The Art of Data Structures

Recursion

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CSC 162: The Art of Data Structures
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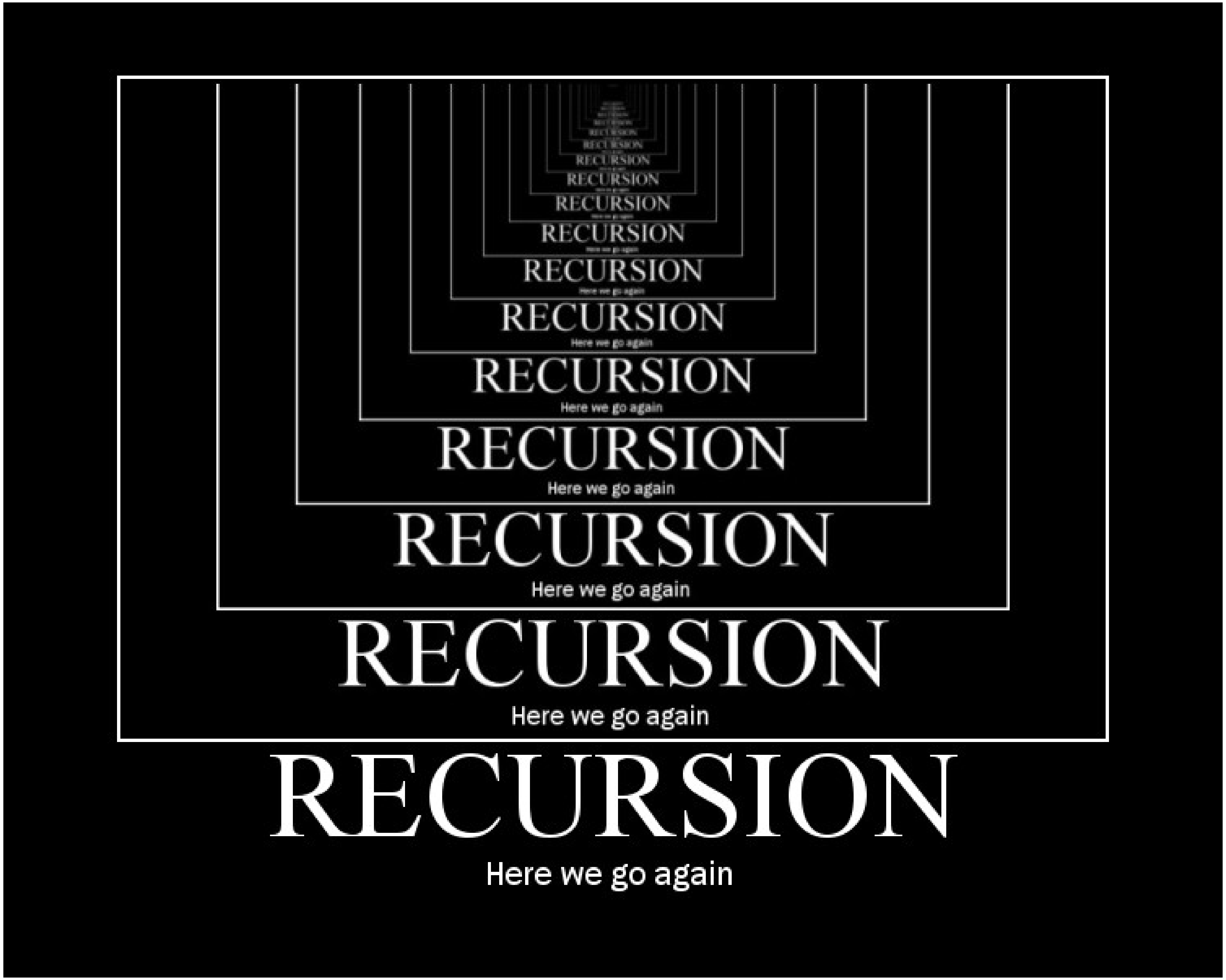
Agenda

- To understand that complex, difficult problems that may have a simple recursive solutions
- To learn how to formulate programs recursively
- To understand and apply the three laws of recursion
- To understand recursion as a form of iteration
- To implement the recursive formulation of a problem
- To understand how recursion is implemented by a computer system

Recursion


RECURSION
Here we go again
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Recursion

- A description of something that refers to itself is called a *recursive definition (like this one for example)*

Recursion

- In mathematics, recursion is frequently used
- The most common example is the factorial:

  For example, 5! = 5(4)(3)(2)(1)

Recursion

- In other words,

$$n! = n(n-1)!$$

- Or

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n(n-1)! & \text{otherwise} \end{cases}$$

- This definition says that `0!` is `1`, while the factorial of any other number is that number times the factorial of one less than that number

Recursion

- Our definition is recursive, but definitely not circular. Consider 4!
  - 4! = 4(4-1)! = 4(3!)
  - What is 3!? We apply the definition again
    4! = 4(3!) = 4[3(3-1)!] = 4(3)(2!)
    4! = 4(3!) = 4(3)(2!) = 4(3)(2)(1!) = 4(3)(2)(1)(0!) = 4(3)(2)(1)(1) = 24

Recursion

- Factorial is not circular because we eventually get to 0!, whose definition does not rely on the definition of factorial and is just 1
- This is called a *base case* for the recursion
- When the base case is encountered, we get a closed expression that can be directly (often, trivially) computed

Recursion

Numbers Sum

- Suppose that you want to calculate the sum of a list of number, e.g. `[1, 3, 5, 7, 9]`

Recursion

Numbers Sum: Iterative

```
def list_sum(num_list):
    the_sum = 0
    for i in num_list:
        the_sum = the_sum + i
    return the_sum

print(list_sum([1,3,5,7,9]))
```

Recursion

Numbers Sum

- total = (1+(3+(5+(7+9))))
  total = (1+(3+(5+16)))
  total = (1+(3+21))
  total = (1 + 24)
  total = 25

- listSum(numList) =
  first(numList) +
  listSum(rest(numList))

Recursion

Numbers Sum: Recursive

```
def list_sum_rec(num_list):
  if len(num_list) == 1:
      return num_list[0]
  else:
      return num_list[0] + list_sum_rec(num_list[1:])

print(list_sum_rec([1,3,5,7,9]))
```

Recursion

- This is inefficient (lots of copying), but Python runs out of stack to keep track of the calls before the cost gets out of hand
- Note that the two implementations actually sum the elements in opposite order
  - We could make them do it in the same order like this:

Recursion

Numbers Sum: Recursive (better)

```
def listsum_rec2(the_sum, l):
    if len(l) == 0:
        return the_sum
    return listsum_rec2(the_sum + l[0], l[1:])

print(listsum_rec2(0, [1,3,5,7,9]))
```

Recursion

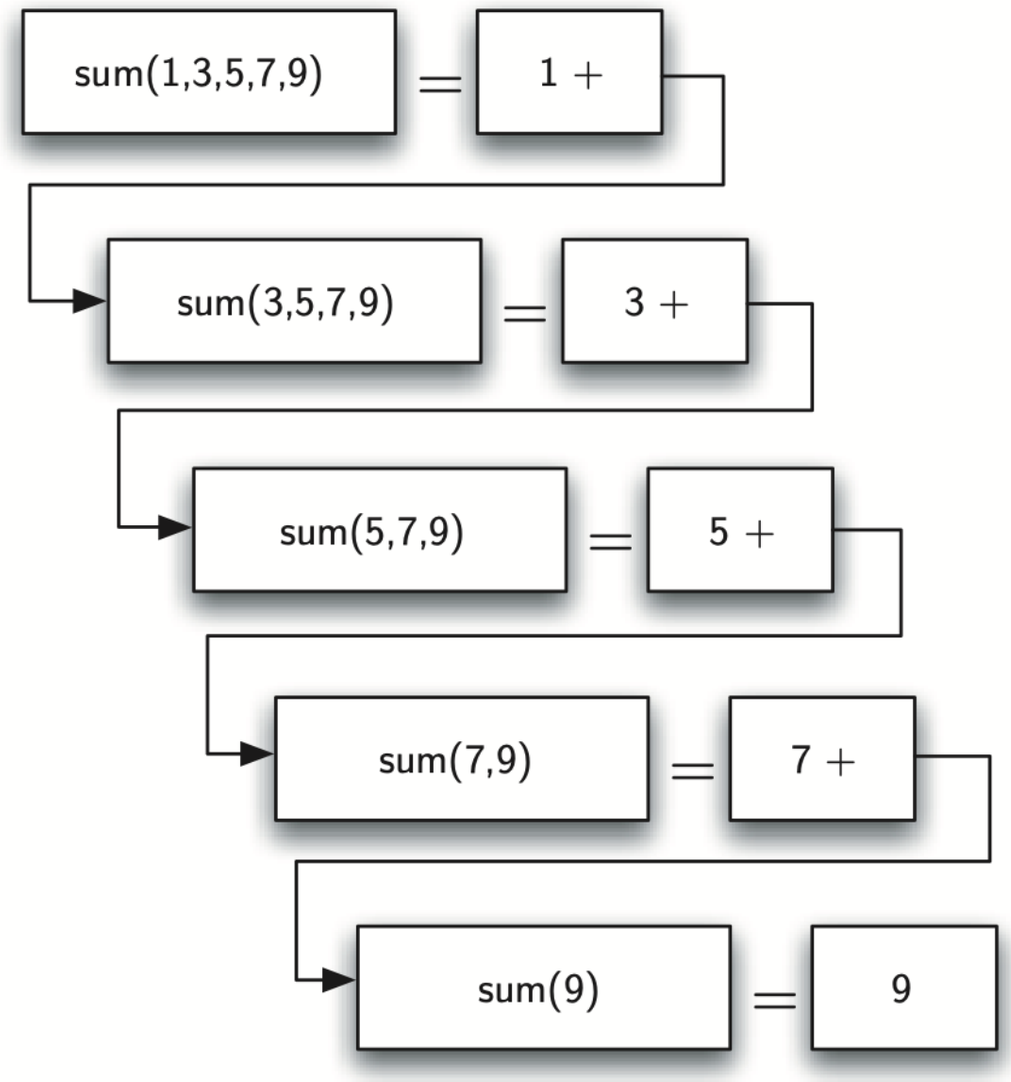
The Recursive Sum Function

```
def helper(sum, l):
    if len(l) == 0:
        return sum
    return helper(sum + l[0], l[1:])

def listsum_rec3(l):
    return helper(0, l)
```

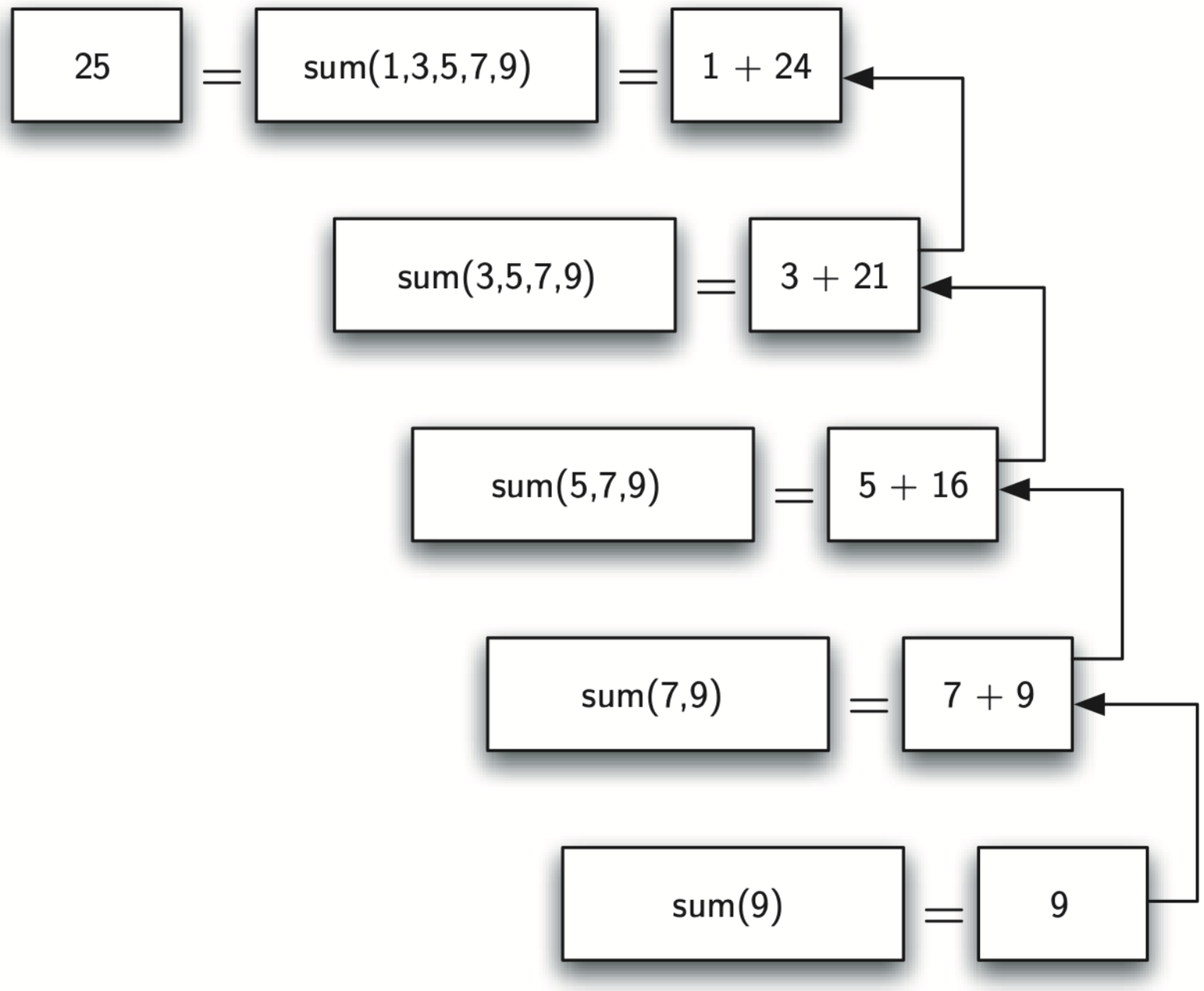
Recursion

Recursive Calls Adding a List of Numbers



Recursion

Recursive Calls Adding a List of Numbers



Recursion

The Three Laws of Recursion

1. A recursive algorithm must have a base case
2. A recursive algorithm must change its state and move toward the base case
3. A recursive algorithm must call itself, recursively

Recursion
Areas of Use

- Two fundamental computational concepts
  - Divide & Conquer: Solve a problem in terms of a smaller version of itself
  - Backtracking: Systematically explore a set of possible solutions

Factorial

Factorial

- We've seen previously that `fact` can be calculated using a loop accumulator
- If `fact` is written recursively...

Factorial

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```

Factorial

- We’ve written a function that calls itself, a *recursive function*
- The function first checks to see if we’re at the base case (`n==0`). If so, return `1`
- Otherwise, return the result of multiplying `n` by the factorial of `n-1`, `fact(n-1)`

Factorial

```
>>> fact(4)
24
>>> fact(10)
3628800
>>> fact(100)
933262154439441526816992388562667004907159682643816214685929638952175999932299156089414639761565182862536979208272237582511852109168640000000000000000000000L
>>>
```

- Remember that each call to a function starts that function anew, with its own copies of local variables and parameters

Factorial

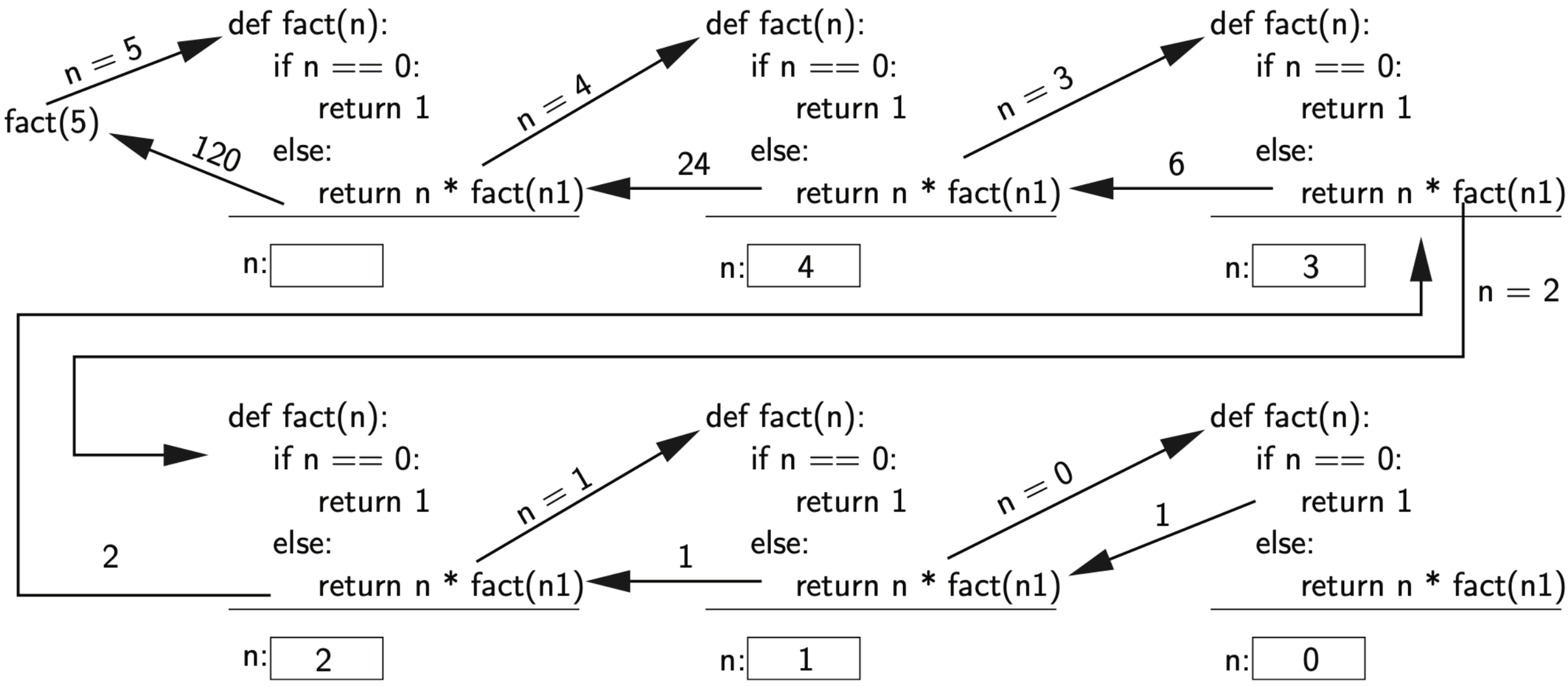


Figure 13.1: Recursive computation of 5!

Fibonacci

Fibonacci

- Sometimes one has to be careful with recursion
- In addition to limits on stack depth [`sys.setrecursionlimit(`*`limit`*`)`], and the cost of argument copying, some naive recursive algorithms are inherently expensive

Fibonacci

Iterative Fibonacci function – *O(n)*

```
def fib_iter(n):
    a = 0
    b = 1
    i = 0

    print(a)
    while i < n:
        t = a+b
        a = b
        b = t
        print(a)
        i += 1

    return a
```

Fibonacci

Naive recursive version, $O(2^n)$

```
def fib_rec_1(n):
if n < 2:
    return 1

return fib_rec_1(n-1) + fib_rec1(n-2)
```

Fibonacci

Good recursive version, O(n)

```
def fib_rec_2(n):
    def helper(a, b, i):
        if i == n:
            return b
        return helper(b, a + b, i + 1)

    return helper(1, 0, 0)
```

Recursion Exercise

Take this recursive function:

```
1  def fun(n):
2      if n == 0:
3          return []
4
5      return fun(n//2) + [n%2]
6
7  fun(25)
```

(a) Identify the line number for the *base case* in `fun(n)`:

(b) Identify the line number, and specific *recursive call* in `fun(n)`:

(c) Use the table below to determine the output of `fun(n)` when it is called with `n = 25` (line 7):

Recursion Level	n	`fun(n)` is called with...	is `n == 0`?	line 5, calculate `n//2`	line 5, calculate `n%2`
0 (start)	25	`fun(25)`	FALSE	12	1
1	12				
2					
3					
4					
5					

(d) The final returned value from calling `fun(25)`:

fun(25)

```
1 def fun(25):
2     if n == 0:
3         return []
4
5     return fun(25//2) + [25%2]
```

```
1 def fun(12):
2     if n == 0:
3         return []
4
5     return fun(12//2) + [12%2]
```

```
1 def fun(6):
2     if n == 0:
3         return []
4
5     return fun(6//2) + [6%2]
```

```
1 def fun(3):
2     if n == 0:
3         return []
4
5     return fun(3//2) + [3%2]
```

```
1 def fun(1):
2     if n == 0:
3         return []
4
5     return fun(1//2) + [1%2]
```

```
1 def fun(0):
2     if n == 0:
3         return []
4
5     return fun(n//2) + [n%2]
```

[1,1,0,0,1]

[] + [1] + [1] + [0] + [0] + [1] ⟶ [1, 1, 0, 0, 1]

[] + [1] + [1] + [0] + [0] ⟶ [1, 1, 0, 0]

[] + [1] + [1] + [0] ⟶ [1, 1, 0]

[] + [1] + [1] ⟶ [1, 1]

[] + [1] ⟶ [1]

[]

recursion level	n	`fun(n)` is called with…	is `n == 0`?	line 5, calc `n//2`	line 5, calc `n%2`
0 (start)	25	fun(25)	FALSE	12	1
1	12	fun(12)	FALSE	6	0
2	6	fun(6)	FALSE	3	0
3	3	fun(3)	FALSE	1	1
4	1	fun(1)	FALSE	0	1
5	0	fun(0)	TRUE	nothing, it returns	nothing, it returns

Int → Str

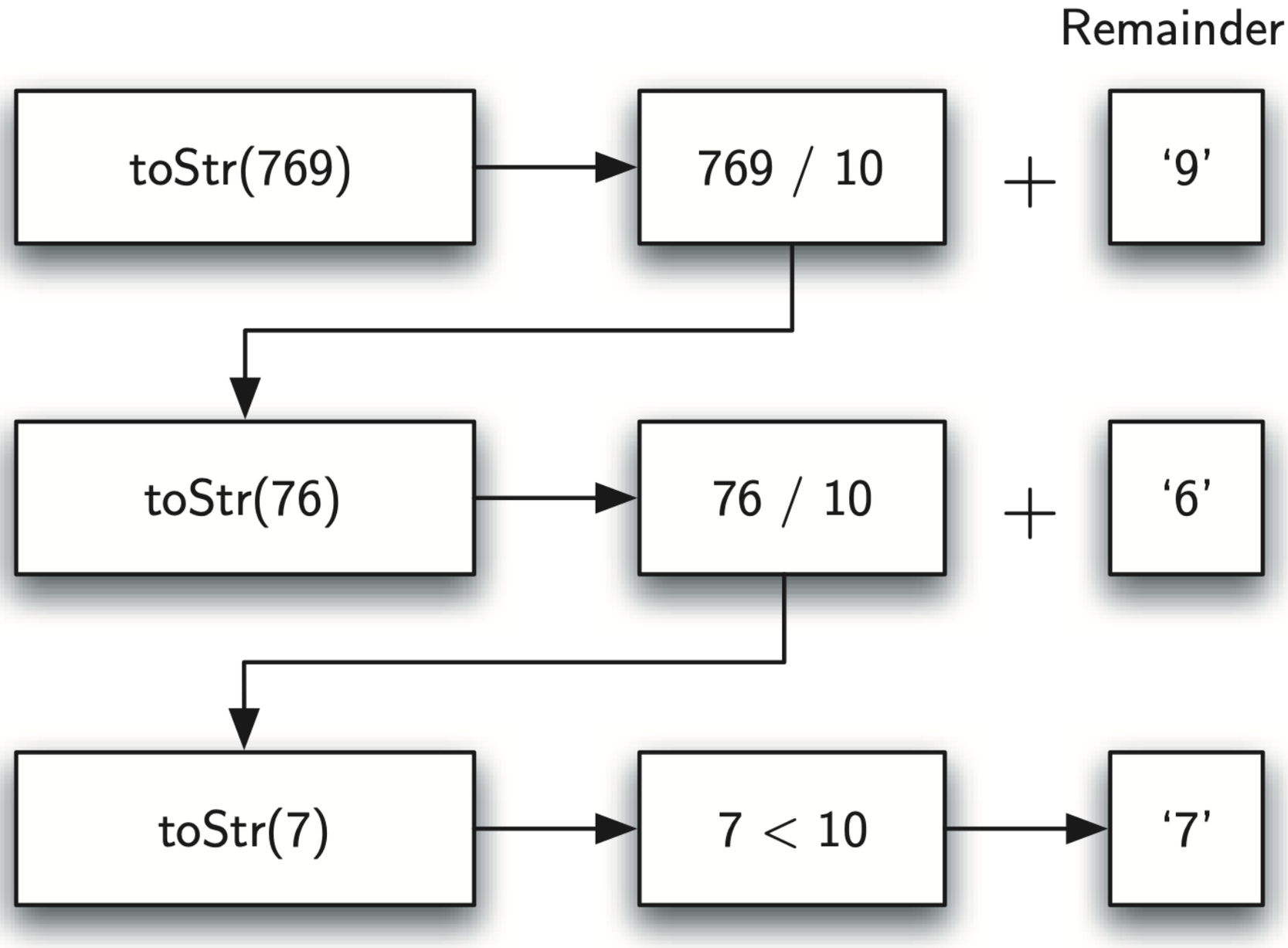
(in any base)

Int → Str
(in any base)

- Reduce the original number to a series of single-digit numbers
- Convert the single digit-number to a string using a lookup
- Concatenate the single-digit strings together to form the final result

Int → Str

in Base 10



Int → Str

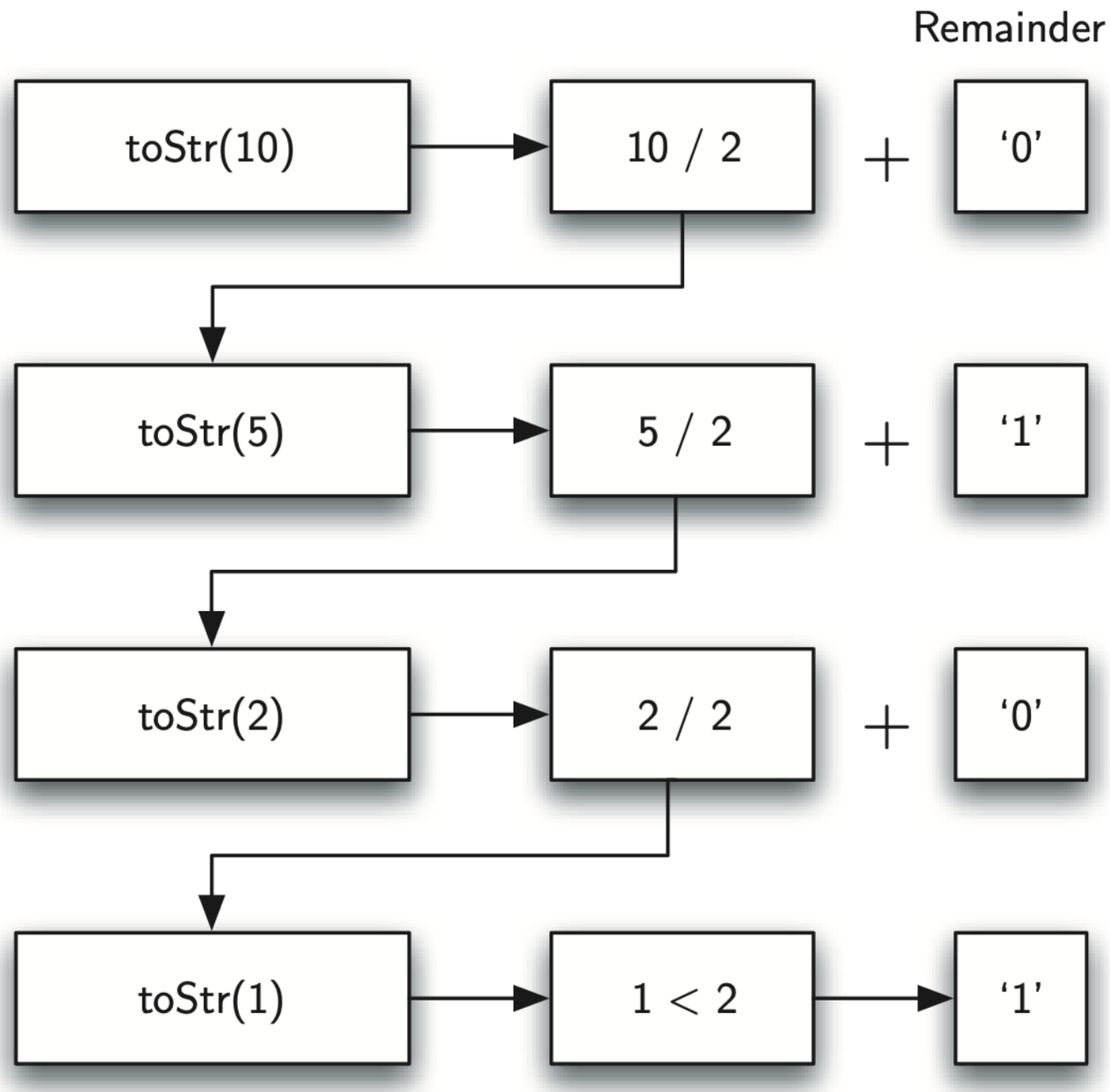
in Base 2–16

```
def to_str(n, base):
   convert_string = "0123456789ABCDEF"
   if n < base:
     return convert_string[n]
   else:
     return to_str(n // base, base) + convert_string[n % base]

print(to_str(1453, 16))
```

Int → Str

Decimal 10 to its Binary String



Int → Str

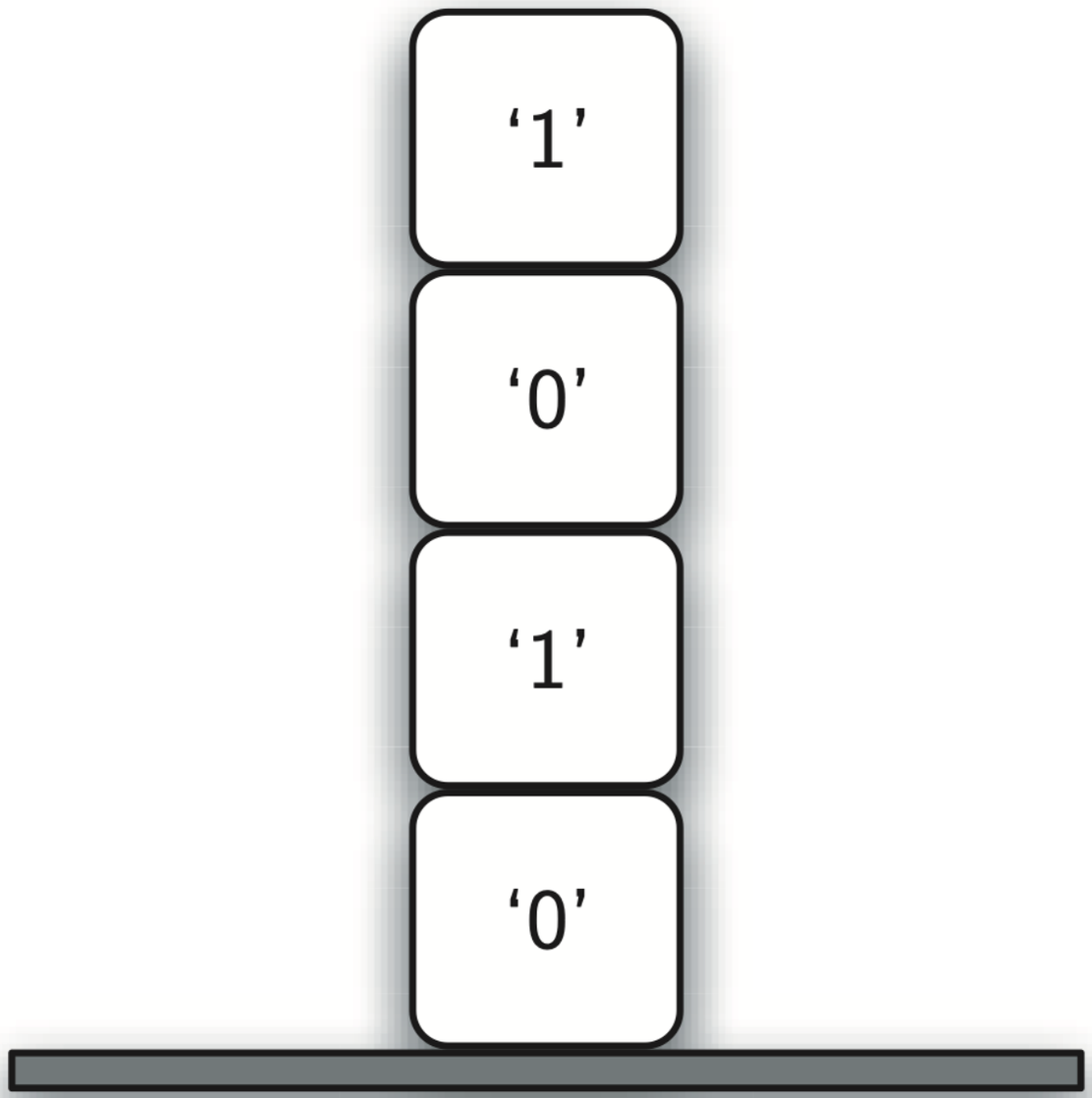
Pushing the Strings onto a Stack

```
r_stack = Stack()

def to_str(n, base):
    convert_string = "0123456789ABCDEF"
    if n < base:
        r_stack.push(convert_string[n])
    else:
        r_stack.push(convert_string[n % base])
        to_str(n // base, base)
```

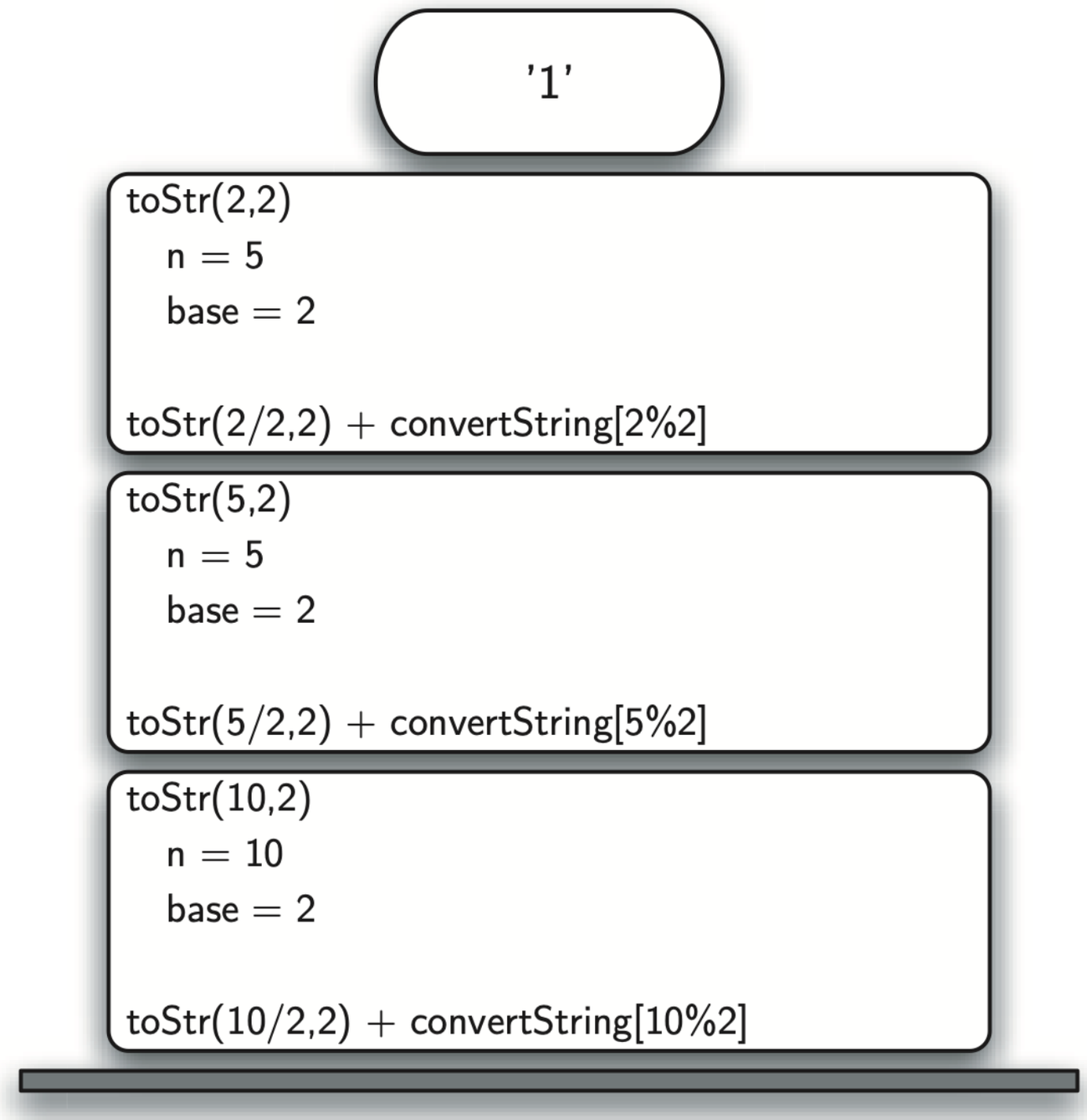
Int → Str

Strings Placed on the Stack



Int → Str

Call Stack: `toStr(10,2)`



String Permutations

Revisit Anagram Tester

String Permutations

Brute Force Anagram Testing

```
def permutation(s, prefix=""):
    n = len(s)
    if (n == 0):
        print(prefix)
    else:
        for i in range(n):
            permutation(s[0:i] + s[i+1:n], prefix + s[i])

permutation("ape")
```

Towers of Hanoi

A Complex Recursive Problem

Towers of Hanoi

Background

- Objective: move N disks from peg A to C can be reduced to three sub problems:

1. Move N-1 disks from peg A to intermediate peg B
2. Move the largest Disk N from peg A to target C
3. Move the N-1 parked disks from B to C

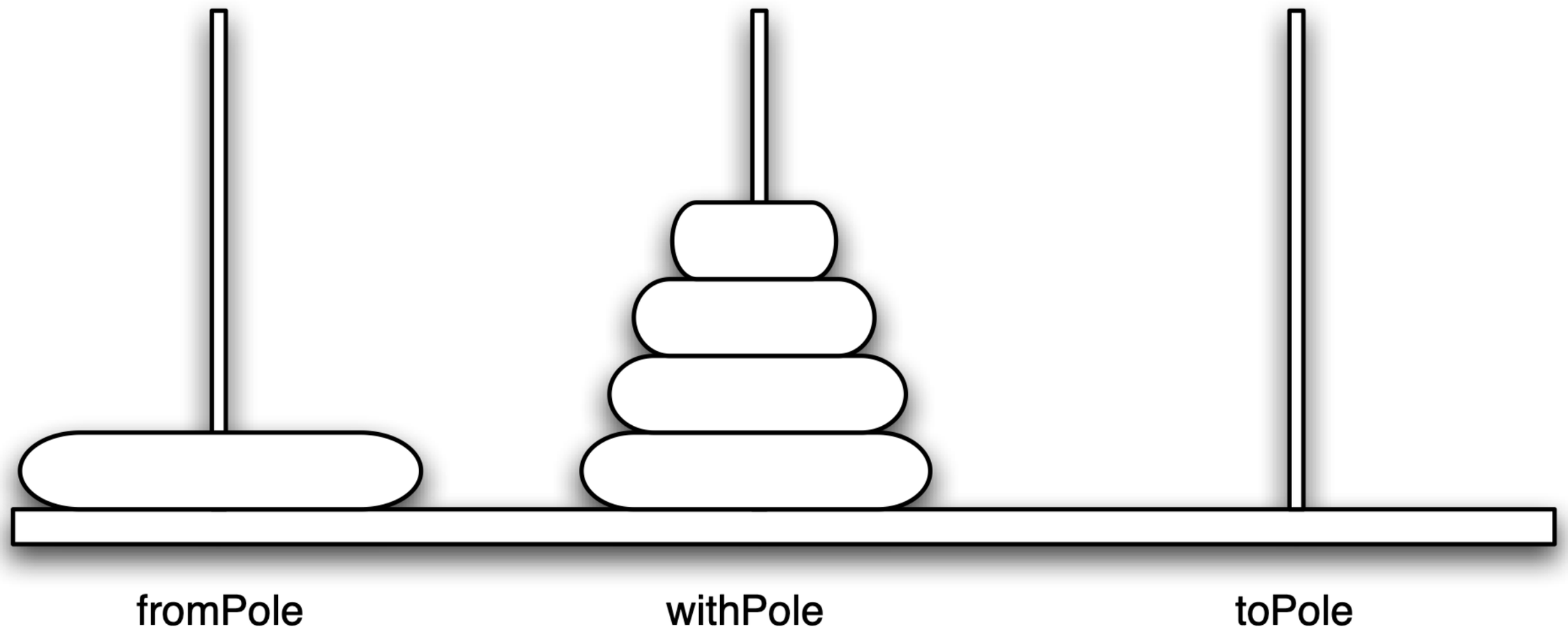
Towers of Hanoi

Background

- Tower of Hanoi (Wikipedia)
- Tower of Hanoi - 5 disks - 31 moves

Towers of Hanoi

An Example Arrangement of Disks



Towers of Hanoi

An Example Arrangement of Disks

- Move a tower of height-1 to an intermediate pole, using the final pole
- Move the remaining disk to the final pole
- Move the tower of height-1 from the intermediate pole to the final pole using the original pole

Towers of Hanoi

Python Code for the Tower of Hanoi

```
def move_tower(height, from_pole, to_pole, with_pole):
    if height >= 1:
        move_tower(height - 1, from_pole, with_pole, to_pole)
        move_disk(from_pole, to_pole)
        move_tower(height - 1, with_pole, to_pole, from_pole)

def move_disk(fp,tp):
    print("moving disk from",fp,"to",tp)

move_tower(3, "A", "B", "C")
```

Towers of Hanoi

An Iterative Version

Interesting secret: there's also an easy iterative solution, but it isn't anywhere near as intuitive

1. On every even-numbered move (starting with zero), move the little disk one pole "clockwise"

   If the total number of disks is even, the first move should be from `from_pole` to `with_pole`; if the total number of disks is odd, the first move should be from `from_pole` to `with_pole`

Towers of Hanoi

An Iterative Version

2. On every odd-numbered move, make the only legal move not involving the smallest disk (there can be only one)

Towers of Hanoi

Python Code for the Tower of Hanoi

```
def hanoi_iter(height, fromPole, toPole, withPole):
    if height % 2 == 0:
        poles = [fromPole, withPole, toPole]
    else:
        poles = [fromPole, toPole, withPole]
    stacks = [range(height, 0, -1), [height], [height]]
    for i in range(2**height-1):
        if i % 2 == 0:  # move little disk
            fd = (i//2)%3
            td = (i//2+1)%3
        else:           # move other disk
            fd = (i//2)%3
            td = (i//2+2)%3
            if (stacks[fd][len(stacks[fd])-1] >
                stacks[td][len(stacks[td])-1]):
                td = (i//2)%3
                fd = (i//2+2)%3
            stacks[td].append(list(stacks[fd]).pop())
        move_disk(poles[fd], poles[td])
```

Recursion Summary

Recursion Summary

- All recursive algorithms must have a base case
- A recursive algorithm must change its state and make progress toward the base case
- A recursive algorithm must call itself (recursively); Recursion can take the place of iteration in some cases

Recursion Summary

- Recursive algorithms often map very naturally to a formal expression of the problem you are trying to solve

Recursion Summary

- Recursion doesn't have to be any more expensive than iteration (though it is in Python)
- It's definitely more expressive: iteration can't capture recursion in the general case without an explicit stack

Questions?