Recall: Truth in a model $M = (D, I)$ for a predication $\Pi(\alpha_1, \ldots, \alpha_n)$, where $\alpha_1, \ldots, \alpha_n$ are constants:

$$\mathcal{F}_M \Pi(\alpha_1, \ldots, \alpha_n) \iff \langle \alpha_1^I, \ldots, \alpha_n^I \rangle \in \Pi^I$$

Interpreting functions & functional terms

e.g., $\text{age}^I = \{ \langle 1, 16 \rangle, \langle j, 14 \rangle, \langle 14, 14 \rangle, \langle 16, 16 \rangle \}$

In general, for $\alpha$ an $n$-place fcns. constant, let $\alpha^I : D^n \to D$, a total fcns., i.e., a single-valued relation on $D^{n+1}$ (& domain ($\alpha^I) = D^n$)

To interpret functional terms $\alpha(\beta_1, \ldots, \beta_n)$ extend $I$ to $I^\alpha$, term assignment for:

$$I^\alpha(\beta) = I(\beta) = \beta^I \text{ for } \beta \text{ a constant}$$

$$I^\alpha(\alpha(\beta_1, \ldots, \beta_n)) = \alpha^I(I^\alpha(\beta_1), \ldots, I^\alpha(\beta_n))$$

e.g., $T^\alpha(\text{age} (\text{Juliet})) = \text{age}^I(T^\alpha(\text{Juliet}))$

$$= \text{age}^I(\text{Juliet}^I)$$

$$= \{ \langle 1, 16 \rangle, \langle j, 14 \rangle, \langle 14, 14 \rangle, \langle 16, 16 \rangle \}(j)$$

$$= 14$$

e.g., $T^\alpha(\text{age}\text{ (age} (\text{Juliet})) = \text{age}^I(T^\alpha(\text{age} (\text{Juliet})))$

$$= \text{age}^I(\text{age}^I(T^\alpha(\text{Juliet})))$$

$$= \text{age}^I(\text{age}^I(\text{Juliet}^I))$$

$$= \text{age}^I(\text{age}^I(j)) = \text{age}^I(14) = 14$$

Truth conditions for equality, given $M = (D, I)$:

$$\mathcal{F}_M (\alpha = \beta) \iff T^\alpha(\alpha) = T^\alpha(\beta)$$

e.g., $T^\alpha(\text{Juliet} = \text{Juliet})$

$$\iff T^\alpha(\text{Juliet}) = T^\alpha(\text{Juliet})$$

$$\iff \text{Juliet}^I = \text{Juliet}^I \ldots \text{TRUE, because } I \text{ is a function!}$$
Truth conditions for predication \( \psi(\alpha_1, \ldots, \alpha_n) \) where \( \alpha_i \): may be functional terms:

\[
\mathcal{E}_M \psi(\alpha_1, \ldots, \alpha_n) \iff \langle T_\mathcal{I}(\alpha_1), \ldots, T_\mathcal{I}(\alpha_n) \rangle \in \psi^I
\]

E.g.

- \( \mathcal{E}_M \text{ Loves}(\text{Romeo, age(Juliet))} \) ??
  - if \( \langle T_\mathcal{I}(\text{Romeo}), T_\mathcal{I}(\text{age(Juliet)}) \rangle \in \text{Loves}^I \)
  - if \( \langle I(\text{Romeo}), \text{age}(T_\mathcal{I}(\text{Juliet})) \rangle \in \text{Loves}^I \)
  - if \( \langle r, \text{age}^I(j) \rangle \in \{ \langle r, j \rangle, \langle j, r \rangle, \langle s, b \rangle, \ldots \} \)
  - if \( \langle r, 14 \rangle \notin \{ \langle r, j \rangle \} \) \( \ldots \) 3 \( \text{FALSE} \)

Truth conditions for sentences with \( \gamma \land \chi \)

\[
\mathcal{E}_M \gamma \land \chi \iff \mathcal{E}_M \gamma \land \mathcal{E}_M \chi
\]

Consider again the FO language containing constants Snoopy, Romeo, Juliet, 14, 16;
Dog, Has-Tail; Loves; age

and the model \( (\mathcal{S}, \mathcal{I}) \), \( S = \{ s, r, j, 14, 16, \ldots \} \)

- Snoopy \( I = s \), Romeo \( I = r \), etc.
- Dog \( I = \{ s, f, \ldots \} \) but not \( r, j, 14, 16 \! \)

Loves \( I = \{ \langle r, j \rangle, \langle j, r \rangle, \langle s, b \rangle, \ldots \} \)

E.g.

\[
\mathcal{E}_M \neg \text{Dog}(\text{Romeo}) ??
\]

- if not \( \mathcal{E}_M \text{Dog}(\text{Romeo}) \) \( (\text{also written } \mathcal{F}_M) \) \( \text{by the truth cond's of } \neg \)
  - if not Romeo \( I \in \text{Dog}^I \)
  - if not \( r \in \{ s, f, \ldots \} \) \( \text{TRUE} \)

E.g.

\[
\mathcal{E}_M (\neg \text{Dog}(\text{Romeo})) \Rightarrow \text{age}(\text{Juliet}) = 16 ??
\]

- if not \( \mathcal{E}_M (\neg \text{Dog}(\text{Romeo})) \) or \( \mathcal{E}_M \text{age}(\text{Juliet}) = 16 \)
  - \( \text{FALSE} \) (see above)
- if \( \text{FALSE} \) or \( T_\mathcal{I}(\text{age(Juliet)}) = T_\mathcal{I}(16) \)
  - \( 14 \) (seen earlier) \( \Rightarrow 16 \)
  - \( \text{FALSE} \)
ALLOWING FOR VARIABLES

e.g., $P(A, x)$ ... “truth” in $\mathcal{M} = (\mathcal{D}, I)$ doesn’t quite make sense.

Introduce variable assignment function $U$

$U : \{ \text{variables of } \mathcal{FOL} \} \rightarrow \mathcal{D}$

e.g., $U(x) = r, U(y) = 16$, etc.

Extend $T_I$ to $T_{IU}$: (term assign. fn)

$T_{IU}(a) = a^I$ if $a$ is an ind. constant

$T_{IU}(x) = x^U$ —— variable

$T_{IU}(\beta_1, ..., \beta_n) = a^I(T_{IU}(\beta_1), ..., T_{IU}(\beta_n))$

e.g., $T_{IU}(\text{Romeo}) = \text{Romeo}^I = r$

$T_{IU}(x) = x^U = r$

$T_{IU}(\text{age(\text{Romeo})}) = \text{age}^I(T_{IU}(\text{Romeo})) = \text{age}^I(r) = 16$

$T_{IU}(\text{age}(y)) = \text{age}^I(T_{IU}(y)) = \text{age}^I(y^U) = \text{age}^I(16) = 16$

For formulas involving variables, we speak of satisfaction conditions rather than truth conditions (but we’ll come back to truth soon!)

Satisfaction conditions for equality and predication

$\text{Fm } (a = b) [U] \quad \text{"v.a. } U \text{ satisfies } (a = b) \text{ relative to model } \mathcal{M}\"

iff $T_{IU}(a) = T_{IU}(b)$

$\text{Fm } \Pi(x, ..., x_n) [U] \quad \text{"v.a. } U \text{ satisfies } \Pi(x, ..., x_n) \text{ relative to model } \mathcal{M}\"

iff $\langle T_{IU}(a), ..., T_{IU}(x_n) \rangle \in \Pi^I$

e.g., for the Snoopy + Romeo & Juliet model $\mathcal{M}$,

and $U(x) = r, U(y) = 16$

$\text{Fm Loves}(\text{Romeo}, x) [U]$

iff $\langle T_{IU}(\text{Romeo}), T_{IU}(x) \rangle \in \text{Loves}^I$

iff $\langle r, r \rangle \in \text{Loves}^I$, which is FALSE
Satisfaction conditions for $\forall, \exists, \land, \rightarrow$:

$\neg \psi[U]$ iff not $\psi[U]$

$\psi \land \psi'[U]$ iff $\psi[U]$ and $\psi'[U]$

$\neg \psi[U]$ etc

Satisfaction conditions for $\forall$

$\psi[U]$ iff for all $u$'s $U'$ identical with $U$ except possibly that $U'(u) \neq U(u)$, $\psi[U']$

Writing $U_{x:d}$ for the assignment function that is the same as $U$ except that $U_{x:d}(x) = d$, we can equivalently say (with $M = (\mathcal{D}, I)$):

$\psi[U]$ iff for all $d \in \mathcal{D}$, $\psi[U_{x:d}]$.

Note: Since $U_{x:d}(x) = d$ by def of $U_{x:d}$,

$T_{U_{x:d}}(x) = U_{x:d}(x)$ by def of $T_I$.

$= d$

e.g., $M = (\mathcal{D}, I)$

$A^I = a$

$B^I = b$

$P^I = \{a, b\}$

$Q^I = \{c\}$

$U = \text{any } u \text{ we like, for at least one variable } x$

$\forall x \ P(x)[U]$ ??

iff for all $d \in \mathcal{D}$, $P_{x:d}(x) \in \mathcal{P} = \{a, b\}$

iff for all $d \in \{a, b\}$, $P_{x:d}(x) \in \mathcal{P} = \{a, b\}$

iff for all $d \in \{a, b\}$, $U_{x:d}(x) \in \{a, b\}$

iff for all $d \in \{a, b\}$, $d \in \mathcal{P}$

TRUE

Note: it didn't matter what $U$ is!
e.g., $\text{Em}\ (\forall x (Q(x) \Rightarrow P(x))[U] ??

iff for all $d \in D$, $\text{Em}\ (Q(x) \Rightarrow P(x))[U_{x:d}]$

iff for all $d \in D$, not $\text{Em}\ Q(x)[U_{x:d}]$ or $\text{Em}\ P(x)[U_{x:d}]$

iff for all $d \in \{a, b, 3\}$, not $d \in Q = \{a, 3\}$ or $d \in P = \{a, b\}$

TRUE

Note: $U$ didn't matter!
(Generally true for closed wffs)

**Satisfaction conditions for $\exists$**

$\text{Em}\ (\exists x (\psi)) [U]$ iff for some $d \in D$, $\text{Em}\ \psi[U_{x:d}]$

E.g., $M$, $U$ as above,

$\text{Em}\ (\exists x Q(x))[U] ??$

iff for some $d \in D$, $\text{Em}\ Q(x)[U_{x:d}]$

iff for some $d \in \{a, b, 3\}$, $T_{U_{x:d}}(x) \in Q = \{a, 3\}$

iff for some $d \in \{a, b, 3\}$, $d \in \{a, 3\}$ TRUE

(use $d = a$)

Now redefine **truth** (at least for wffs involving variables) in terms
of satisfaction:

$\text{Em} \varphi \iff$ for all v.a.'s $U$, $\text{Em} \varphi[U]$

E.g., in the previous model, we found

$\text{Em} (\forall x P(x))[U]$

$\text{Em} (\forall x (Q(x) \Rightarrow P(x))[U]$

$\text{Em} (\exists x Q(x))[U]$

in each case independently of the choice of $U$. So for this $M$ we have

$\text{Em} (\forall x P(x))$

$\text{Em} (\forall x (Q(x) \Rightarrow P(x))$

$\text{Em} (\exists x Q(x))$