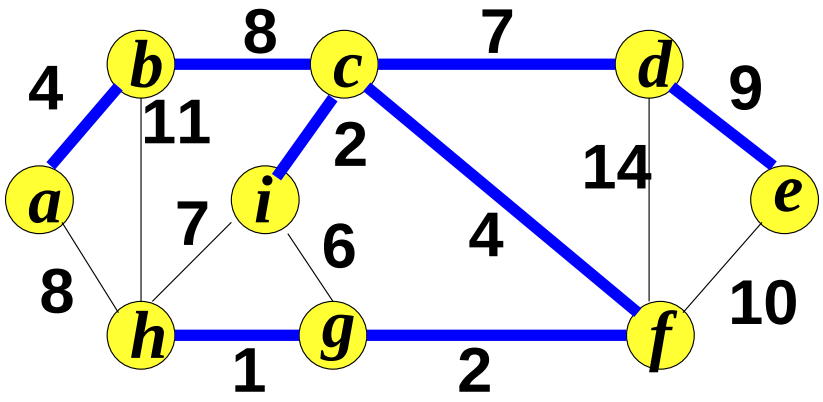
Comparison of Efficiency

Procedure	Binary (worst-case)	Binomial (worst-case)	(amortized)
Make-Heap	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
Insert	$\Theta(\lg n)$	$O(\lg n)$	$\Theta(1)$
Minimum	$\Theta(1)$	$O(\lg n)$	$\Theta(1)$
Extract-Min	$\Theta(\lg n)$	$\Theta(\lg n)$	$O(\lg n)$
Union	$\Theta(n)$	$O(\lg n)$	$\Theta(1)$
Decrease-Key	$\Theta(\lg n)$	$\Theta(\lg n)$	$\Theta(1)$
Delete	$\Theta(\lg n)$	$\Theta(\lg n)$	$O(\lg n)$

Chapter 23: Minimum Spanning Tree

Let $G = (V, E)$ be a connected (undirected) graph. A spanning tree of G is a tree T that consists of edges of G and connects every pair of nodes.

Let w be an integer edge-weight function. A minimum-weight spanning-tree is a tree whose weight weight respect to w is the smallest of all spanning trees of G.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Safe edges and cuts

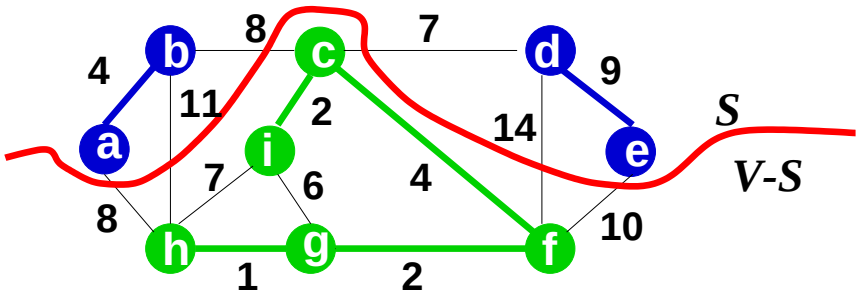
A : expandable to an MST

$e \in E - A$ is **safe** for A if $A \cup \{e\}$: expandable to an MST or an MST already

a **cut** of G : a partition $(S, V - S)$ of V

an edge e **crosses** $(S, V - S)$ if e connects a node in S and one in $V - S$

$(S, V - S)$ **respects** $A \subseteq E$ if no edges in A cross the cut



For any edge property Q, a **light edge** w.r.t. Q is one with the smallest weight among those with the property Q

Theorem A Let $G = (V, E)$ be a connected (undirected) graph with edge-weight function w. Let A be a set expandable to an MST, let $(S, V - S)$ be a cut respecting A, and let $e = (u, v)$ be a light edge crossing the cut. Then e is safe for A.

Proof Let T be an MST containing A and not containing e. There is a unique path ρ in T from u to v. ρ has an edge crossing $(S, V - S)$. Pick one such edge d. Then $T' = T \cup \{e\} - \{d\}$ is a spanning tree such that $w(T') = w(T)$ so T' is an MST and e is safe. ∎

Corollary B Every light edge connecting two distinct components in $G_A = (V, A)$ is safe for A.

Kruskal's Algorithm

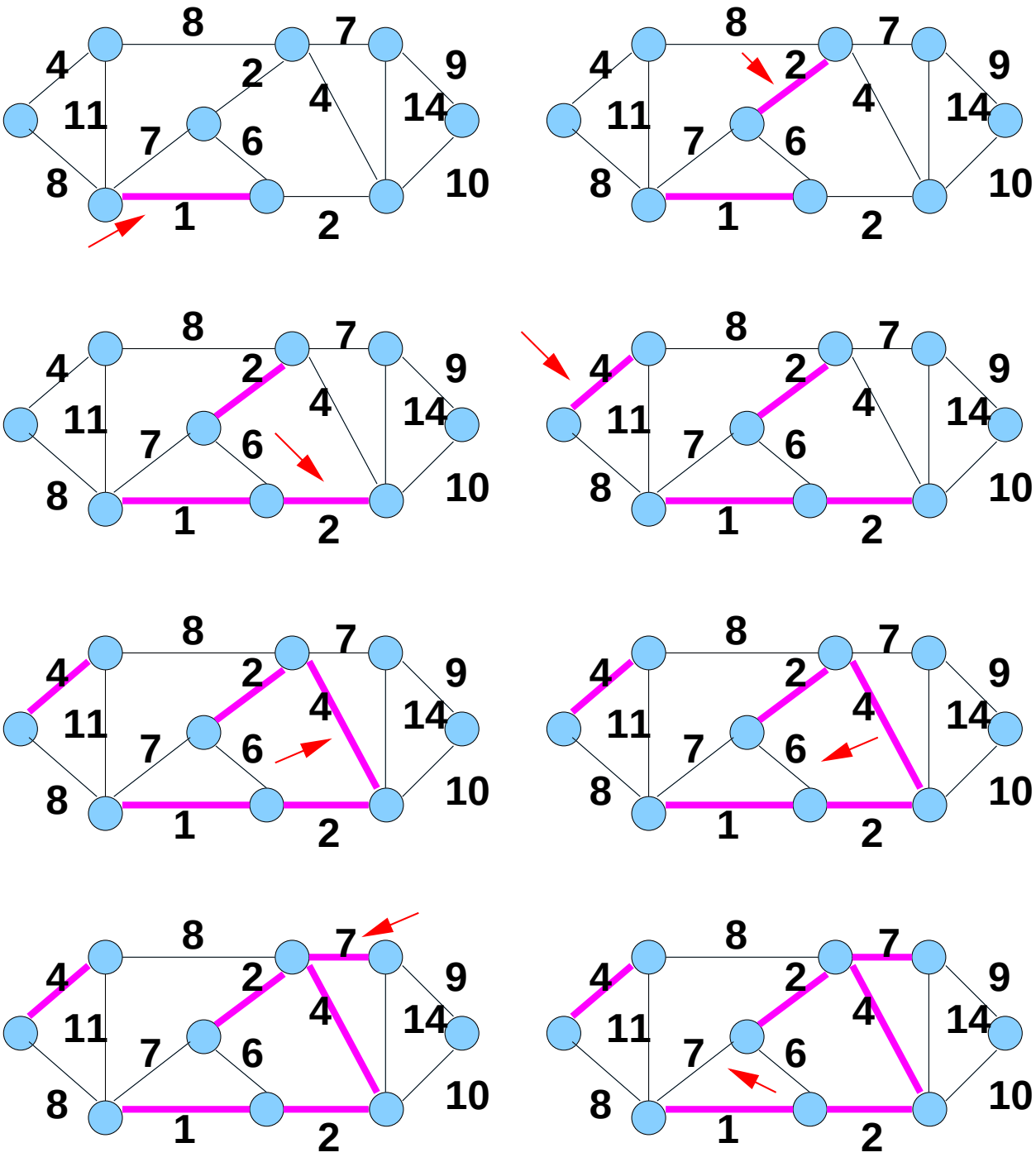
Maintain a collection of connected components and construct an MST A.

Initially, each node is a connected component and $A = \emptyset$.

Examine all the edges in the **nondecreasing order of weights**.

- If the current edge connects two different components, add e to A to unite the two components.

The added edge is a light edge; otherwise, an edge with smaller weight should have already united the two components.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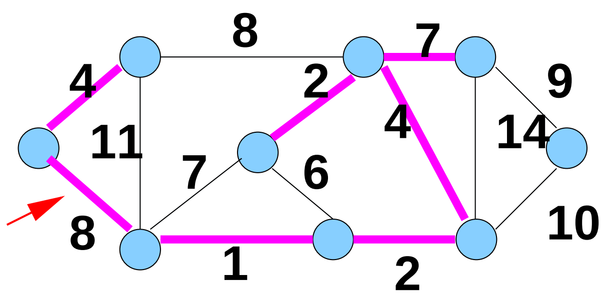
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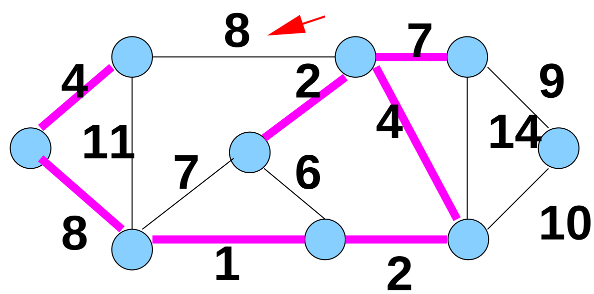
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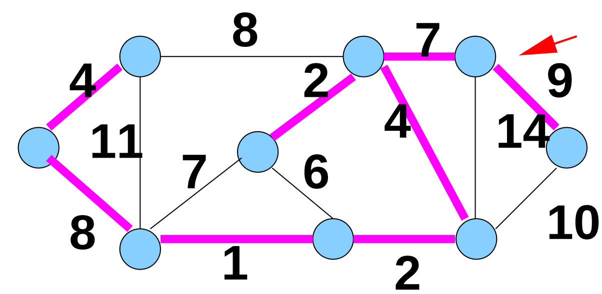
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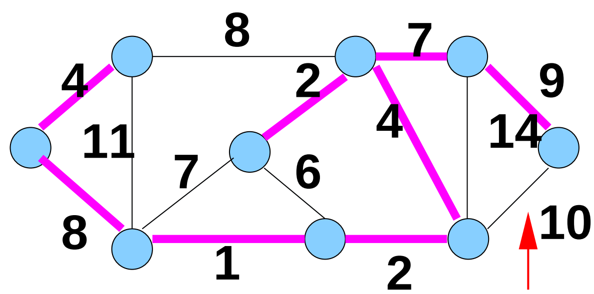
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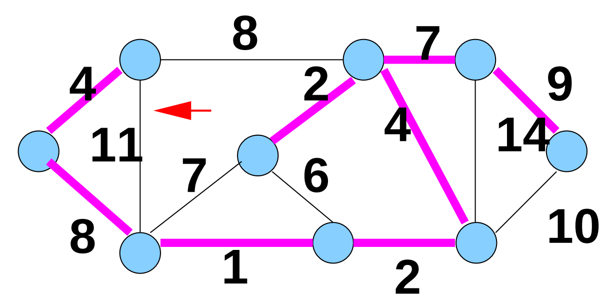
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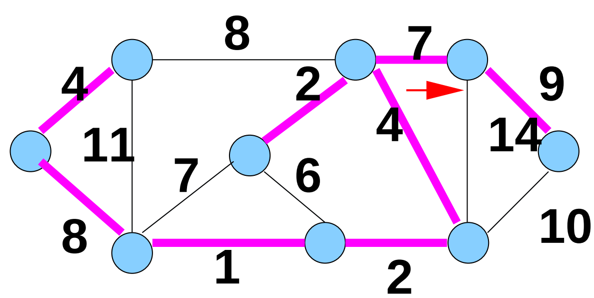
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Implementation with "disjoint-sets"

```
1   A ← ∅
2   for each vertex v ∈ V do
3       Make-Set(v)
4   reorder the edges so their weights are
        in nondecreasing order
5   for each edge (u, v) ∈ E in the order do
6       if Find-Set(u) ≠ Find-Set(v) then
7           A ← A ∪ {(u, v)}
8           Union(u, v)
9   return A
```

The number of disjoint-set operations that are executed is $2E + 2V - 1 = O(E)$, out of which V are **Make-Set** operations.

What is the total running time?

The total cost of the disjoint-set operation is $O(E \lg^* V)$ if the union-by-rank and the path-compression heuristics are used.

Sorting the edges requires $O(E \log E)$ steps.

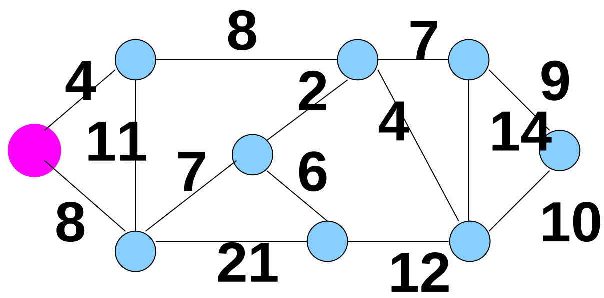
We can assume $E \geq V - 1$ and $E \leq V^2$.

So, it's $O(E \log V)$ steps.

Prim's algorithm

Maintain a set of edges A and a set of nodes B. Pick any node r as the root and set B to $\{r\}$. Set A to $\emptyset$. Then repeat the following $V - 1$ times:

- Find a light edge $e = (u, v)$ connecting $u \in B$ and $v \in V - B$.
- Put e in A and v in B.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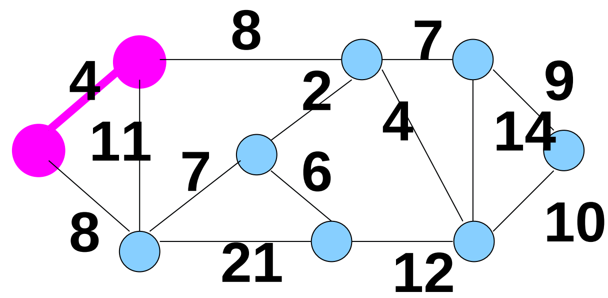
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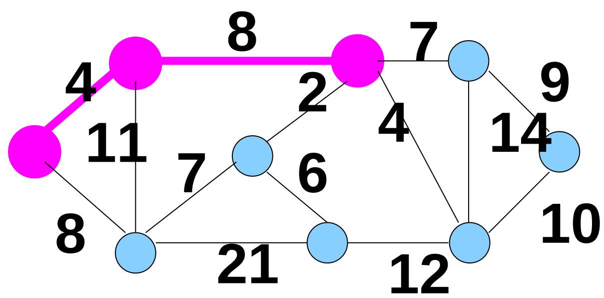
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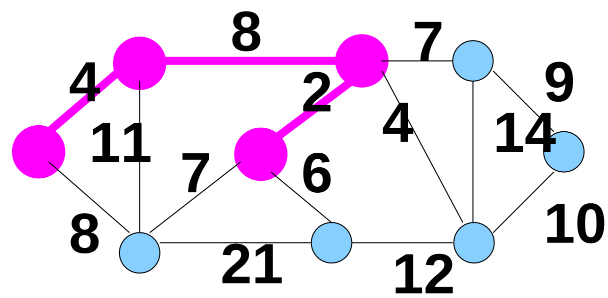
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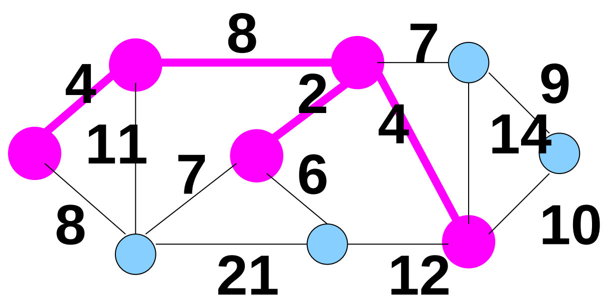
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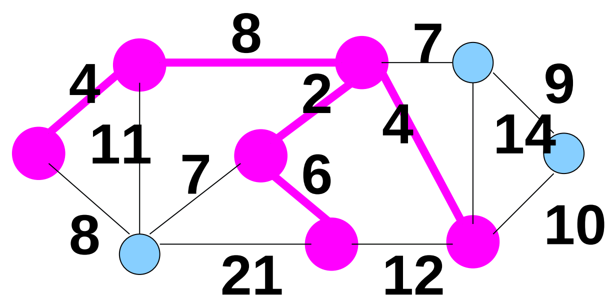
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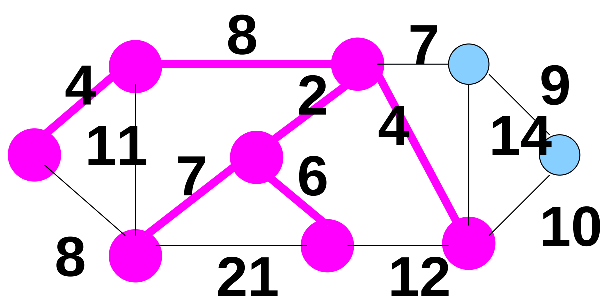
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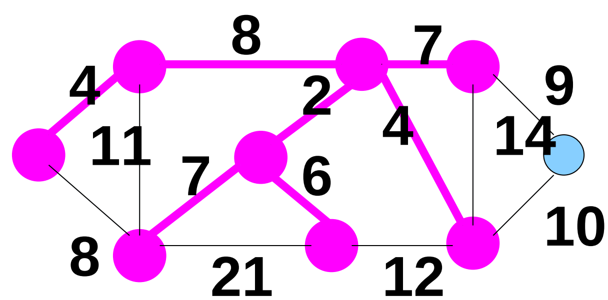
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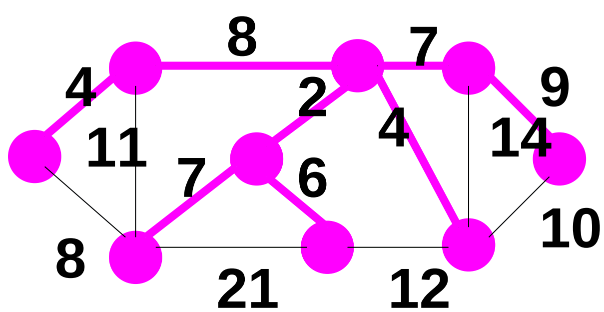
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Implementation Using a Priority Queue

For each node in Q, let $key[v]$ be the **minimum edge weight connecting** v to a node in B. By convention, $key[v] = \infty$ if there is no such edge.

For each node v record the parent in the field $\pi[v]$. This is the node u such that (u, v) is the light edge when v is added to B.

An implicit definition of A is

$$\{(v, \pi[v]) \mid v \in V - \{r\} - Q\}.$$

```
1    Q ← V
2    for each u ∈ Q do key[u] ← ∞
3    key[r] ← 0
4    π[r] ← nil
5    while Q ≠ ∅ do
6        u ← Extract-Min(Q)
7        for each v ∈ Adj[u] do
8            if v ∈ Q and w(u, v) < key[v] then
9                π[v] ← u
10               key[v] ← w(u, v)
```

Line 3 forces to select r first. Lines 7-10 are for updating the keys.

Implement Q using a heap. The running time is

$$
\begin{aligned}
& V \cdot (\text{the cost of } \textbf{Build-Heap}) \\
+ \;& (V-1) \cdot (\text{the cost of } \textbf{Extract-Min}) \\
+ \;& E \cdot (\text{the cost of } \textbf{Decrease-Key}).
\end{aligned}
$$

If either a binary heap or a binomial heap is used, the running time is:

$$\begin{aligned} & V \cdot O(1) \\ + \;& (V-1) \cdot O(\lg V) \\ + \;& E \cdot O(\lg V) \\ = \;& O((E+V) \lg V) = O(E \lg E), \end{aligned}$$

which is the same as the running time of Kruskal's algorithm.

If a Fibonacci heap is used, the running time is:

$$\begin{aligned} & V \cdot O(1) \\ + \;& (V-1) \cdot O(\lg V) \\ + \;& E \cdot O(1) \\ = \;& O(V \lg V + E), \end{aligned}$$

which is better than the running time of Kruskal's algorithm.