

PROPOSAL FOR WORK ON HEDETNIEMI'S CONJECTURE

ANDREW READ-MCFARLAND

1. INTRODUCTION

Hedetniemi's conjecture is a 40 year old open problem introduced by Stephen Hedetniemi in [1] while he was a graduate student at the University of Michigan [2]. Formally the conjecture reads as follows:

Conjecture 1.1. $\min\{\chi(G), \chi(H)\} = \chi(G \times H)$.

This if shown this conjecture is surprisingly powerful, as it provides a way to find the chromatic number of a graph by calculating the chromatic number of a graph with much fewer vertices. Since graph k -colorability is a well known NP-complete problem [3], the ability to reduce the size of the input heavily reduces run time.

In this paper we will discuss background knowledge on the topic, the methods and goals of our research, as well as more a detailed explanation of our researches' benefits.

2. PREVIOUS WORK AND BACKGROUND

When we say $G \times H$ we are referring to the tensor product of the graphs G and H that has the vertex set $V(G) \times V(H)$ where the vertex (g, h) is adjacent to (g', h') if and only if g is adjacent to g' in G and h is adjacent to h' in H . From this definition it comes naturally to show $\min\{\chi(G), \chi(H)\} \geq \chi(G \times H)$, any coloring of G or H can be used as a valid coloring of $G \times H$. That is, for any vertex (g, h) give it the color g had in G (or h had in H if using a coloring from H). Thus, we already know that the tensor product will not be any harder to color, but we need to show it isn't any easier.

It has been shown that the conjecture actually holds for $\chi(G), \chi(H) \leq 4$. The proofs for chromatic number of 1 and 2 are fairly trivial. Duffus, Sands, and Woodrow showed that the conjecture holds for chromatic number up to 3 [4], and El-Zahar and Sauer show it holds for up to chromatic number of 4 in [5]. The proof given by Duffus, Sands, and Woodrow is fairly extensive, so we have already greatly simplified it.

Hajnal showed that the conjecture is false for graphs with an uncountably infinite chromatic number [6] and Rinot extended this result to countably infinite chromatic numbers [7]. While these results are noteworthy, they are likely not of interest for practical usage.

3. BENEFITS

As hinted at earlier, Hedetniemi's conjecture if proven (even for up to a larger chromatic number) could improve many algorithms. Finding the chromatic number of a large graph is a very long process, so by creating graphs G and H such that the original graph is $G \times H$ would drastically reduce the graph size, and so finding $\chi(G)$ and $\chi(H)$ would be much faster than computing $\chi(G \times H)$. It's also important to note that this process could again be applied to G or H , and any coloring found in any of these smaller parent graphs can be used to induce a coloring in the original graph.

As far as the practical uses of graph colorings in general, their main use is to determine how to make what is being represented not have conflicting qualities. For instance, we could represent radio towers as nodes, and have an edge denote that the towers were in range of each other. Then a coloring would be a way to assign frequencies to the towers such that no frequencies overlapped [8]. Akamai also released a case study about how they use graph colorings to push out updates to their servers. They found that they had to update their 75000 servers in 8 waves, but if a better coloring (requiring only 7 or fewer colors) would cut down on their number of waves, and thus allow them to push out updates faster and easier.

4. METHODS AND GOALS

We seek to make advancements centered around Hedetniemi's conjecture. Certainly we would like to prove it true for all finite graphs, or even those with finite chromatic number. However this is unlikely, but simply proving that it holds for $\chi \leq 5$ would be a large and extremely helpful result.

We have not seen any real work done making use of the planarity of graphs in trying to tackle the $\chi(G) = \chi(H) = 5$ case. The tricky part of higher chromatic number graphs is that there are fewer properties to rely on. However, since every planar graph is 4 colorable (as shown by the 4 color theorem)[9], if a graph has chromatic number 5, then it is not planar. This is very useful information as there are many results shown about planarity. We hope to use this new approach to at least solve this case, and push further if possible.

REFERENCES

- [1] Stephen Hedetniemi. Homomorphisms of graphs and automata. 01 1966.
- [2] Claude Tardif. [gtn liv:6] hedetniemis conjecture, 40 years later. 2008.
- [3] M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified np-complete problems. In *Proceedings of the Sixth Annual ACM Symposium on Theory of Computing*, STOC '74, pages 47–63, New York, NY, USA, 1974. ACM.
- [4] D. Duffus, B. Sands, and R. E. Woodrow. On the chromatic number of the product of graphs. *J. Graph Theory*, 9(4):487–495, 1985.
- [5] M. El-Zahar and N. W. Sauer. The chromatic number of the product of two 4-chromatic graphs is 4. *Combinatorica*, 5(2):121–126, 1985.
- [6] A. Hajnal. The chromatic number of the product of two \aleph_1 -chromatic graphs can be countable. *Combinatorica*, 5(2):137–139, 1985.
- [7] Assaf Rinot. Hedetniemi's conjecture for uncountable graphs. *J. Eur. Math. Soc. (JEMS)*, 19(1):285–298, 2017.
- [8] Tom Leighton and Marten Dijk.
- [9] K. Appel and W. Haken. Every planar map is four colorable. part i: Discharging. *Illinois J. Math.*, 21(3):429–490, 09 1977.