

Tiling in Perspective: the ST Method

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This paper available at:

<http://www.cs.rochester.edu/u/brown/bio/hobby.html>

Section 0. Table of Contents and Reader's Guide

1. Goals, The Idea, Abbreviations

We'll explore tiling on a plane in 0, 1, and 2- pt. perspective for parallelograms, squares, hexagons and octagons.

- a. Tile extension in 1-Pt Perspective and a classical graphical solution.
- b. A simple model of imaging yields both graphical and numerical Similar Triangle (ST) methods for this and other tiling problems.

2. Parallelogram (PG) Tiling in 1-Pt Perspective (1PP)

How the ST method gives a graphical construction and formulae for required divisions.

3. PG Tiling in 2-Pt Perspective (2PP)

The same ST formulae, done for each vanishing point (VP) and initial tile size, yield a 2PP tiling.

4. PG Tiling in Orthographic Projection (OPP)

- a. no perspective, but foreshortening.
- b. all tiles look alike: arbitrary parallelograms.

5. Square Tiles

How to draw square tiles consistent with VPs and the constraint of equal tile sides:

- a. The relation of side lengths in the world determines the image length of one side given the image length of the other.
- b. Use the 1PP ST method to find A_i . The B_i are simply scaled versions of A_i .

6. Hexagonal (Octagonal, etc) Tiles

- a. Irregular and Regular Hexagons
- b. Regular Hexagon Dimensions and underlying PG.
- c. Diagonal lines are determined from a PG tiling.
- d. PG sides and diagonals create hexagons and octagons.

7. Historical Note

This stuff is not original with me.

Goals: Fig. 1. shows a probably-familiar problem and graphical solution. We need to extend the given image of a parallelogram (PG) tile in one-point perspective (1PP). The graphical solution of Fig.1 uses both spatial dimensions: it requires knowing A1, B1, and the vanishing point VP: from them follow L2 and all four lines of the first tile. Our goal is to find A1, A2,... Fig. 1 shows how the construction of A2 needs A1, then A3 needs A2, etc.

The ST Method is based on Fig. 2. We see a side view of the horizontal ground plane (thus of the infinite line on the ground that produced L1 in Fig. 1)). Its features (regularly spaced points) are imaged from the viewpoint onto the image line (corresponding to L1 in Fig.1). The viewpoint is at height H and at a distance f behind the image line. The imaged points are at distances of A_1, A_2, \dots up the image line. They converge at the vanishing point VP , which is at the same image height H as the viewpoint and the horizon of the ground line. The ground-line feature distances from the image have not changed, nor have heights of image points. In Section

1 we apply the Similar Triangles (ST) property (coming up) to this diagram. That is all the ST method amounts to.

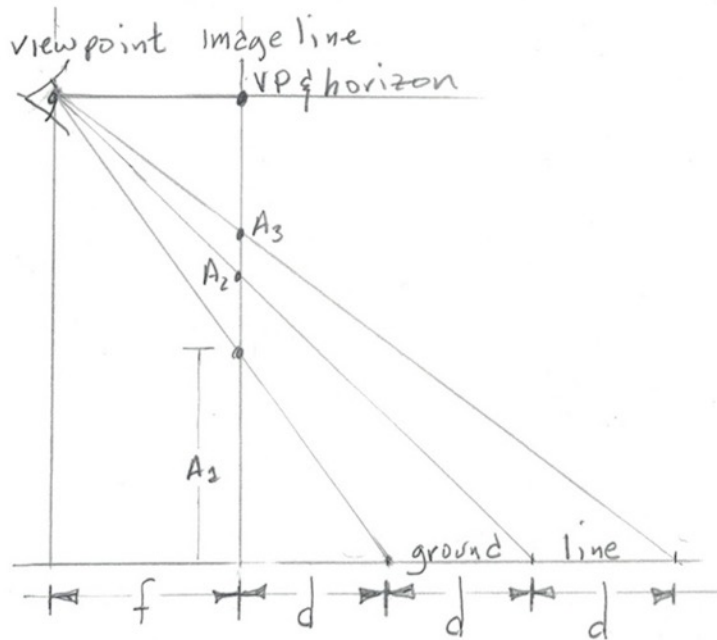


Fig. 2. Basis of the ST method. Tile boundaries on the ground are equally-spaced by a distance d , which we often set to 1 for simplicity. A_i are the distances along image line L_1 of tile boundaries.

The ST method uses the formulation shown in Fig.2, and follows from the powerful idea of Point Projection (PP) that is the basis for WPCF. I wrote “Tiling in Perspective” because the ST method is elegant and general. I’ve never seen anything like this treatment (not that I was actively looking). I didn’t steal it (honest). I’m sure it’s re-invented — if you know an earlier version, let me know..

There are many tutorial introductions to perspective lore, among them “Where Perspective Comes From” (WPCF), available in PDF at <http://hdl.handle.net/1802//31142> and <http://www.cs.rochester.edu/u/brown/bio/hobby.html>

We’ll use two basic facts about triangles. We appeal very briefly to the *Pythagorean Theorem*: in a triangle with one right angle, call the short sides (making the right angle) R and S , and the long side, or hypotenuse, T . Here’s the theorem:

Pythagoras: $T^2 = R^2 + S^2$.

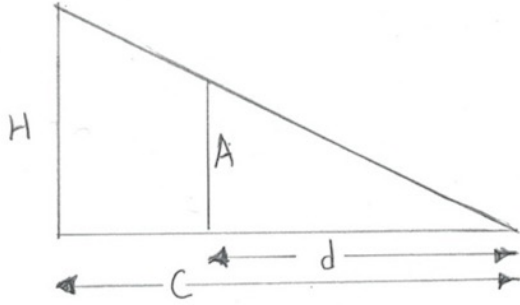


Fig. 3. Two similar triangles as we see them in this paper: With lengths H , C , d , and A , we know $A/d = H/C$ (see text).

Second, *similar triangles* (ST). Two triangles are similar if their angles are the same. That means that one is just a scaled-up or -down version of the other (Fig. 3). So in the figure, sides A and H correspond, as do d and C , so $A = kH$ and $d = kC$ for some value of k . So $A/H = k$ and $d/C = k$, and we get the

Similar Triangles (ST) Property: $A/H = d/C$.

As in Fig. 2, our future C s will be the sum of two other lengths, f and some d . So when we use the ST argument, it often looks like $A/H = d/(d+f)$, which we can solve for A : $A = Hd/(d+f)$.

Abbreviations:

PP: parallel projection.

OPP, 1PP, 2PP, ...: zero-, one-, two-point perspective.

VP; VP1, VP2, ...: vanishing point(s).

DP, DP1, DP2, ...: distance point(s).

O the Origin point; the corner of “first tile” from where A_i and B_i are measured.

H , H_1 , H_2 , ...: the distance from O to VP(s).

L_1 , L_2 , ...: lines from O to VPs or DPs.

A_1 , A_2 , ..., A_i , ..., A_n ...: the image length of the row of 1, 2, ..., i , ..., n ... tiles from O, on line L_1 .

B_1 , B_2 , ..., B_i , ..., B_n , ...: Like A_i above but with L_2 .

PG: parallelogram (4-sided polygon with opposite sides parallel).

UPG: underlying PG of a tiling with diagonal lines.

ST: Similar Triangles.

WPCF: “Where Perspective Comes From”.

Section 2: PG Tiling, 1PP

Recall Fig. 1. The ST method is another way to produce A_1, A_2, \dots , indeed A_i for any i , from two image measurements.

In Fig. 1, L_1 is the image of a line on the ground plane. In Fig. 2, we see it running from 0 to H up the vertical image line. L_1 is the image of a line of regularly-spaced tile-boundaries on the ground plane, which we see in Fig. 2 along the horizontal ground line, with tile boundaries arbitrarily set to one unit of distance. We'll use the ST property to find the A_i .

Graphical ST Construction: No Measuring, No Math:

Make a "life size" diagram like Fig. 2 using actual image distances H and A_1 . Choose any f we like. Projecting A_1 from the viewpoint at height H yields d on the ground line. Add more d -lengths along the ground line (their length depends on f , so it was OK to pick any f), project them through the viewpoint, and the image line is what we want on L_1 in Fig. 1, i.e. our extended tile side distances $A_1, A_2, \dots, A_i, \dots$. We're done.

Measurement ST Construction: No Drawing:

Measure H and A_1 in the drawing whose tile we want to extend. Use them as values and Fig. 2 to identify STs, which tell us the sizes we want since every diagonal projection line is (and contains) the hypotenuse of a pair of STs that tell us the relation of A_i to B_i . This time we can choose a convenient $d = 1$, which dictates a value for f by the relation above. Let's start in Fig. 2 with the similar triangles with corresponding sides H and A_1 . If $d=1$, then $A_1/1 = H/(f+1)$ so $A_1 = H/(f+1)$, our first ST result.

Now we don't know f , but we can convert our result into a formula for f using known quantities: $f = (H/A_1) - 1$. We could substitute f into the formula for A_1 but it's less confusing to treat f as a constant in formulae and use its value for any further numerical evaluation.

The next highest image point is the boundary of the second tile. That boundary is at 2 on the ground (in our arbitrary d units) and its image is on the image at A_2 . L_1 is the image, remember. Thus with a new pair of STs, read off the figure and infer by the ST property:

$(A_2)/2 = H/(f+2)$, so $A_2 = 2H/(f+2)$. A pattern is developing from our first result, and we correctly predict $A_3 = 3H/(f+3)$. Using ST property on Fig. 2, it is easy to see the

ST General Rule: The n th tile is at height $nH/(f+n)$,
and its side length on L_1 is
 $[nH / (f+n)] - [(n-1)H / (f + n - 1)]$.

Sanity check: Divide top and bottom of n -tile-height $nH/(f+n)$ by n to get the equivalent $H/(f/n+n/n) = H/(f/n + 1)$. If we extend the image to an infinite row of tiles, then n goes to infinity, f/n goes to 0, and the top of the "tile at infinity" is at H , the height of the VP, as we expect.

Section 3: PG Tiling, 2PP:

Here we assume an image of two sides of a PG tile in 2PP and knowledge of VP1 and VP2.(Fig. 4). One beauty of the ST method is that the 2PP case is solved just by doing the 1PP method twice and independently. No need for new ideas as in the classical graphical method.

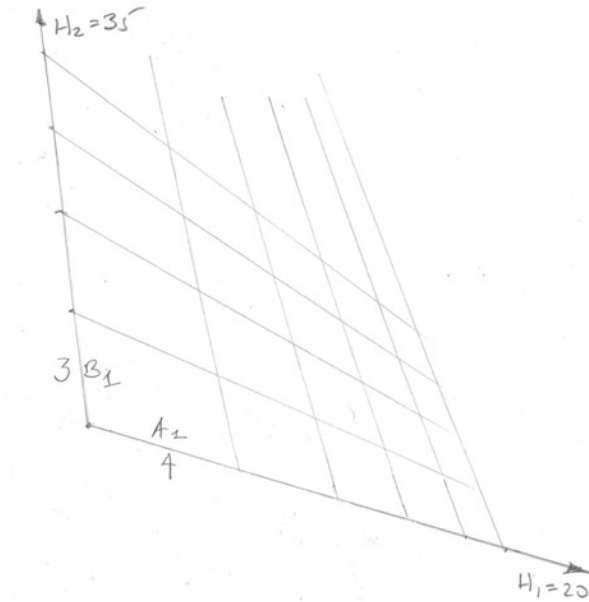


Fig 4. 2PP of PG tiling with first tile sides A_1 and B_1 (4 and 3), and VPs for which $H_1 = 20$ and $H_2 = 35$).

The ST method calculates the A_i as before from A_1 and H_1 , and the B_i analogously (and independently) from B_1 and H_2 . Transfer the A_i and B_i lengths to the drawing and make the connections.

We did not need to compute the B_i with 1PP; we derived them from the A_i and the parallelism of the “L2” sides. Exercise for the reader: why does the classical graphical method work in 1PP, and how can it be generalized to PG tilings in 2PP and 3PP drawings?

Section 4: OPP for PG Tiling

OPP is a consistent name for Orthographic projection (PP with an infinite focal length). OPP is common in technical and architectural drawings and is (along with 1PP and 2PP) just another aspect of Point Projection (see WPCF: spoiler: the answer to “where perspective comes from” is “point projection”).

Considering Fig. 2 as f goes to infinity reveals that lines projecting a world point to an image point are *all parallel* and horizontal, heading “East and West”. One consequence is that the entire ground line projects to one point.

But looking down from (projecting up in) a different direction (say from the NorthWest instead of West) will make an (infinite) image of the whole ground line. In fact, seen from straight up, the image of the ground plane is exactly the ground plane itself. In a 3-D world with a 2-D ground plane and image, convince yourself that square tiles will project in OPP to identical (size and shape) PGS, no matter their distance from the image plane. The tiles are foreshortened but not in perspective (WPCF explains the difference). Without knowing the direction of projection relative to the ground plane, we have no idea of the actual shapes of PG tiles on the ground: e.g. the choice of viewpoint and view direction can image a square tile to any specified PG.

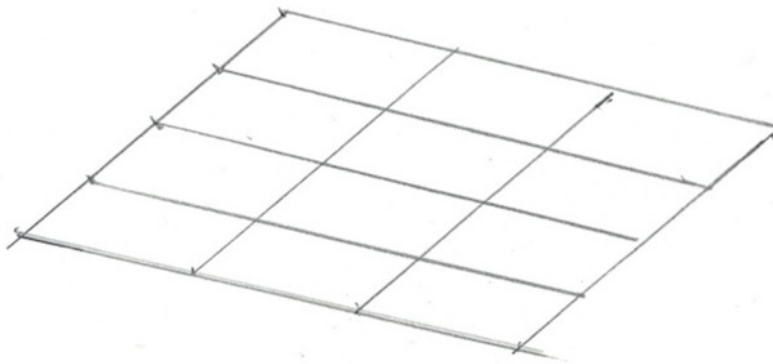


Fig. 5. Identical (forgive the drawing) PG tile images in orthographic projection (OPP) of a tiling with PGs of arbitrary and unknowable shape .

Section 5. Square Tilings in 2PP

Assume we have two VPs and an origin point O (near corner) so we have $H1$ and $H2$. On $L1$ the length of the image of a square tile side is $A1$ (Fig. 6a). If we knew the correct $B1$ we could use ST to extend them both to get $A2, A3, \dots$ and $B2, B3, \dots$ and we'd be done. (We'll discover a shortcut later in the section). *The new idea here is to derive $B1$ from $A1$ and known constraints* between them (here, they both originate from equal tile side lengths — we'll add a bit more complexity in Section 6.). The answer for square tiles is given below, preceded by its rationale and derivation.

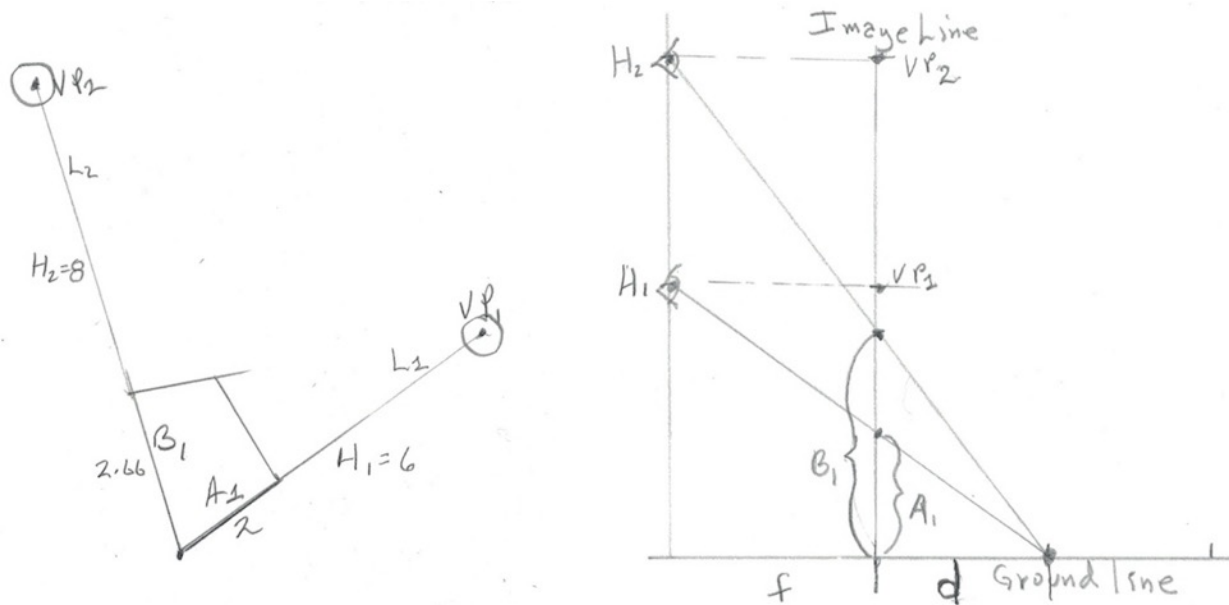


Fig. 6. a. To start, we know only VP1, VP2, (circled) and we pick an A1 (dark line) that we choose to be the image of a square's side on L1 (A1 is a free artistic choice.) We want to know B1 on L2. b. Here image lines L1 and L2 are superimposed. The relation of A1 to B1 for equal lengths (sides of a square of length 1 here) is determined by H1 and H2, using the ST property.

We get B1 using Fig. 6b, which shows both H1 and H2 situated, as usual, on a line at distance f to the left of the image line (we won't need f 's value, you'll see). We know A1 on L1 so we can draw the line through H1 and A1 to get d , the world size (given our f) of the side of the square tile.

That same d is being imaged on L2 to the length B1. We can use the ST property (Section 1) or read off Fig. 6b that

$$A1/d = H1/(f+d). \text{ Likewise,}$$

$$B1/d = H2/(f+d).$$

We can solve for A1 and B1 and then divide the solutions to get first $A1 = dH1/(f+d)$, $B1 = dH2/(f+d)$, and then $B1/A1 = H2/H1$, so we get the

ST Square Tiling Rule: $B1 = A1(H2/H1)$,

which is computable from things we know. We've just discovered that *the images of square sides are proportional to their VP distances*. Thus we do not need to use the "raw ST method" of Section 2 to extend the Bi: we compute them from Ai and $(H2/H1)$.

Section 6. Hexagonal and Octagonal Tilings in 2PP

Hexagons and “Octagons and diamonds” are popular tilings (Fig. 7). Let’s figure out how to draw regular hexagons in 2PP (hence in 1PP). The question is harder to pose than for the square since we need to know about hexagon dimensions (coming up soon).

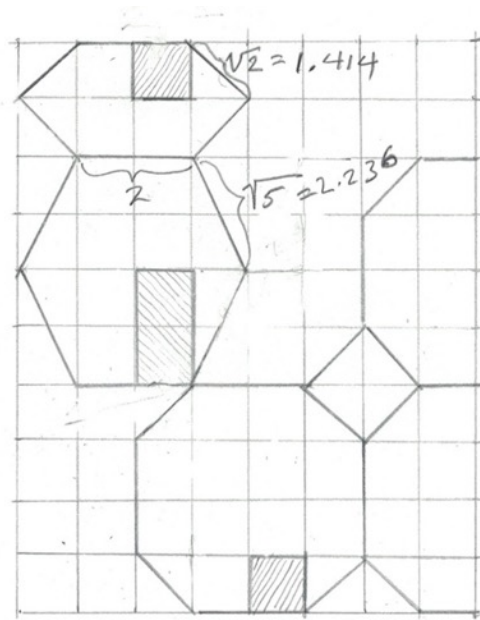


Fig. 7: Plan view (not perspective) of the “regular octagon and diamond” tile pattern and two irregular hexagonal tiles; UPGs are shaded. Hexagon side lengths are shown: the $\sqrt{2}$ and $\sqrt{5}$ are from Pythagoras. These tilings are easy to draw on a square grid, but don’t yield regular hexagonal tiles. Spoiler: if the distance between horizontal grid lines were to shrink the right amount, the tall hexagon could be squished into being regular (Fig. 8).

In Fig. 7, Clearly the plane can be tiled with any of the three patterns. The *underlying PG* (UPG) is the building block of a tiling: tile sides are parallel to UPG sides or are diagonals of UPGs, which are called *diagonal lines*. In Fig. 7, the skinny hexagon and octagon have a square UPG and the tall hexagon has a UPG that is one square wide and two squares high. The “body” of both hexagons is two UPGs wide and two wide. The body of the octagon is a cross of 12 UPGs. To put the tiling into 1PP or 2PP, see Sections 2 and 3: we only need to create the Ai and Bi for the UPGs: then the diagonal lines of the perspective UPGs are “carried along” in the same perspective.

Diagonal lines are an important concept (known since the mid 1400’s — see Section 7.). They aren’t part of the ST method but they expand our repertoire of tile shapes without adding

technical or computational difficulties. Diagonal lines parallel in the world must have their own vanishing points in a perspective drawing. But these are not “first-class VPs”, and so need a different name, which is *distance points* (DPs).

2PP has two VPs, period. In tilings, they are naturally associated with the (horizontal and vertical) sides of a tile’s UPG. We probably remember (or see WPCF or most any drawing book) that *VP1 and VP2 determine a horizon* toward which the whole ground plane is vanishing. That means any two lines parallel on the ground plane *meet at the horizon*, not just in any random new VP. So given VP1 and VP2, the distance points of other parallel lines (like diagonal lines) are constrained to be collinear with VP1 and VP2, on the horizon.

But say we feel creative and daring and introduce another VP3 for parallel lines in our (hexagon, say) tiling that has equal status with VP1 and VP2. It “don’t need no stinkin’ horizon”, and indeed is not on their horizon (is not a DP). Well, we quickly get in trouble. It’s not guaranteed we can even draw a hexagon using independent VP1, VP2, and VP3. If we do, it could look superficially like a hexagon in perspective, but it is fatally malformed: it cannot arise from PP (can’t occur “in nature”), and using the ST method on its three sides gives incoherent results.

We’ll use the geometrical facts of Fig. 8 and the idea of constraints between sides (see Section 5) to produce regular hexagonal tilings that can be extended using diagonal lines (coming up next).

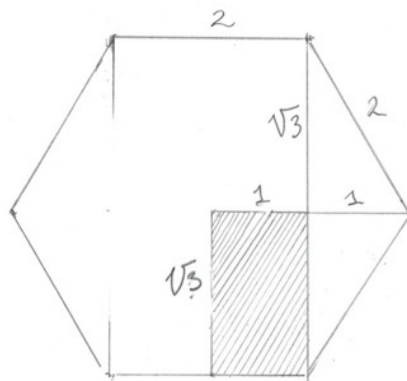


Fig. 8. Basic distances in a regular hexagon. It is like the tall hexagon of Fig. 8, but its (hatched) UPG measures 1 by $\sqrt{3}$ (Pythagoras again) or 1 by about 1.732.

Regular Hexagonal Tiling with ST in 2PP:

Here is my recipe: it works “in general”. In particular the value $\sqrt{3}$ comes from our desire for a regular hexagon.

1. Know your UPG dimensions: It's simple and general to let one dimension be 1. For the regular hexagon we want (from Fig. 8) a UPG of 1 by $\sqrt{3}$. Find or choose O, VP1 and VP2, measure H1 and H2, and decide how long one of the UPG sides (called A) should be in the drawing (Fig 9a). In my example H1 = 32.5cm, H2= 27.5cm, and I want A = 3cm on L1 to be the imaged size of the UPG side of size 1. Fig. 9a shows our drawing at this point: we need to find the right B on L2, then to extend the tiling using ST (Section 3), then to draw diagonals and find hexagon tile images.

2. It helps to make a familiar diagram or two just to keep our head straight (Fig 9b). They (or it) needn't be to scale, the idea is just to associate lengths with names so we can calculate B. We know (or choose) all lengths except B, which as usual we need to find.

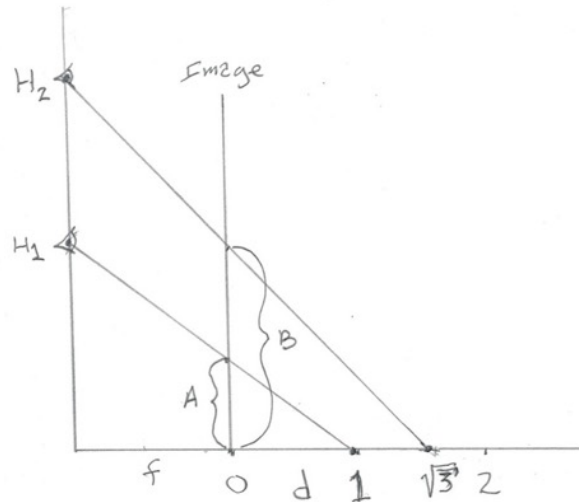
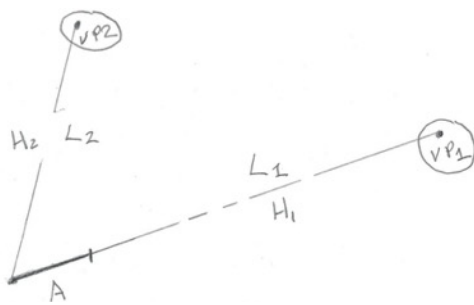


Fig. 9. a: The initial choices for a regular hexagonal tiling: VP1, VP2, H1, H2, and A, the image along L1 we choose for the UPG side of length 1. b: Two of our usual pairs of similar triangles showing H1, H2, f, d=1 and d= $\sqrt{3}$, A and B. The image lines are superimposed here, but are independent; each depends on its H. What's important is the ST relationship of lengths we can read off the diagrams.

Convince yourself that Fig.9a embodies what we know: PPG lengths of 1 and $\sqrt{3}$ on the ground line, H1 and H2 the viewpoint heights respectively, A the imaged side length we chose in step 1. There's no value for B; we need to calculate it. In Fig 6b, H2 is bigger than H1 to separate the two sets of similar triangles, but only the relationships and stated lengths in the triangles matter, not their scale.

3. So, let's deal with the relationships:

a. Directly from the diagram and ST: $A/1 = H1/(f+1)$, $B/\sqrt{3} = H2/(f + \sqrt{3})$.

b. Multiplying first equation by 1, second by $\sqrt{3}$: $A=H1/(f+1)$, $B = \sqrt{3} H2/(f + \sqrt{3})$.

c. Dividing, $B/A = [\sqrt{3} H2 (f+1)] / [H1(f + \sqrt{3})]$.

d. So multiplying both sides by A, we get

$$B = A [\sqrt{3} H2 (f+1)] / [H1(f + \sqrt{3})]$$

e. You recall from Section 2 or just read off Fig. 9b and use ST:

$$(f+1)/H1=1/A, \text{ so}$$

$$f = (H1/A) - 1.$$

Now we

f. Substitute the value of f from step e into the formula of step d to get B from A. Calculate B and proceed as in Section 3, or just notice that, as for square tiles, B_i is proportional to A_i : $B_i = A_i(B/A)$. Extend A (renamed $A1$) and B ($B1$). Draw diagonals, Find Hexagons.

4. We're done. See Fig. 10 (after Section 7.)

7. Insultingly Brief Historical Note

Filippo Brunelleschi (1377-1446) was a designer, architect, and maybe the first modern engineer, an influential figure in the renaissance. He is credited with developing linear perspective (around 1413) and using viewing grids (WPCF, Fig. 1) but the first paintings with accurate linear perspective may be attributed to Ambrogio Lorenzetti (Annunciation, 1344).

Distance points in the tiling context show up in the treatise of Jean Pelerin (1445 – 1522), also known as the Viator. Entitled *De Artificiali Perspectiva*, it was first published in Toul in 1505 and later a in a 2nd (pirated?) edition (Nuremberg, 1509). It produces the same results as those of Leon Battista Alberti (1407? – 1472), but constructs the tiles differently. Alberti's construction was written in 1435-36 and titled *Il Trattato della Pittura e Cinque Ordini Architettonici*, or *De Pictura*, or *On Painting*.

Answer to Exercise at end of Section 3: Classical method in 2- or 3PP.

This method works because of the parallel lines in a 1PP tiling (verify). In 2PP or 3PP drawing (we're on a plane so even in a 3PP drawing we're in 2PP), given $L1, L2, A1, B1$: make a meta-construction by drawing a line $L1a$ through A's VP next to $L1$ and duplicate some 1PP tiling with parallel lines (any direction) from O and $A1$ over to $L1a$. Do the classical construction on the meta-construction Repeat for $L2$ and $B1$, and $L2$'s VP. Erase all (meta-) construction lines.

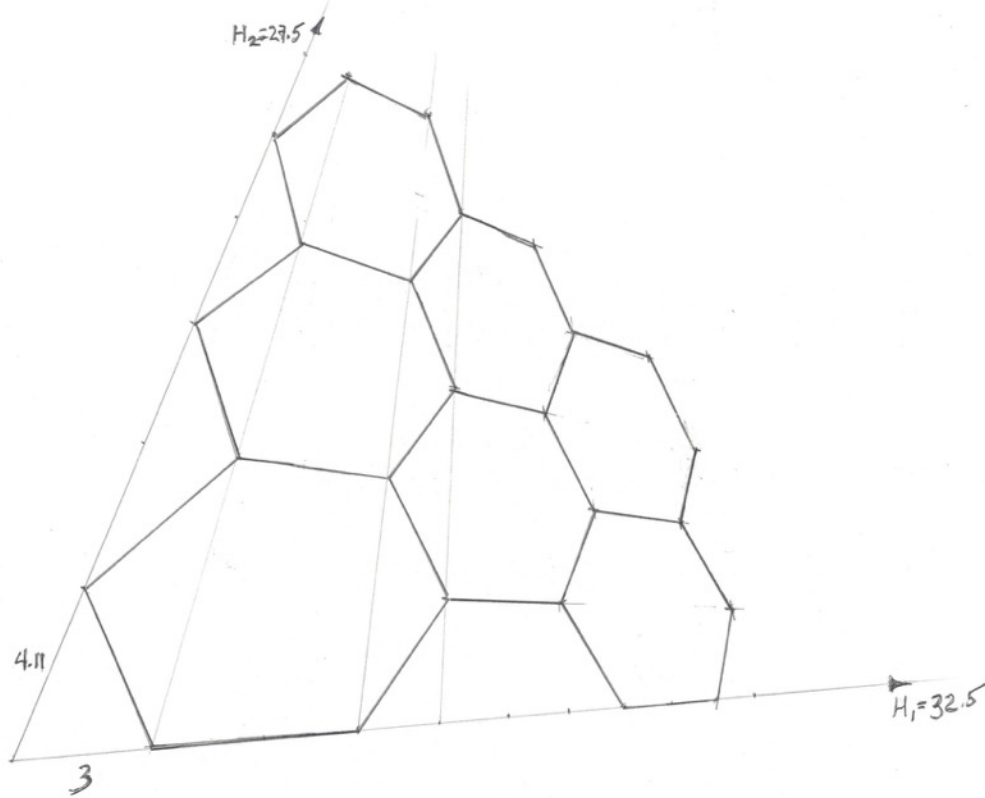


Fig. 10b

Table 1

Inputs	const sqrt(3)	Intermed Results	index	Ai	Bi
H1	32.5	1.73205080 f	9.83333333	1	3 4.118444
H2	27.5	B/A numerator	516.006803	2	5.49295 7.540814
A	3	B/A denominator	375.874984	3	7.59740 10.42982
		B/A	1.37281496	4	9.39759 12.90115
		B	4.11844488	5	10.9550 15.03926
				6	12.3157 16.90730
				7	13.5148 18.55339
				8	14.5794 20.01487
				9	15.5309 21.32115
				10	16.3865 22.49570

Fig. 10.

- a: A regular hexagonal tiling in 2PP using lengths chosen in step 1 above (see text).
- b: The spreadsheet for the calculations.