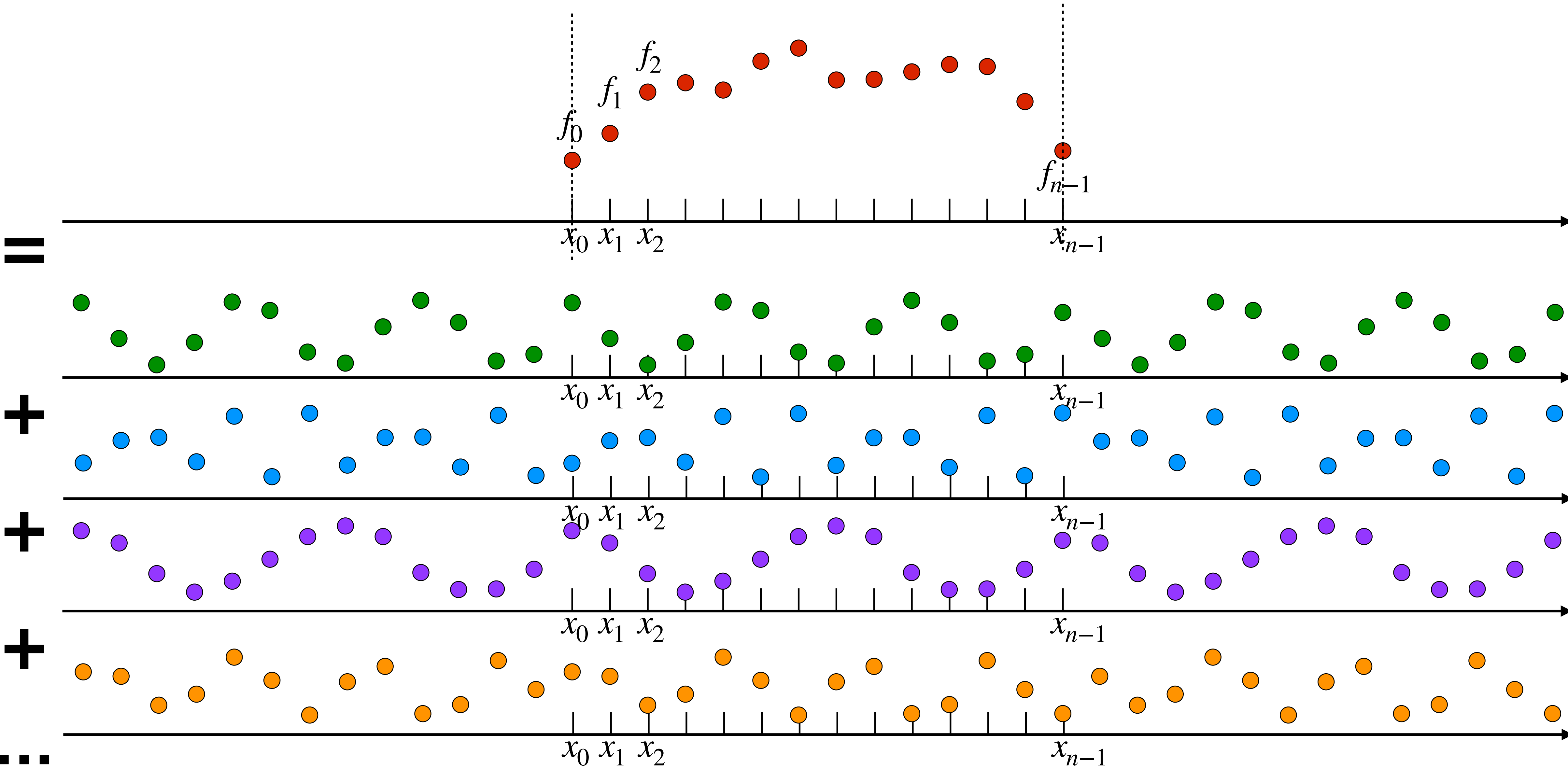
$$\left\{ \dots, e^{\frac{i0k}{n}}, e^{\frac{i2\pi k}{n}}, e^{\frac{i4\pi k}{n}}, e^{\frac{i6\pi k}{n}}, \dots \right\} \quad k \in [0, 1, \dots, n-1]$$

For instance:
$$\sum_{k=0}^{n-1} e^{\frac{i2\pi k}{n}} e^{\frac{i4\pi k}{n}} = 0$$

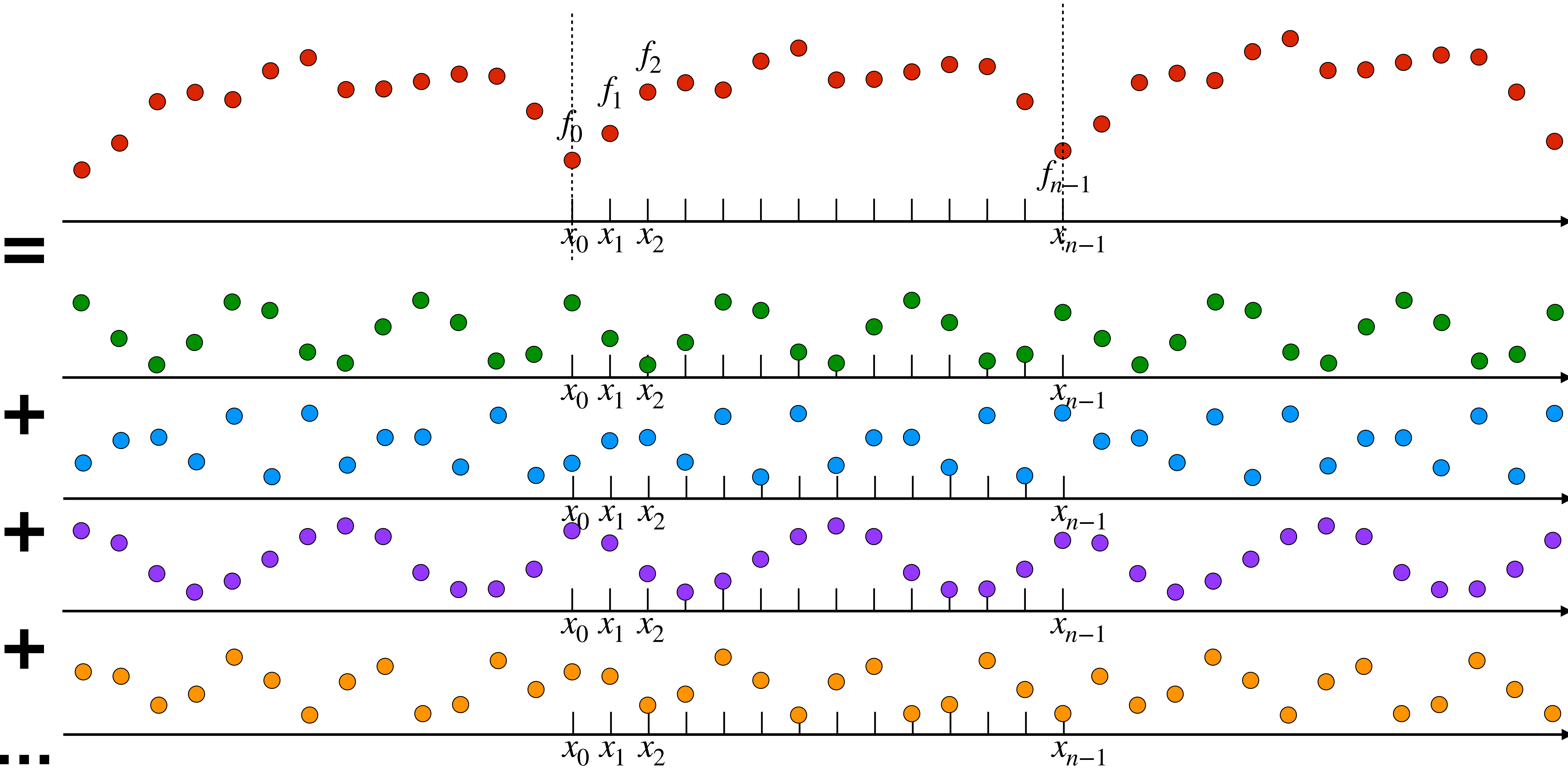
Intuition



Intuition



Intuition



Discrete Fourier Transform (Series)

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_k e^{\frac{i2\pi jk}{n}} \quad \hat{f}_k = \sum_{j=0}^{n-1} f_j e^{\frac{-i2\pi jk}{n}} \quad k \in [0, 1, \dots, n-1]$$

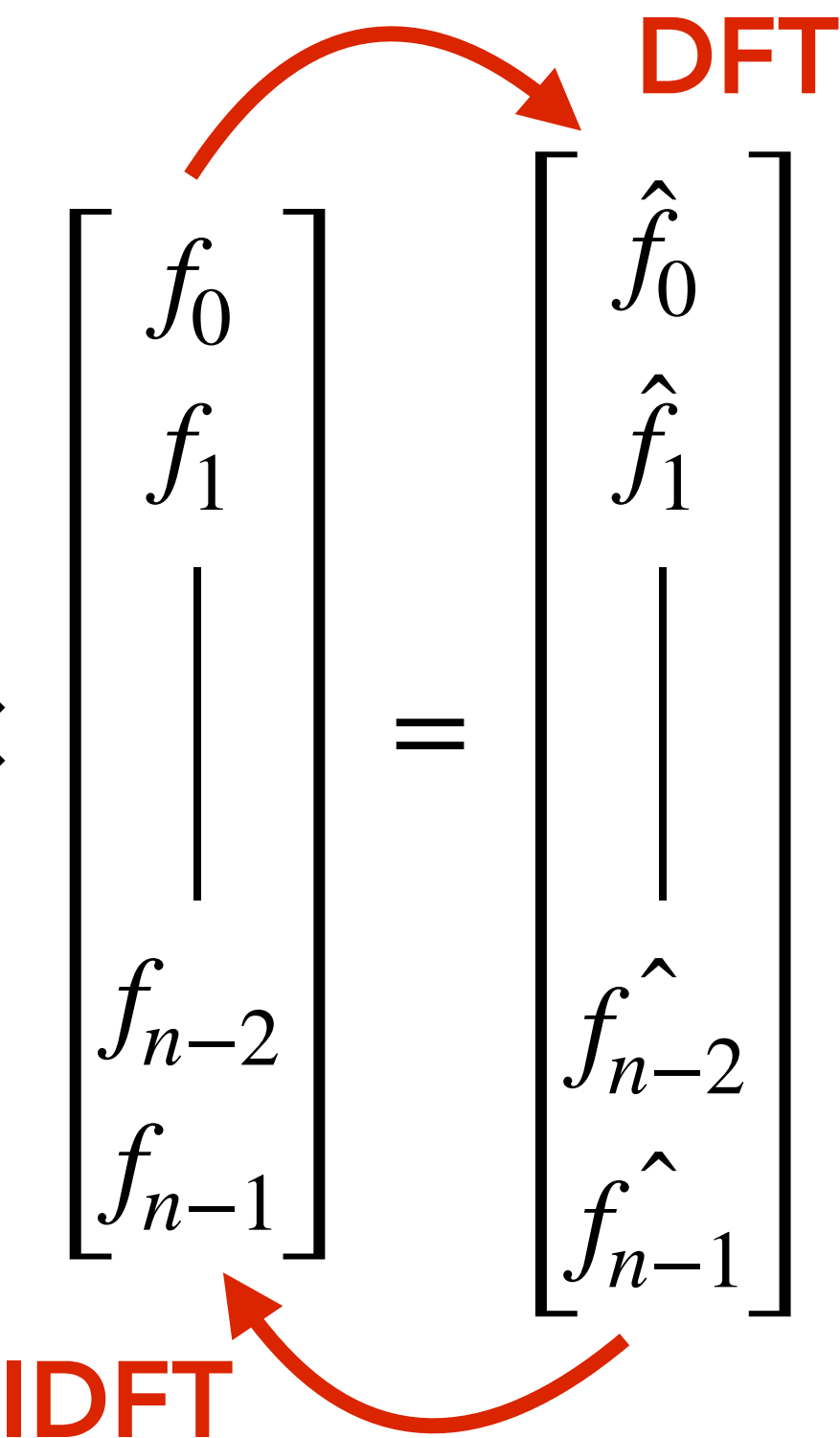
- This is called Discrete Fourier Transform
 - But it's a misnomer; should really be called Discrete Fourier Series
- Fast Fourier Transform (FFT) is a fast way to calculate DFT
- Project to the basis vectors and calculate the DFT coefficients
- Note that these basis functions are not a subset of the those used in the continuous case!

DFT Matrix

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_k \omega_n^{-jk} \quad \hat{f}_k = \sum_{j=0}^{n-1} f_j \omega_n^{jk} \quad \omega_n = e^{\frac{-i2\pi}{n}}$$

The DFT Matrix

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)^2} \end{bmatrix} \times \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{n-2} \\ \hat{f}_{n-1} \end{bmatrix}$$



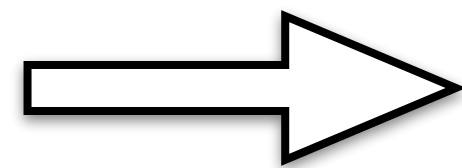
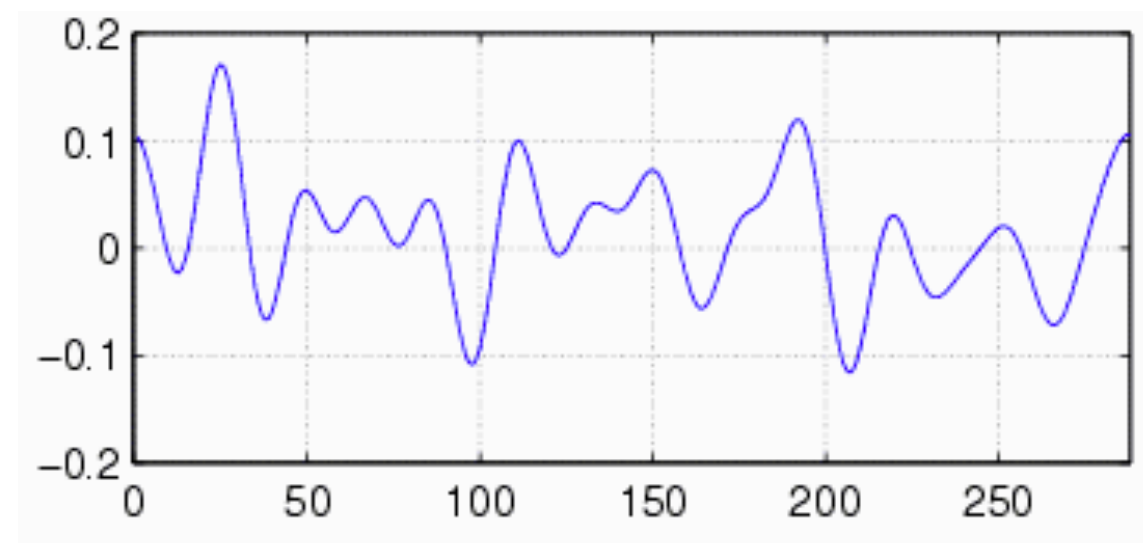
DFT Interpretation

- f now is a N-dimensional vector; it can be represented by projecting to N mutually orthogonal vectors.
- DFT transforms a N-dimensional vector to another N-D vector.

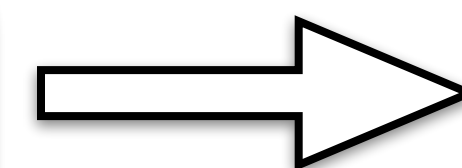
$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)^2} \end{bmatrix} \times \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{n-2} \\ \hat{f}_{n-1} \end{bmatrix}$$

Discrete Fourier Transform

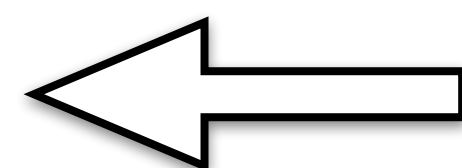
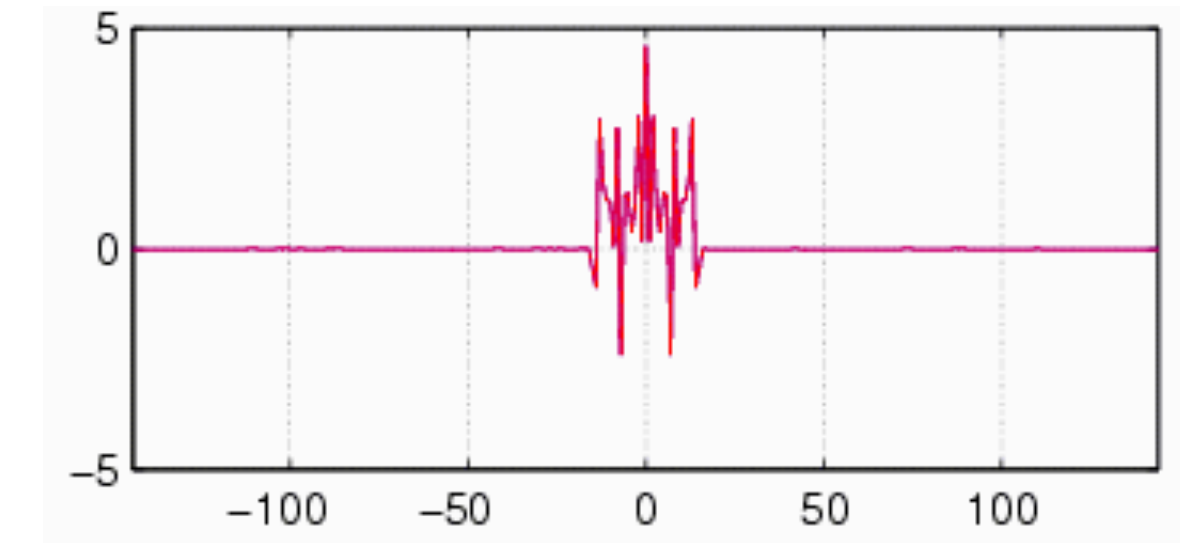
f_k



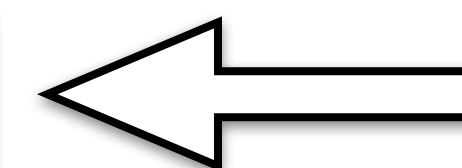
DFT



\hat{f}_k



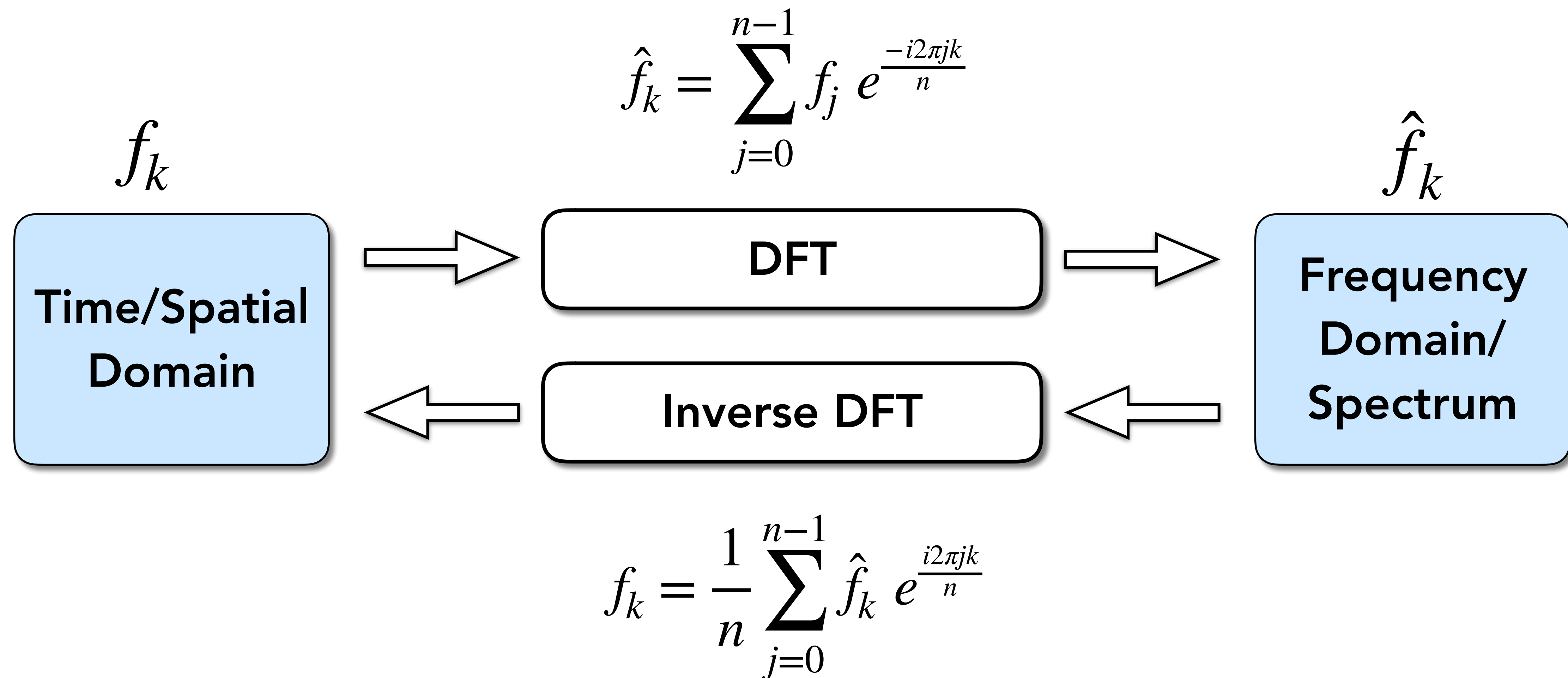
Inverse DFT



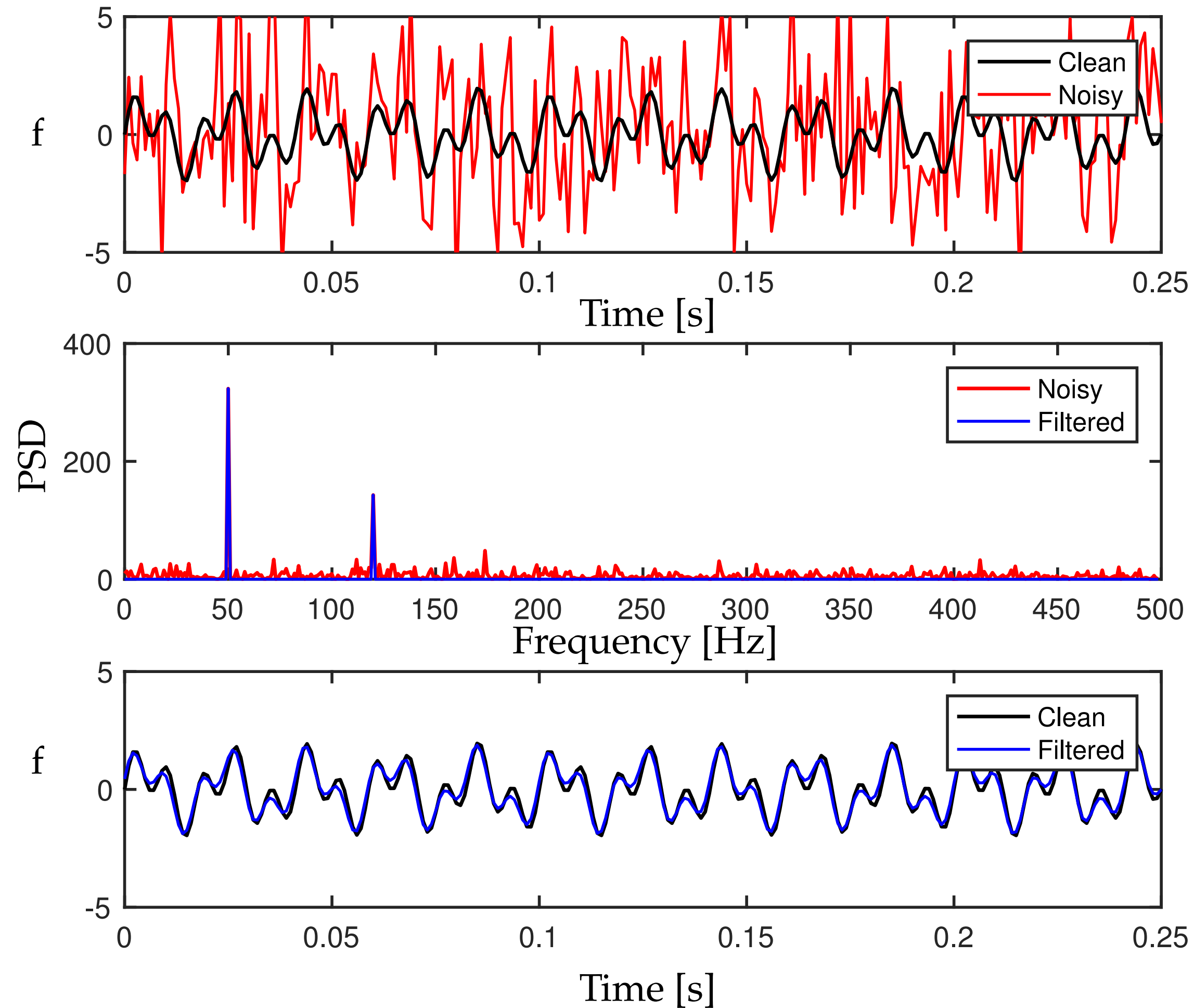
$$\hat{f}_k = \sum_{j=0}^{n-1} f_j e^{\frac{-i2\pi jk}{n}}$$

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_k e^{\frac{i2\pi jk}{n}}$$

Discrete Fourier Transform



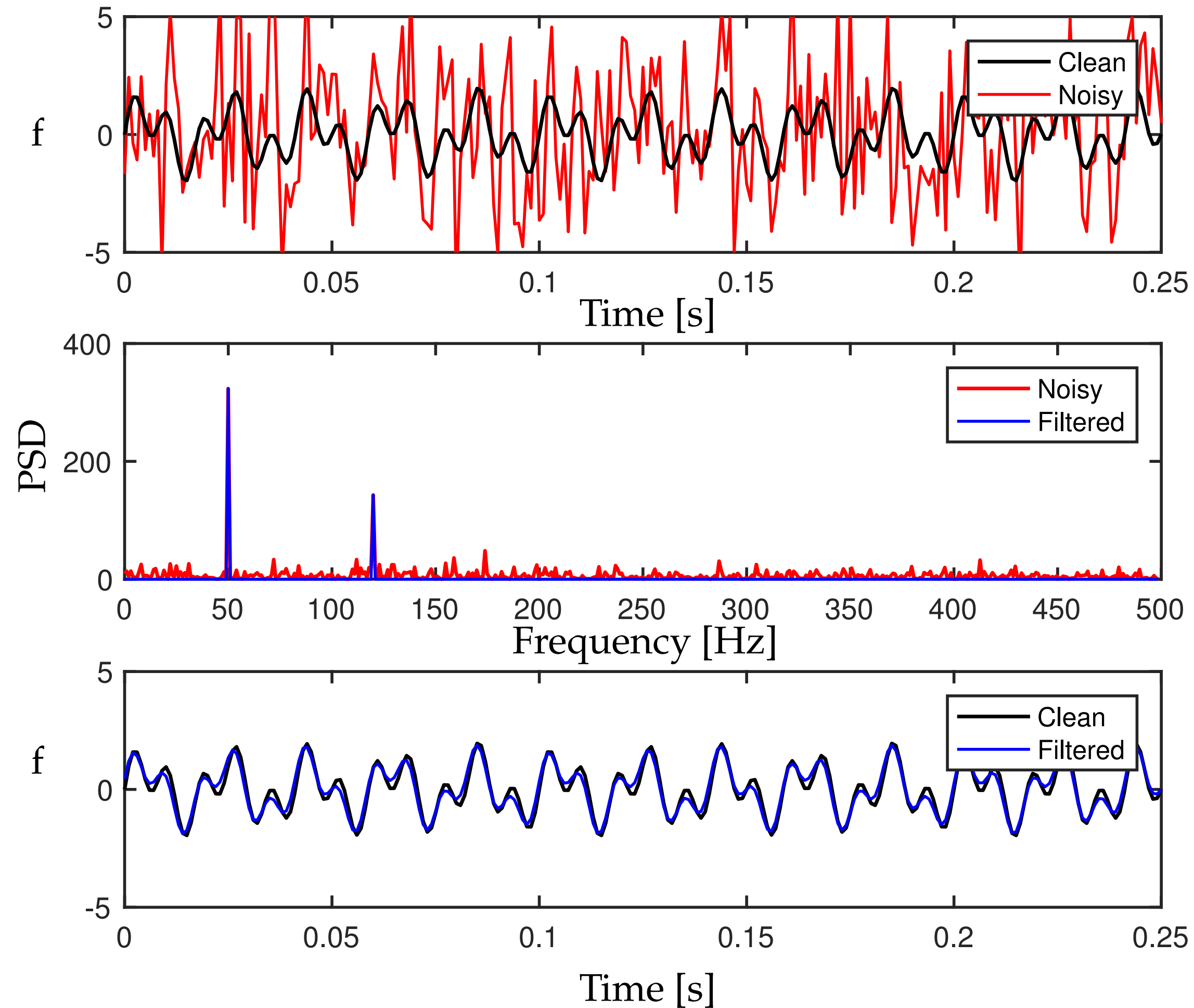
Signal Denoising



$$f(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t) + \text{noise}$$

$$f_1 = 50 \quad f_2 = 120$$

Signal Denoising

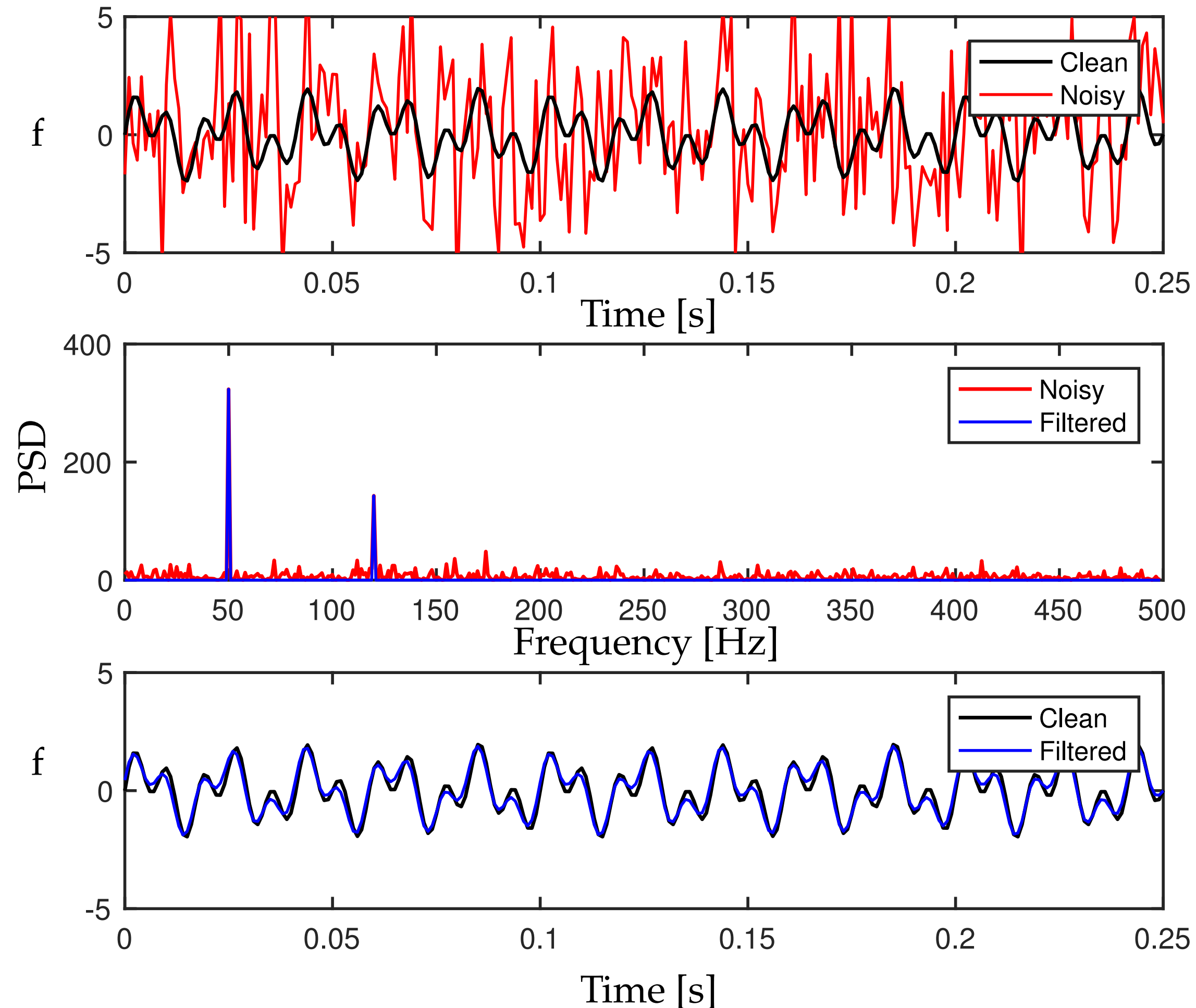


$$f(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t) + \text{noise}$$

$$f_1 = 50 \quad f_2 = 120$$

$$\hat{f}(\omega) = DFT(f(t))$$

Signal Denoising



$$f(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t) + \text{noise}$$

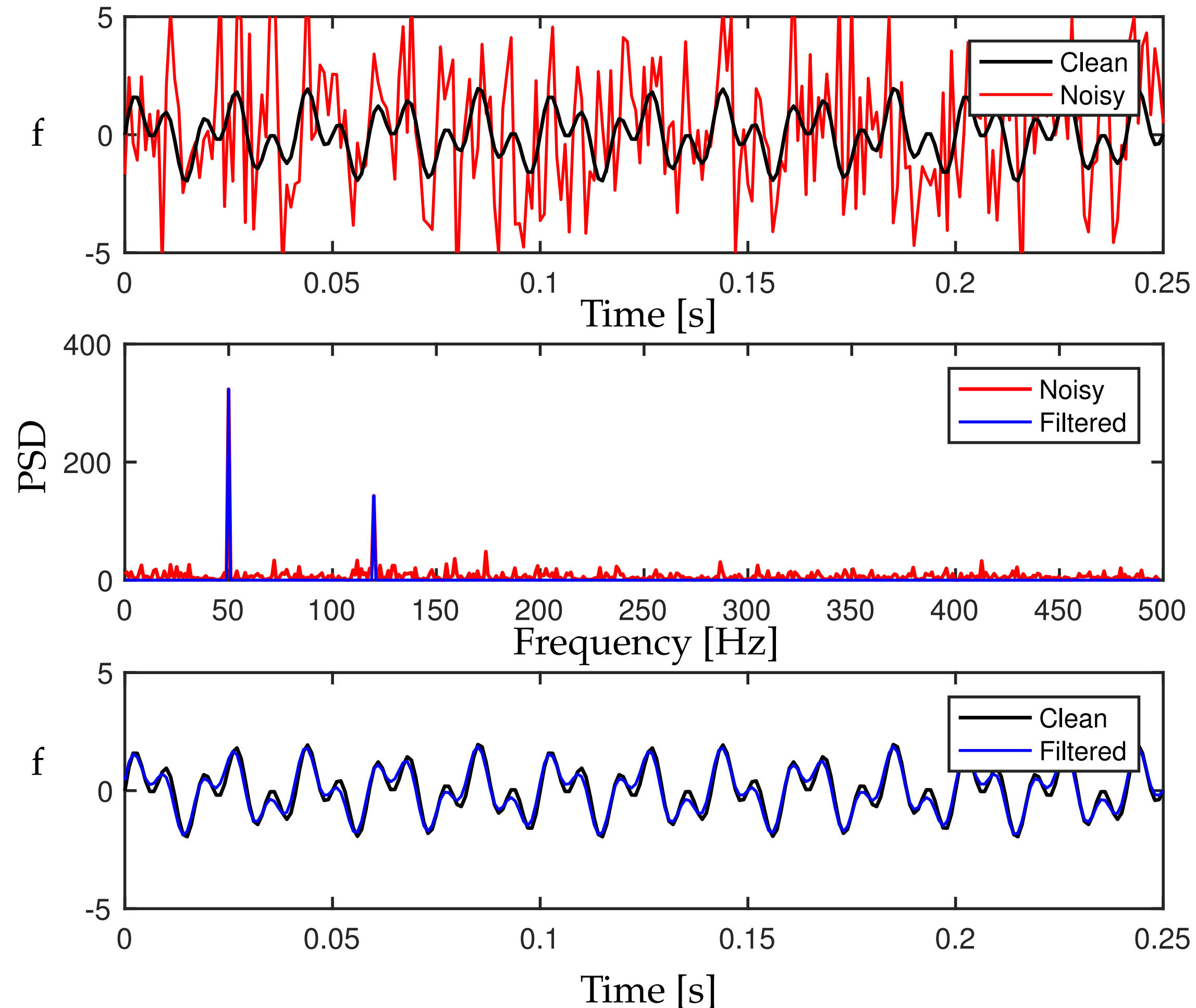
$$f_1 = 50 \quad f_2 = 120$$

$$\hat{f}(\omega) = DFT(f(t))$$



Remove low-magnitude frequency components $\hat{f}'(\omega)$

Signal Denoising



$$f(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t) + \text{noise}$$

$$f_1 = 50 \quad f_2 = 120$$

$$\hat{f}(\omega) = DFT(f(t))$$

Remove low-magnitude
frequency components $\hat{f}'(\omega)$

$$f'(t) = IDFT(\hat{f}'(\omega))$$

2D DFT

2D Basis Functions

Real Bases

$$\left\{ \sin(ux + vy), \cos(ux + vy) \right\}, (u, v) \in \mathbb{Z}$$

Complex Bases

$$e^{i(ux+vy)}, (u, v) \in \mathbb{Z}$$

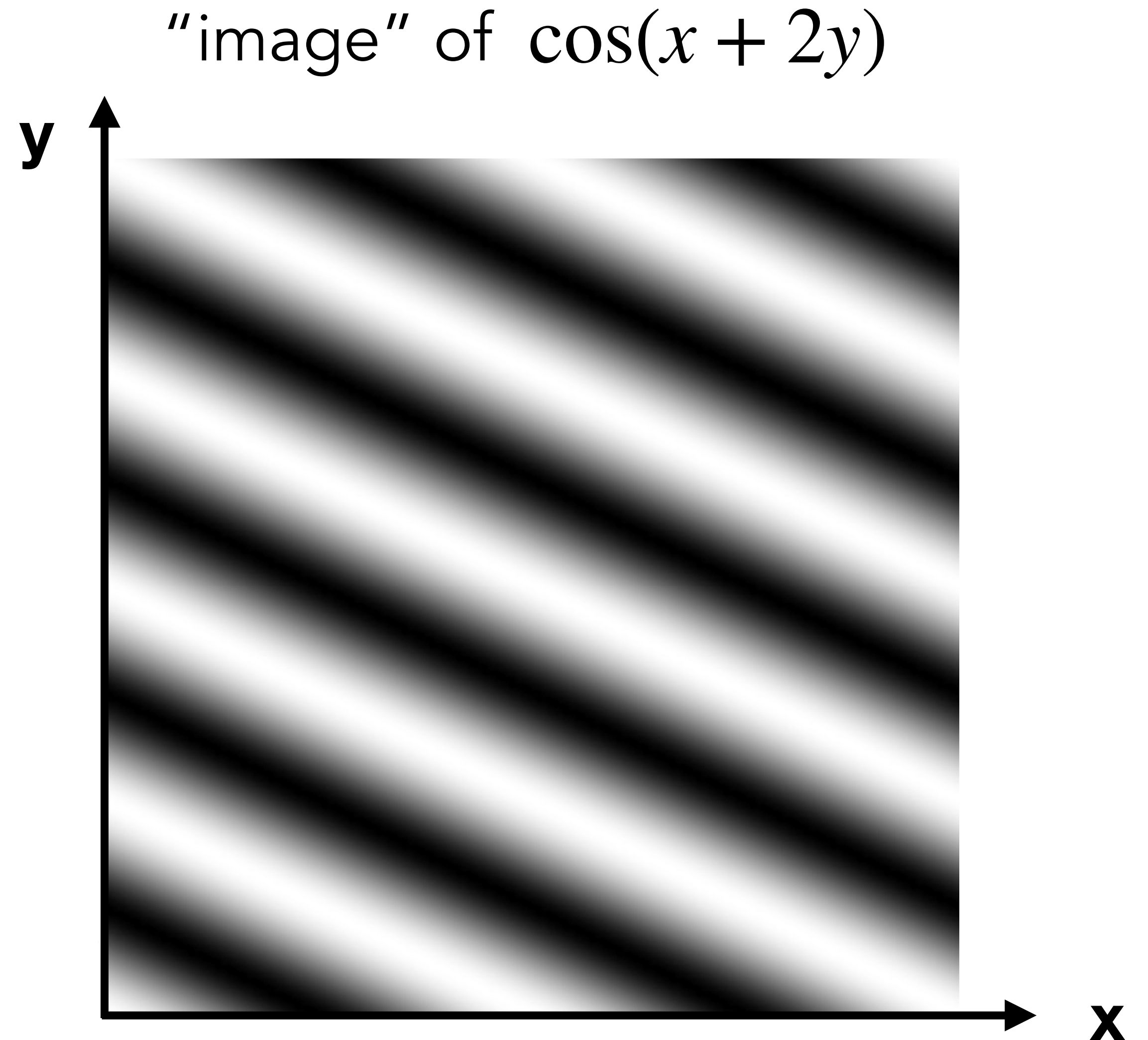
2D Basis Functions

Real Bases

$$\left\{ \sin(ux + vy), \cos(ux + vy) \right\}, (u, v) \in \mathbb{Z}$$

Complex Bases

$$e^{i(ux+vy)}, (u, v) \in \mathbb{Z}$$



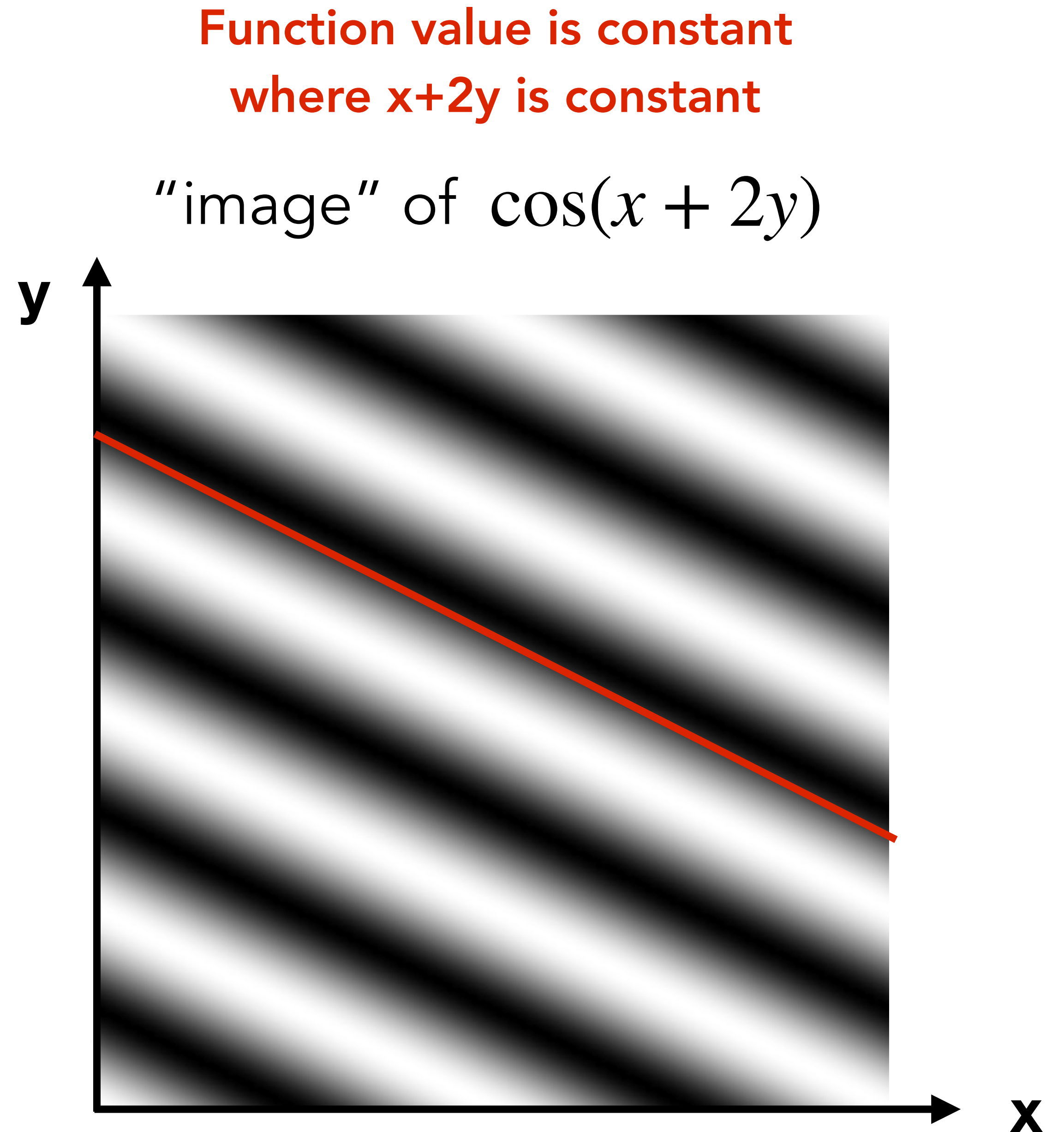
2D Basis Functions

Real Bases

$$\left\{ \sin(ux + vy), \cos(ux + vy) \right\}, (u, v) \in \mathbb{Z}$$

Complex Bases

$$e^{i(ux+vy)}, (u, v) \in \mathbb{Z}$$



2D Discrete Fourier Transform

1D DFT

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_k e^{\frac{i2\pi jk}{n}}$$

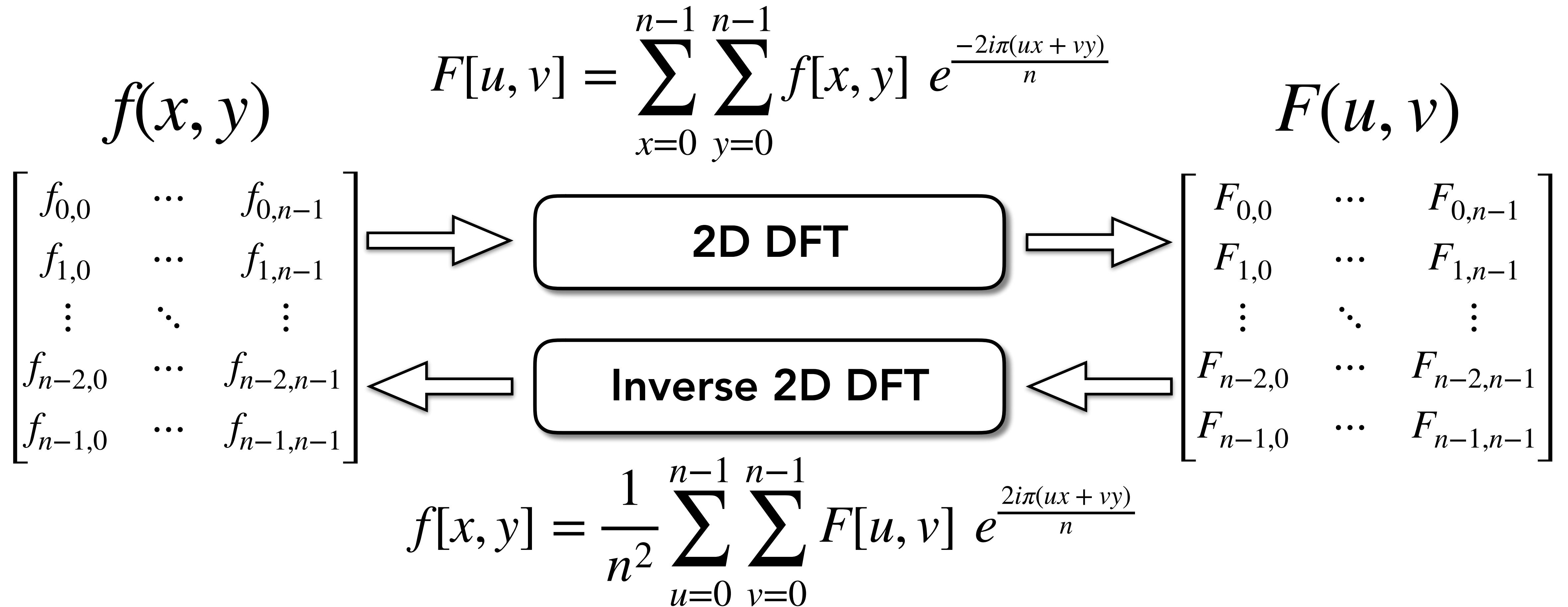
$$\hat{f}_k = \sum_{j=0}^{n-1} f_j e^{\frac{-i2\pi jk}{n}}$$

2D DFT

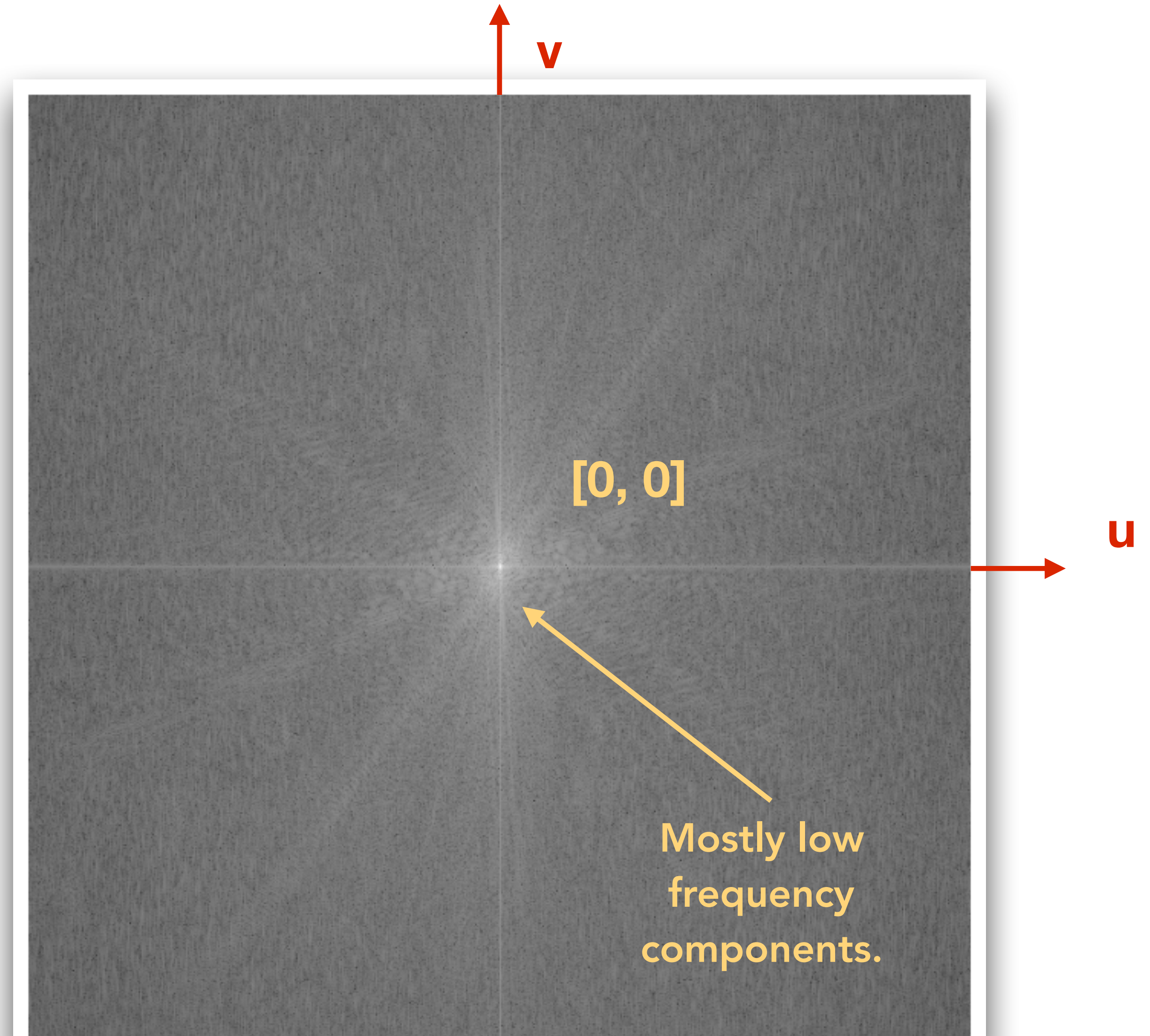
$$f[x, y] = \frac{1}{n^2} \sum_{u=0}^{n-1} \sum_{v=0}^{n-1} F[u, v] e^{\frac{2i\pi(ux + vy)}{n}}$$

$$F[u, v] = \sum_{x=0}^{n-1} \sum_{y=0}^{n-1} f[x, y] e^{\frac{-2i\pi(ux + vy)}{n}}$$

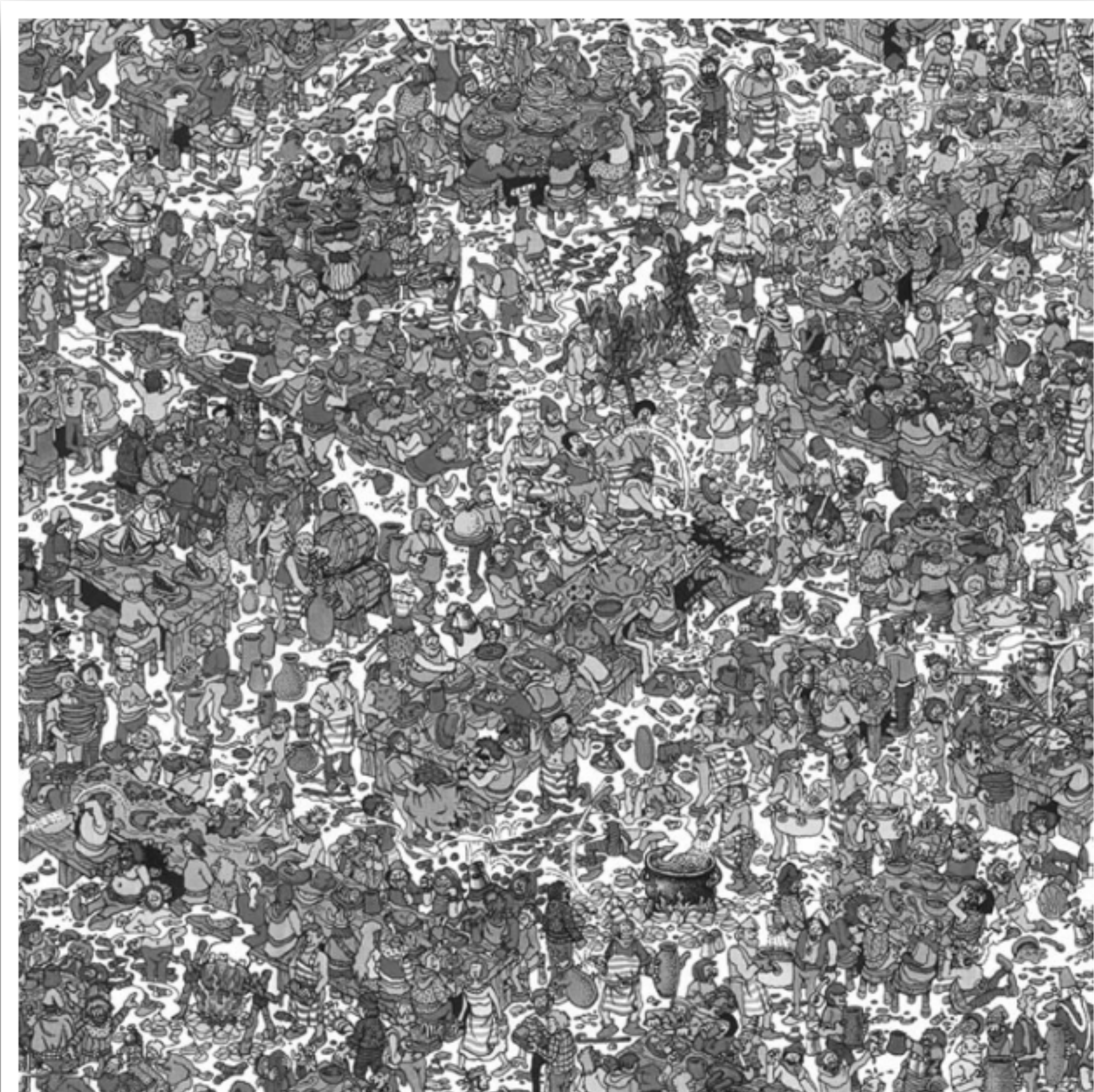
2D DFT



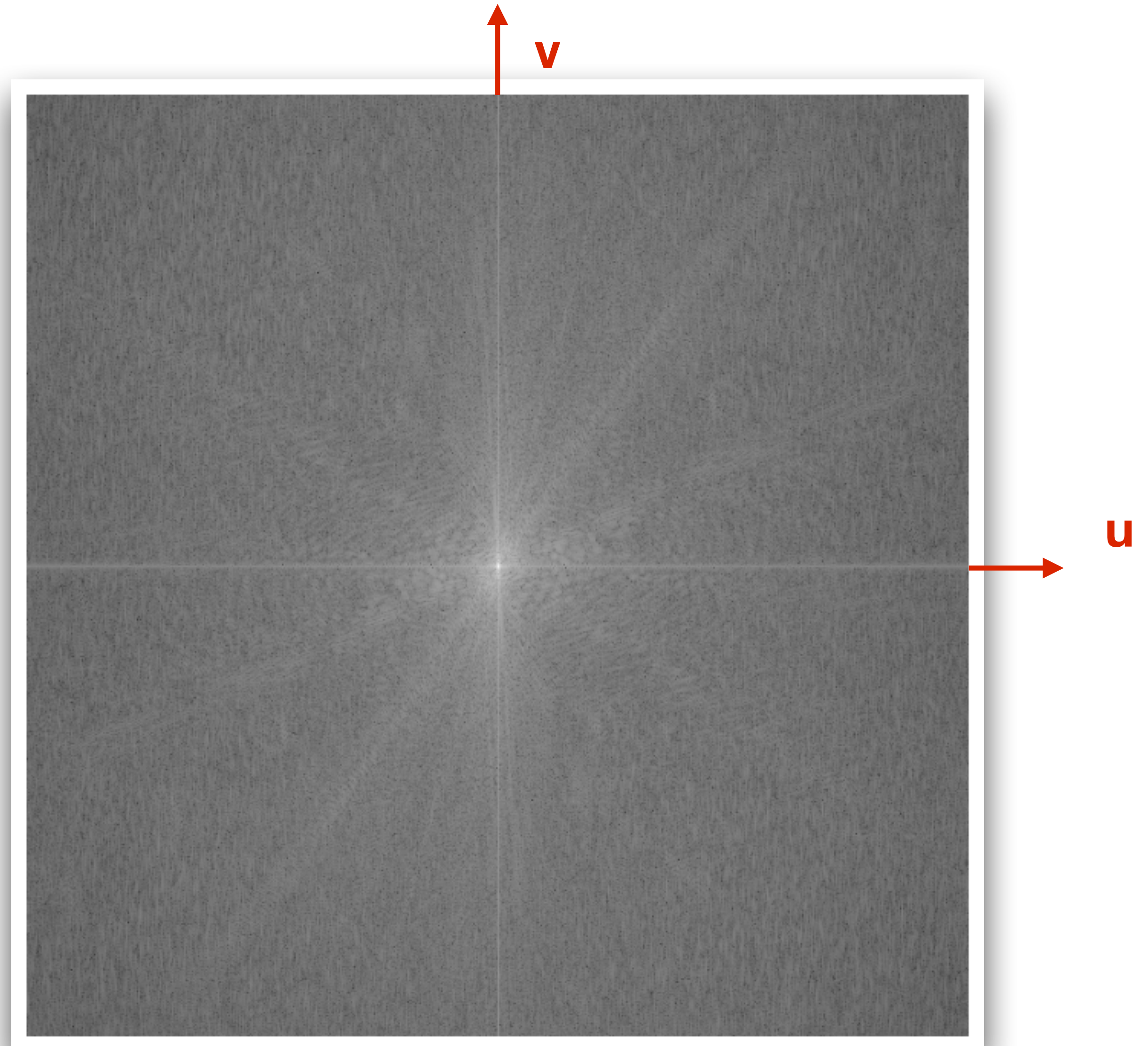
A Typical Photo



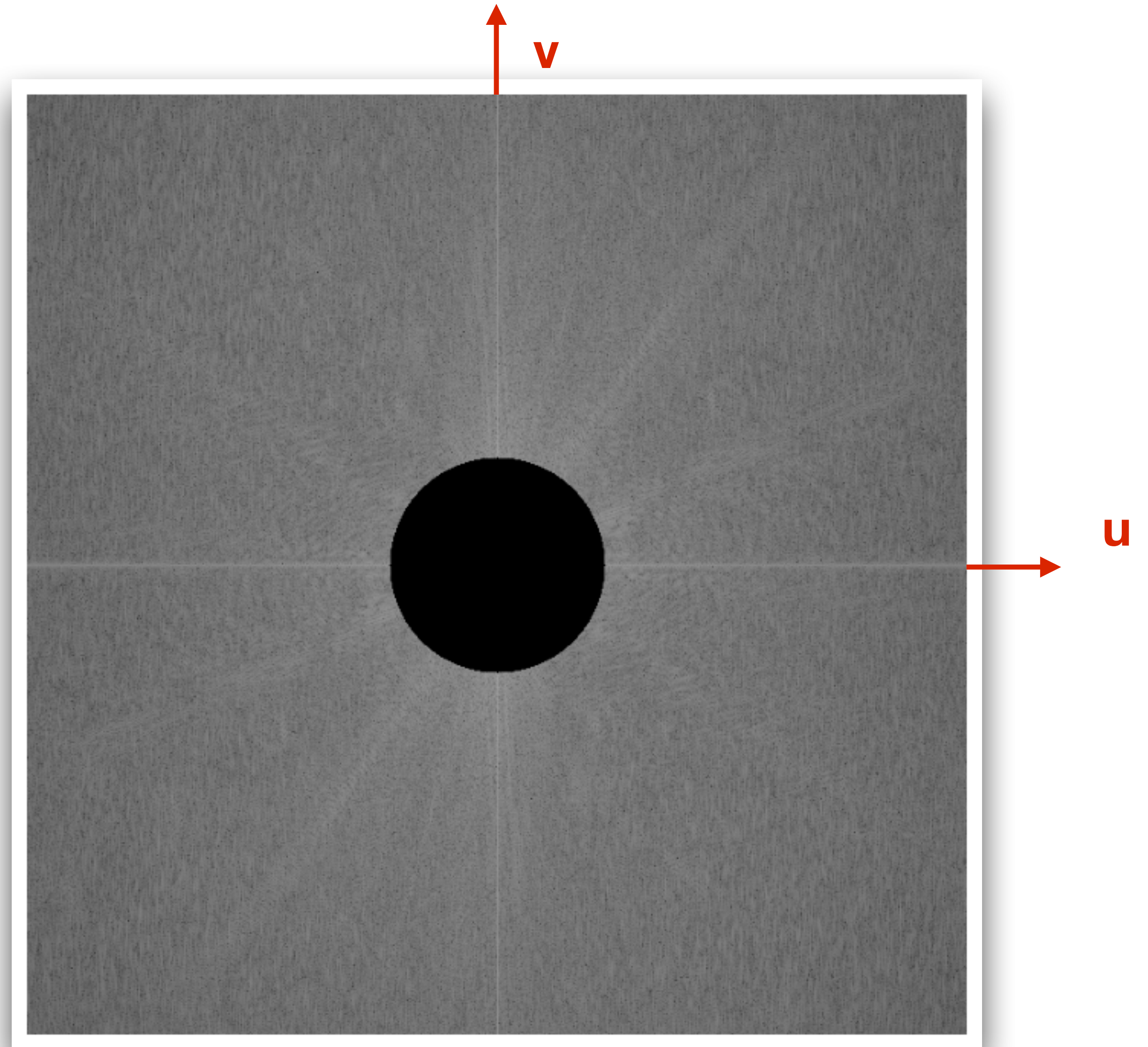
A Photo With Many High-Frequency Components



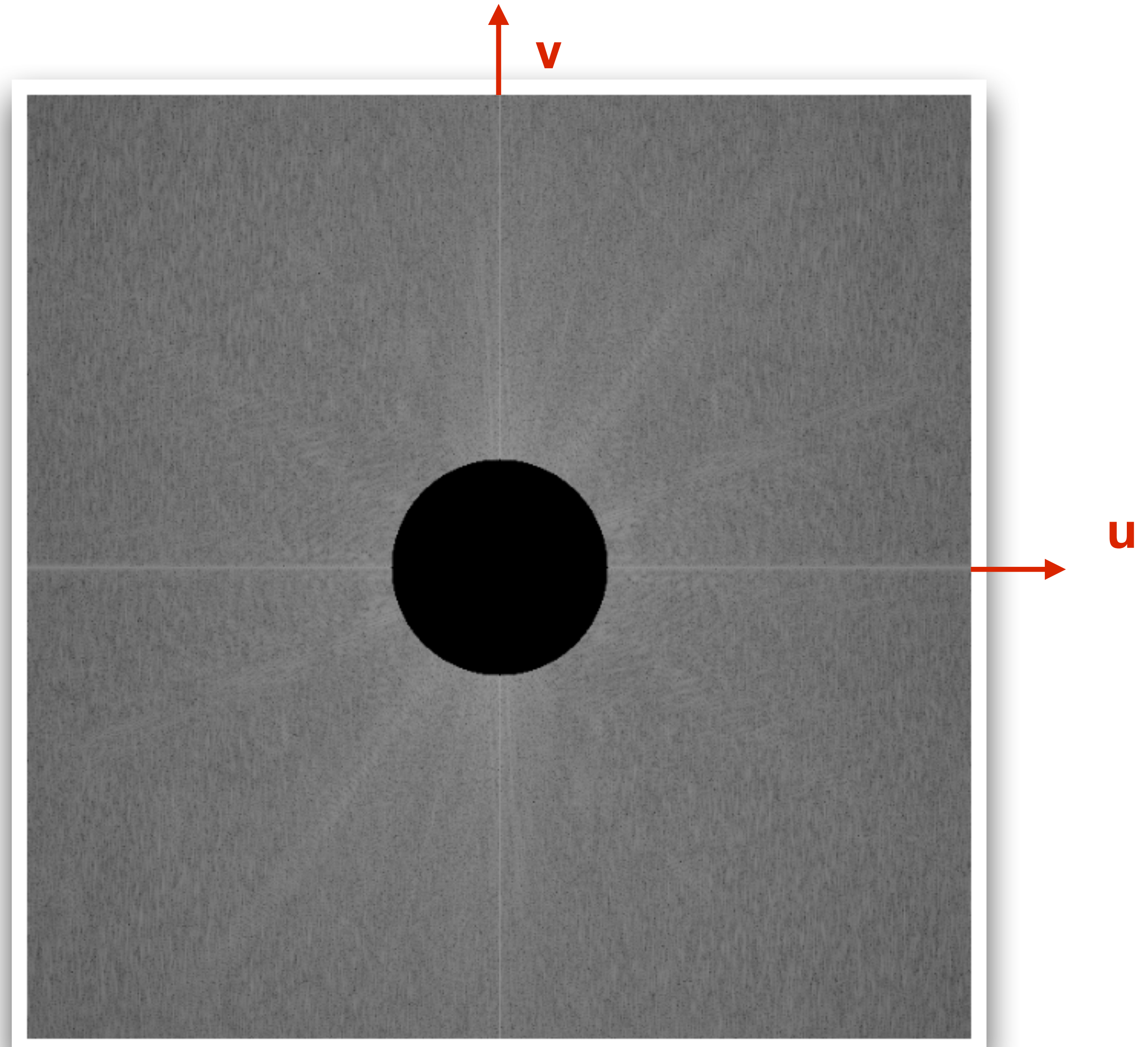
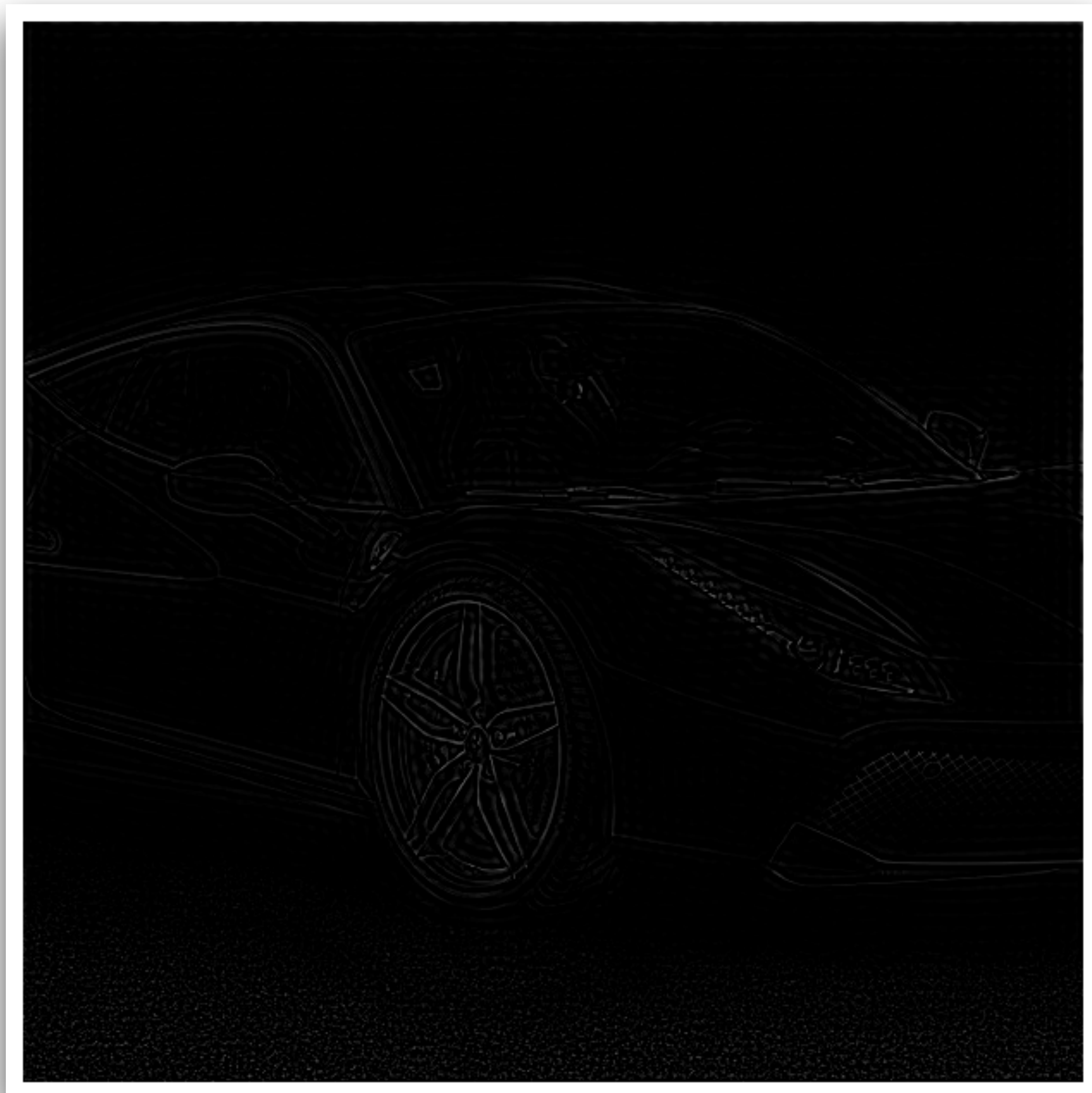
Original Photo



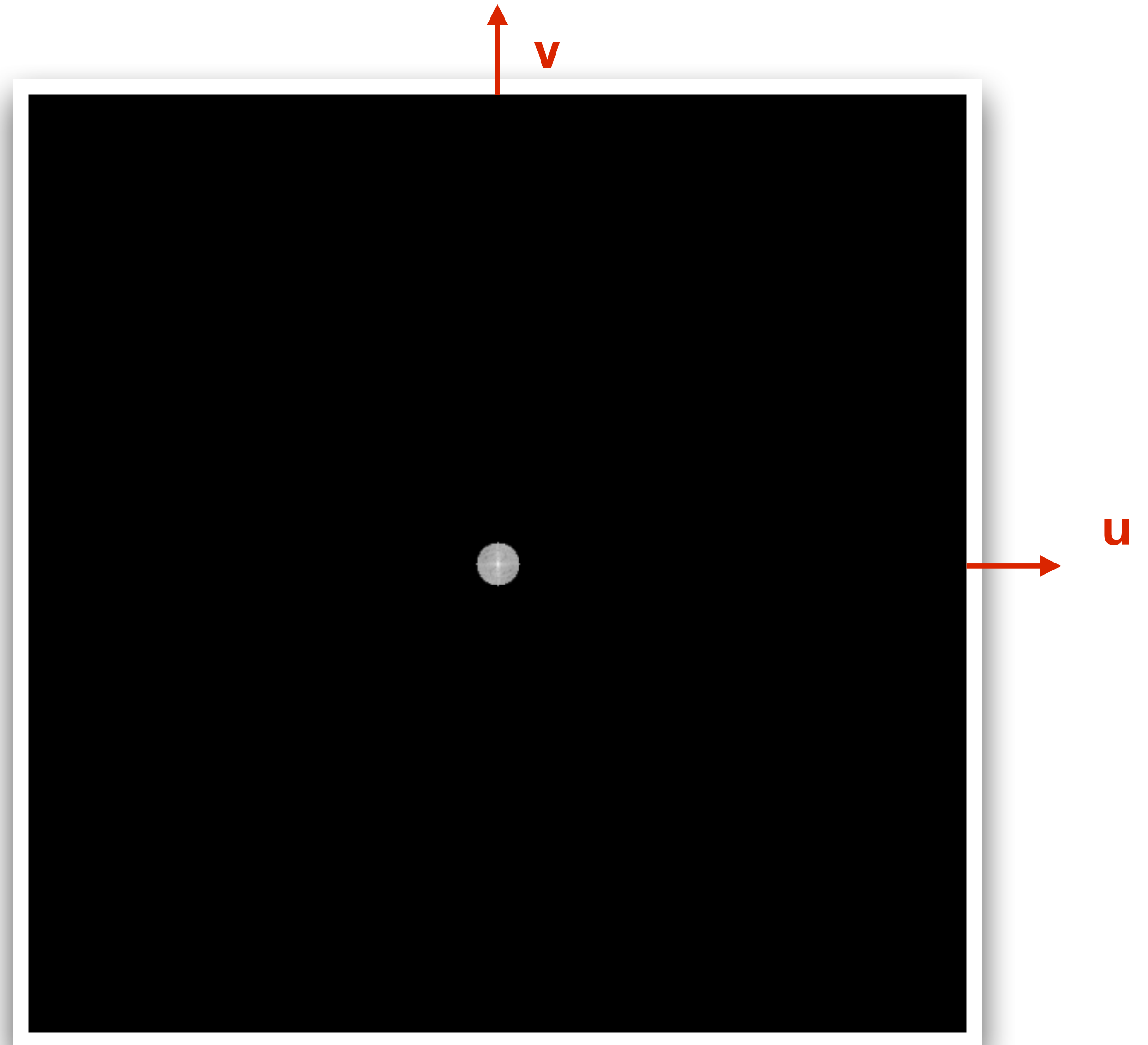
High-Pass Filter (Removing Low-Freq Components)



High-Pass Filter (Removing Low-Freq Components)



Low-Pass Filter (Removing High-Freq Components)



Low-Pass Filter (Removing High-Freq Components)

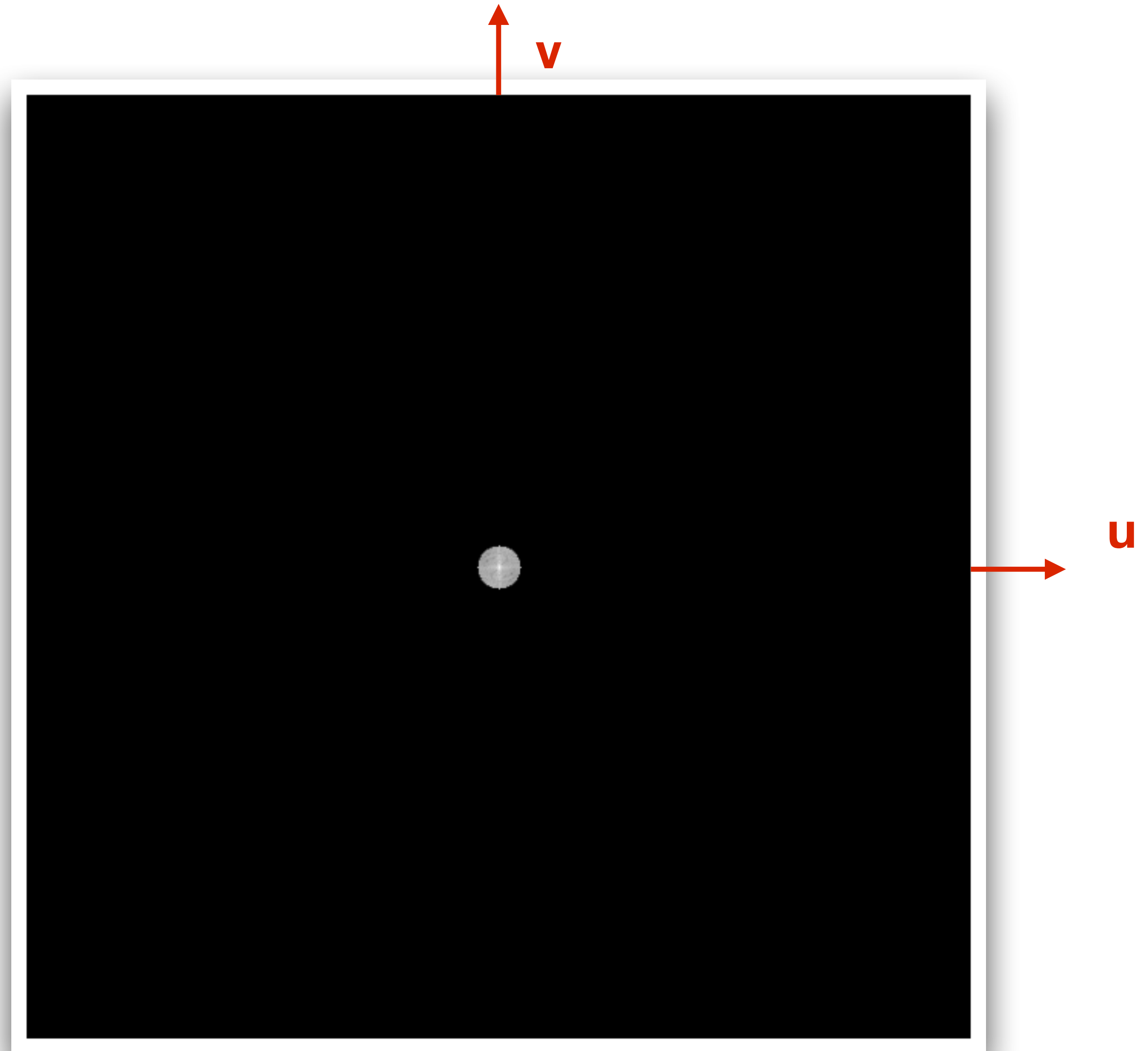


Image Compression with 2D DFT

- A precise DFT would turn an $M \times N$ image into another $M \times N$ image (each pixel represents a Fourier coefficient).
- Zero-out small Fourier coefficients to compress image.
 - Uncompress through inverse DFT
- JPEG uses something very similar.

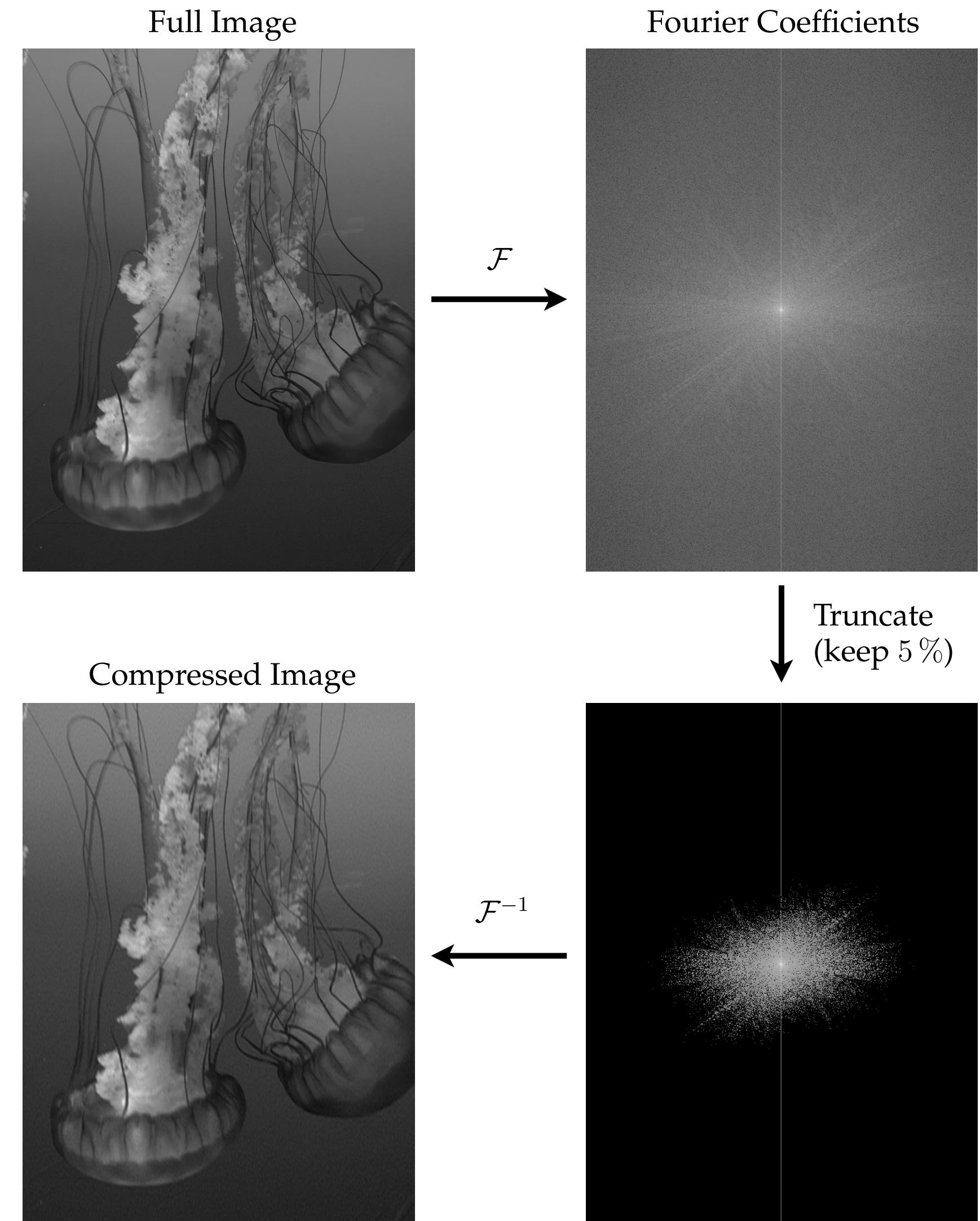


Image Compression with 2D DFT

- A precise DFT would turn an $M \times N$ image into another $M \times N$ image (each pixel represents a Fourier coefficient).
- Zero-out small Fourier coefficients to compress image.
 - Uncompress through inverse DFT
- JPEG uses something very similar.

Full image



5.0% of FFT



1.0% of FFT



0.2% of FFT



Fourier Transform

What About Non-Periodic Functions?

- Fourier series is concerned with L -periodic functions.
- What about a general, potentially non-periodic function? Does the Fourier theory still apply to it?
- Idea: let L approach infinity and see how the Fourier series change!

As L Approaches Infinity

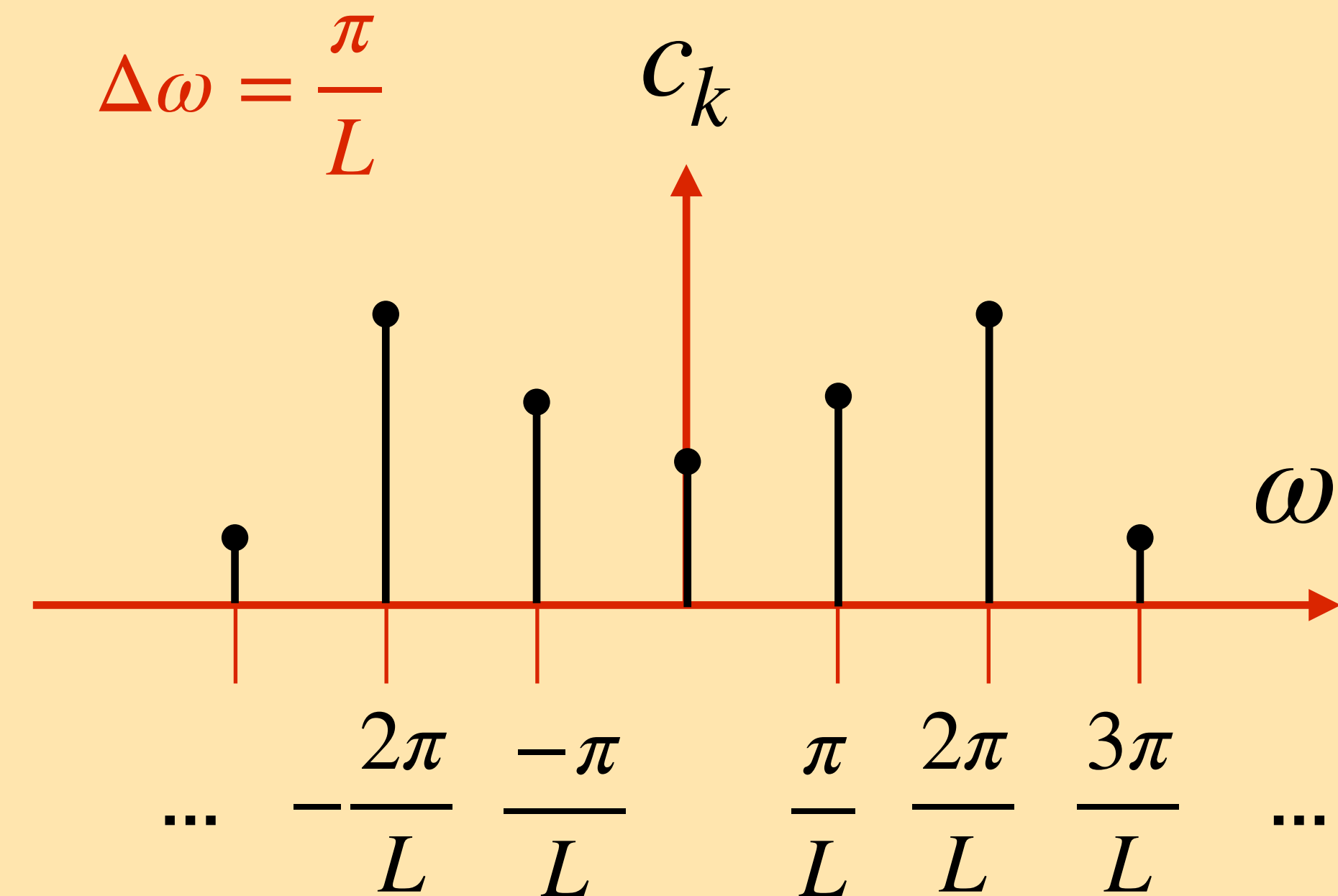
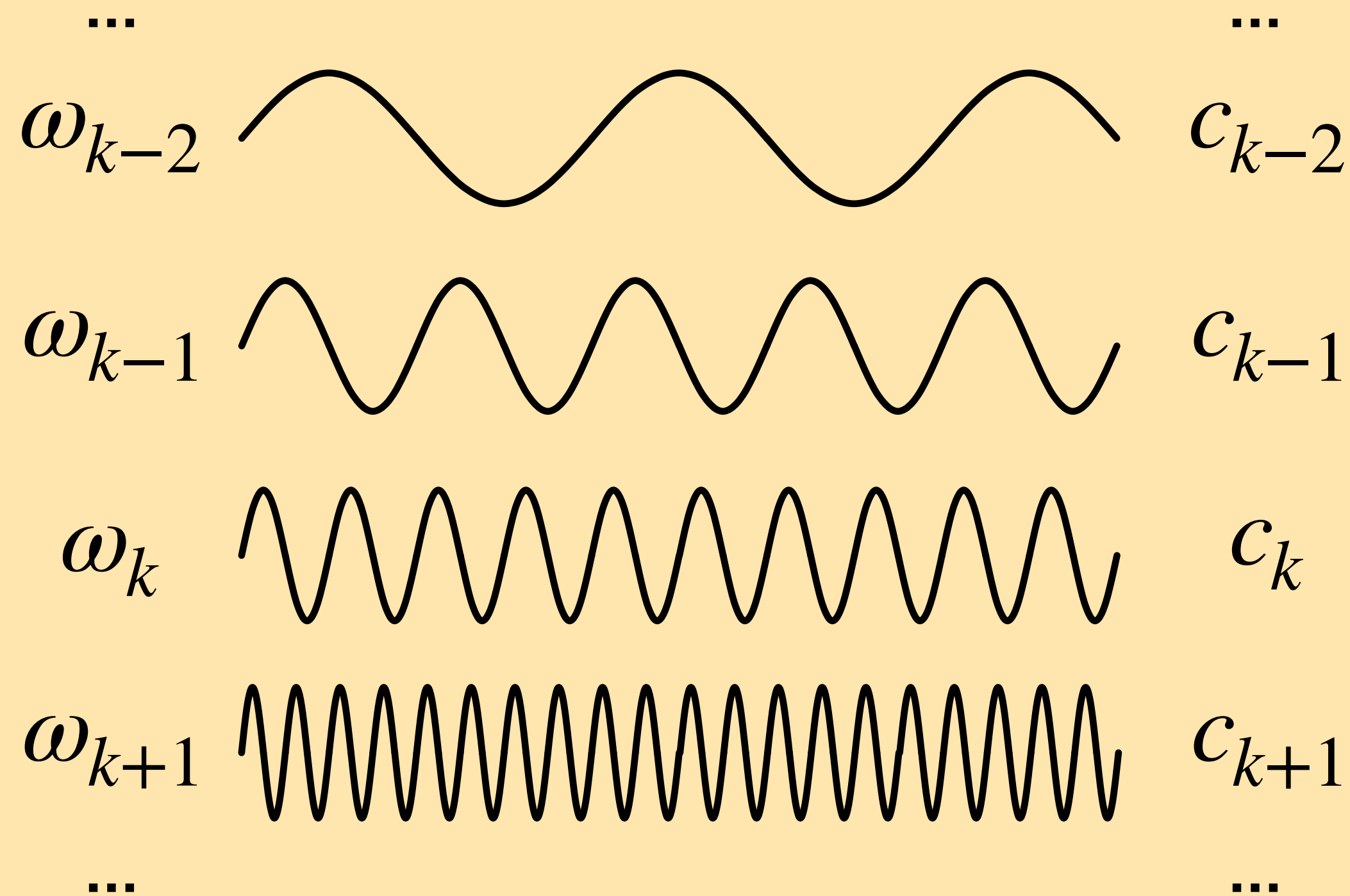
$$\omega_k = \frac{k\pi}{L}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x}$$

$$c_k = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\omega_k x} dx$$

Frequency

Coeff.



As L Approaches Infinity

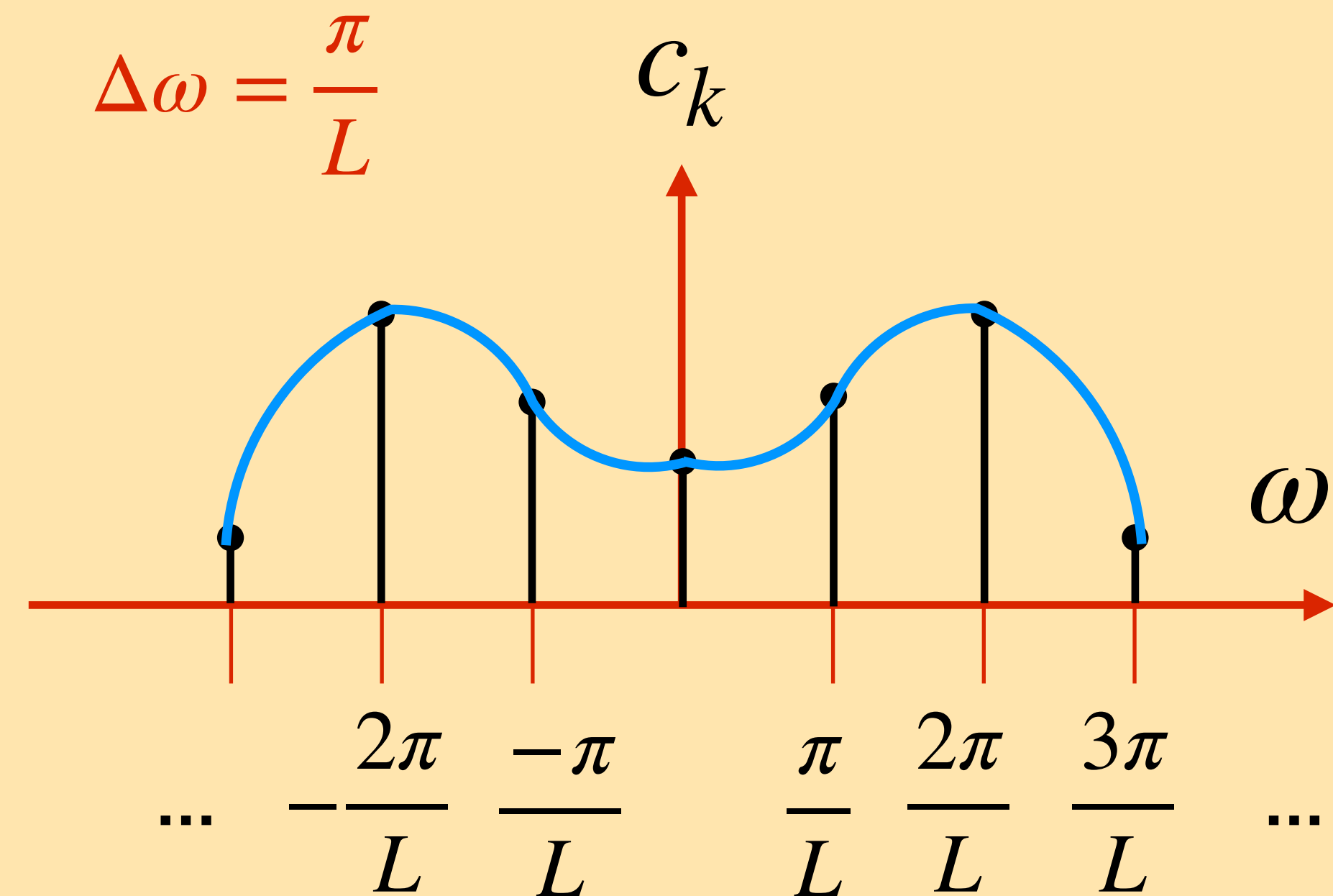
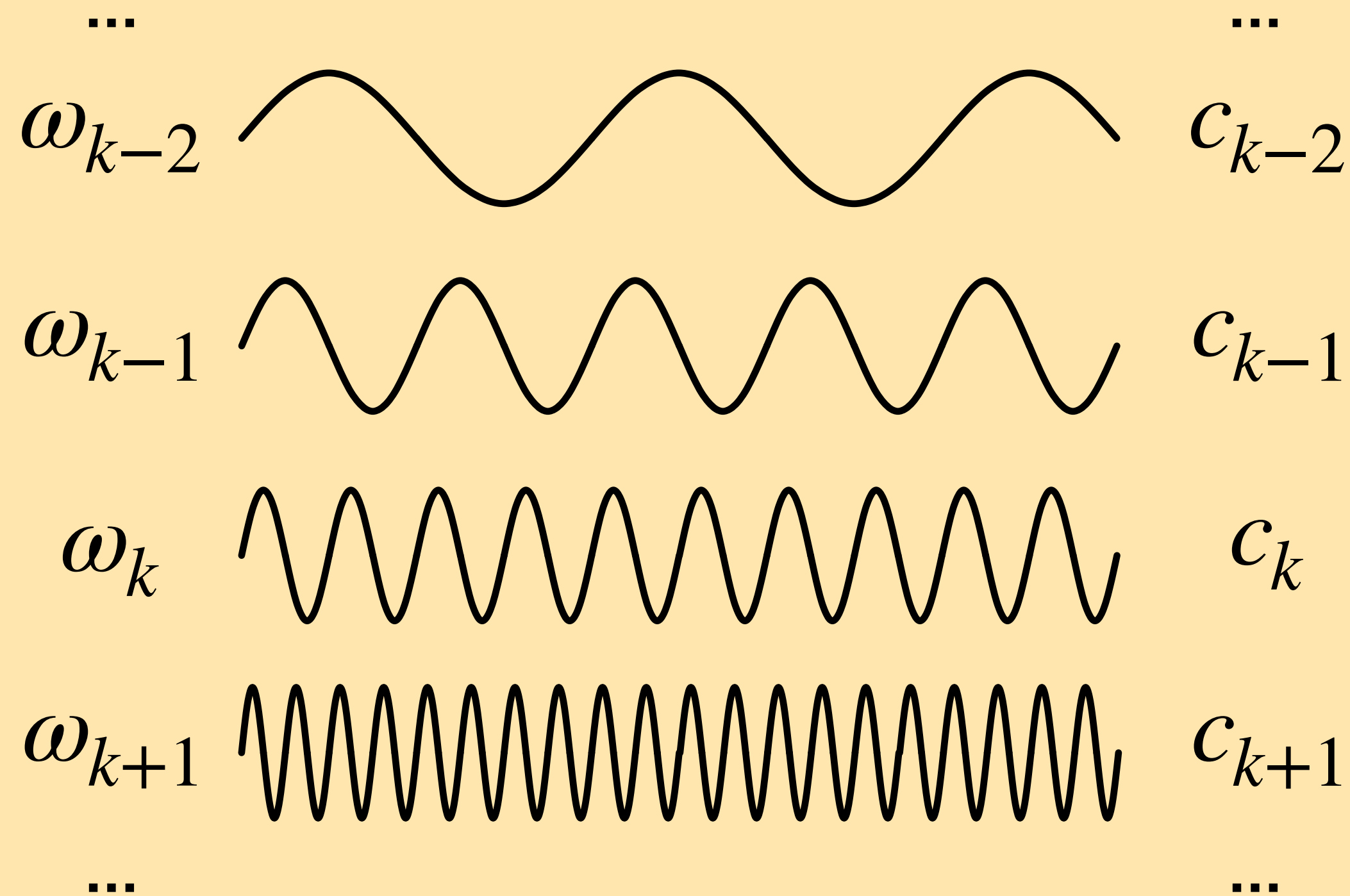
$$\omega_k = \frac{k\pi}{L}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x}$$

$$c_k = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\omega_k x} dx$$

Frequency

Coeff.



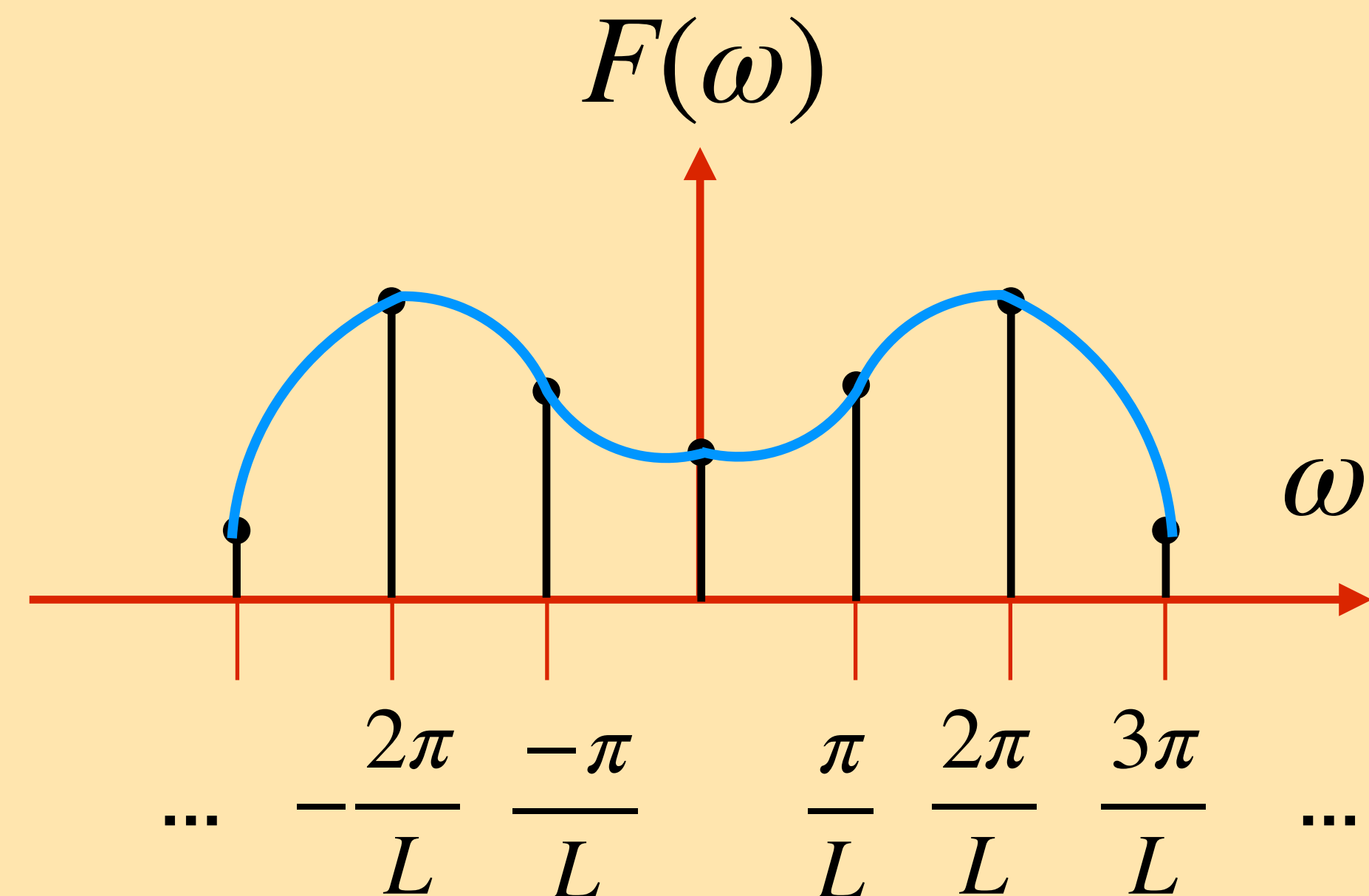
As L Approaches Infinity

$$\omega_k = \frac{k\pi}{L}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x}$$

$$c_k = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\omega_k x} dx$$

- As L approaches ∞ , the infinite sequence of Fourier coefficients becomes a continuous function $F(\omega)$.

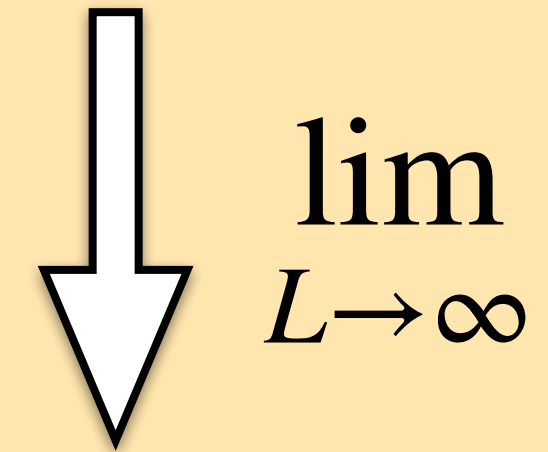


As L Approaches Infinity

$$\omega_k = \frac{k\pi}{L}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x}$$

$$c_k = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\omega_k x} dx$$



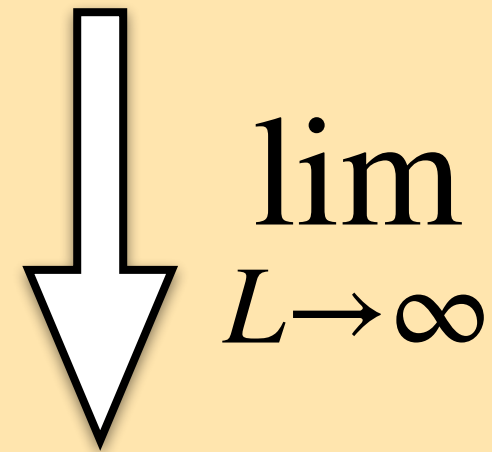
$$F(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx$$

As L Approaches Infinity

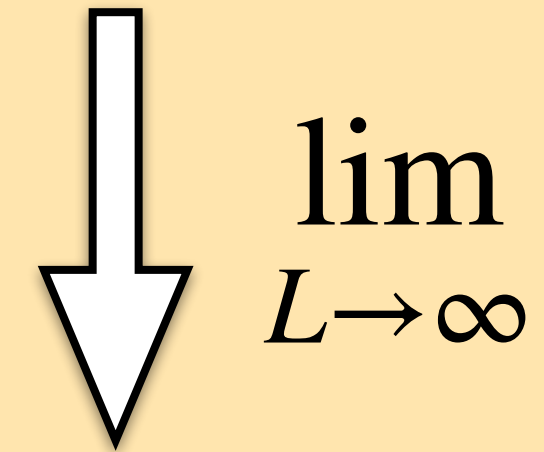
$$\omega_k = \frac{k\pi}{L}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x}$$

$$c_k = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\omega_k x} dx$$

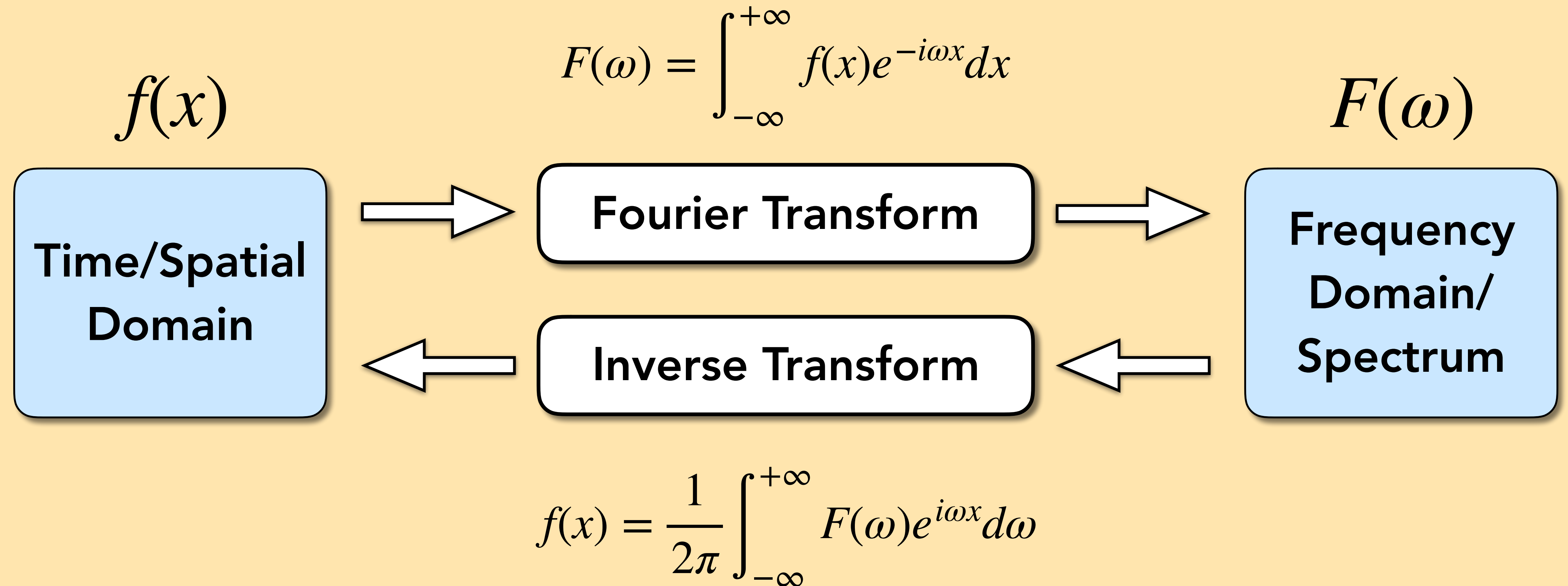


$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega x} d\omega$$



$$F(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx$$

Fourier Transform



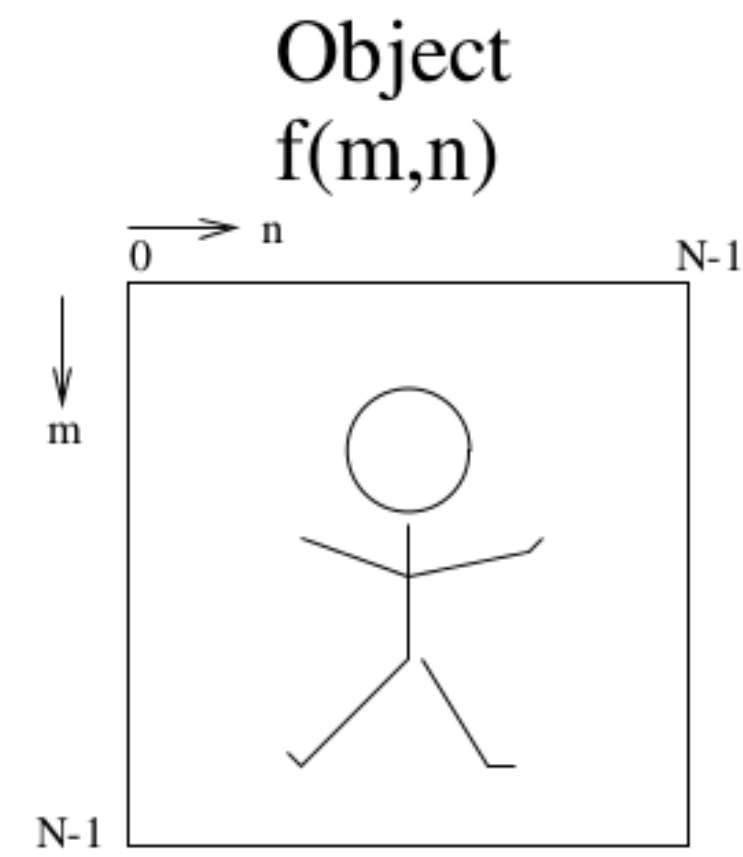
Key Things to Take Away

- Signals
 - Audio waves are discrete 1D signals (in the time domain)
 - Images are discrete 2D signals (in the spatial domain)
- Any time/spatial domain signals can be converted to frequency domain
 - Frequency domain representation of signal is called **spectrum**
- Fourier series applies to continuous L-period signals
- Discrete Fourier transform applies to discrete L-period signals
- Fourier transform applies to non-periodic continuous signals

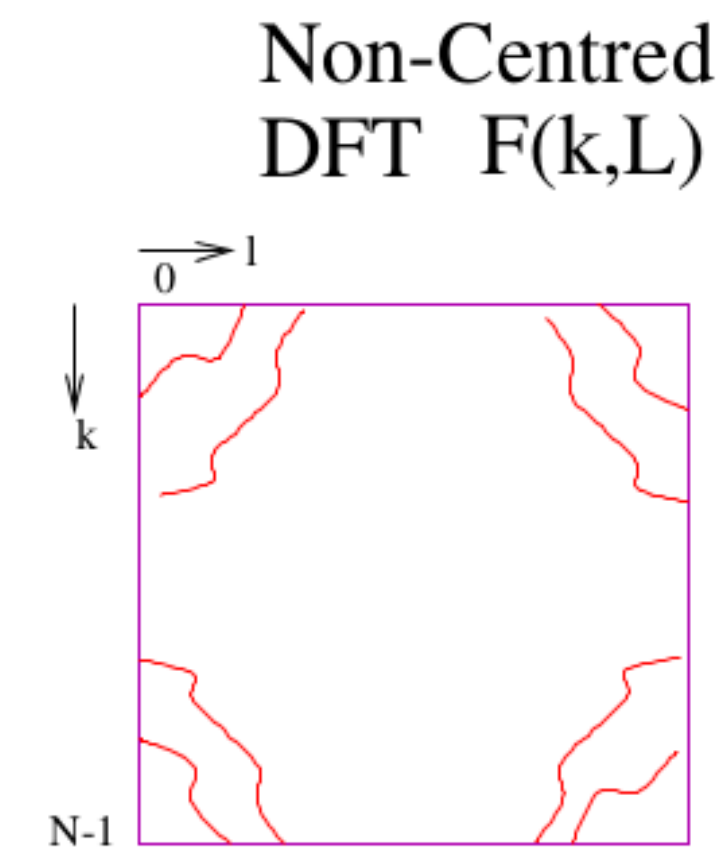
Key Things to Take Away

- How to tell frequency from images?
 - “Busy” areas or sudden changes (e.g., edge) correspond to high frequency components
 - Smooth areas are generally of lower frequency
- Removing low/high frequency components from the spectrum corresponds to sharpening/blurring images in spatial domain

Centering: Looking at DFTs



↓ DFT



Extended DFT Calculated at all k,l

